Типичные перестройки неявных обыкновенных дифференциальных уравнений

Typical transitions of implicit ODEs

Ilya Bogaevsky

Moscow State University

International conference “ANALYSIS and SINGULARITIES”
dedicated to the 75th anniversary of Vladimir Igorevich Arnold
Moscow, Russia, December 17—21, 2012

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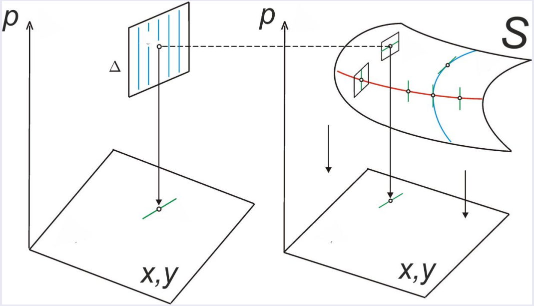

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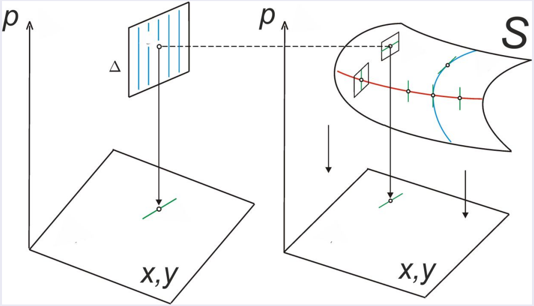

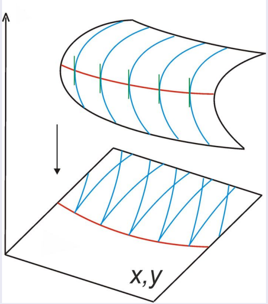

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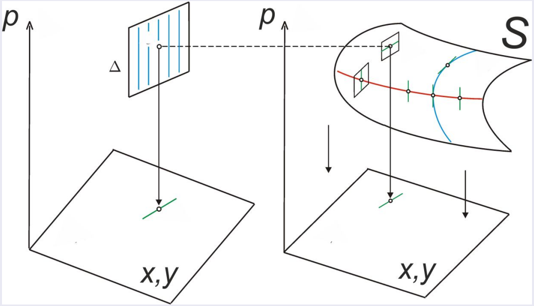

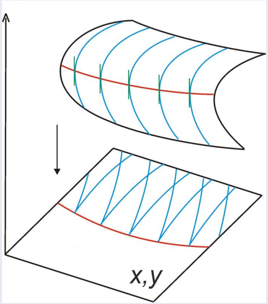

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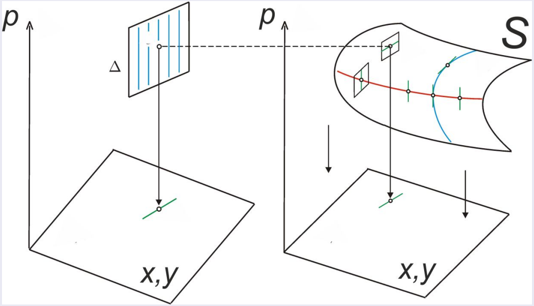

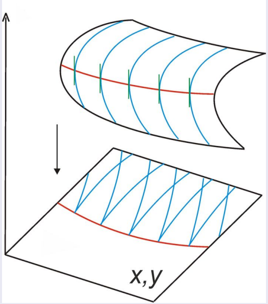

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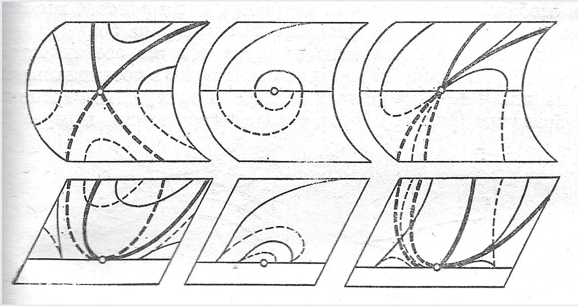
Folded saddle, folded focus, and folded node.



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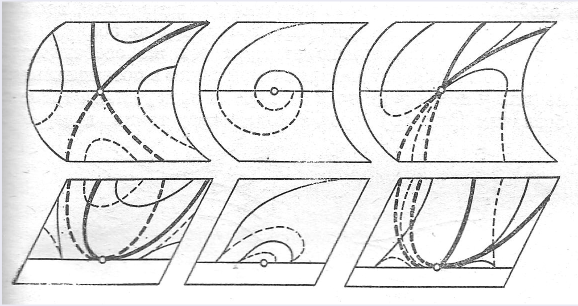
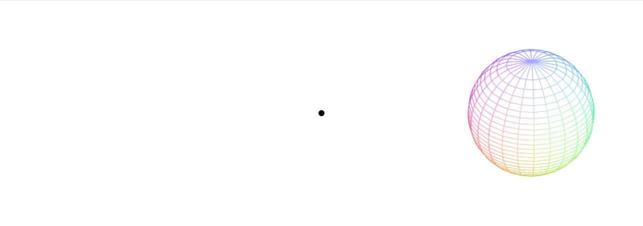
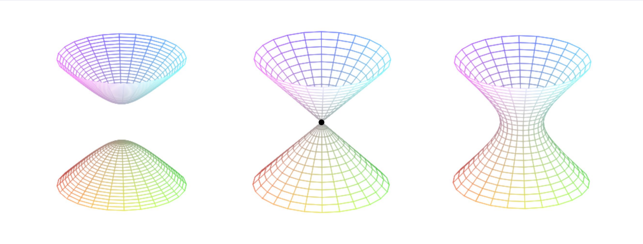


Normal forms: $y'^2 = y - kx^2$ where k is a continuous invariant.

Transitions of S:

The following 2 transitions occur in a typical smooth family of surfaces $S_\varepsilon = \{F(p, x, y, \varepsilon) = 0\}$ depending on a parameter $\varepsilon \in \mathbb{R}$ (if F is smooth and generic):





Main result:

Let

$$F(p,x,y,\varepsilon)=0,\quad p=\frac{dy}{dx},\quad F\in\mathbb{R}[[p,x,y,\varepsilon]]$$

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Then it is reduced to one of the following 4 normal forms:

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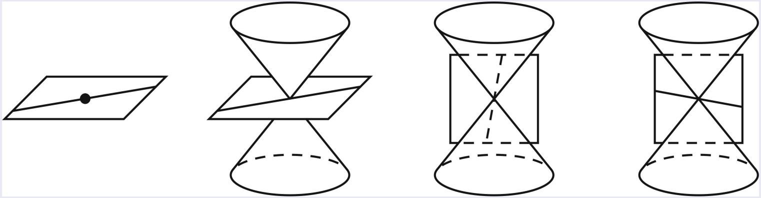
where $c\in\mathbb{R}[[\varepsilon]]$ is a functional invariant.

The 4 cases:



The surface $F(p,x,y,0)=0$, the contact plane, and the kernel of $\pi:(p,x,y)\mapsto(x,y)$.

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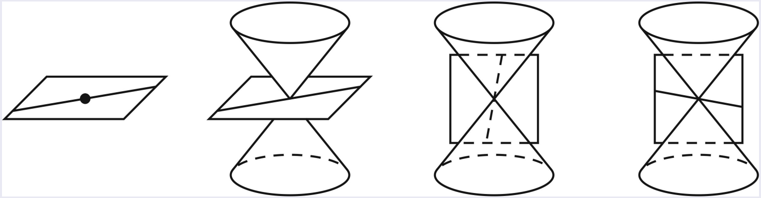

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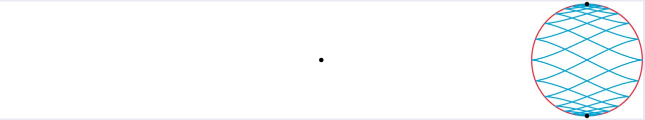
Symmetric normal forms:

$$\pm p^2 \pm x^2 + z^2 = \varepsilon + c(\varepsilon)z^3, \quad z = y - \frac{1}{2}px,$$

$$p\,dx - dy = \frac{1}{2}p\,dx - \frac{1}{2}x\,dp - dz$$

Transition $p^2 + x^2 + z^2 = \varepsilon$



$\varepsilon < 0$ $\varepsilon = 0$ $\varepsilon > 0$

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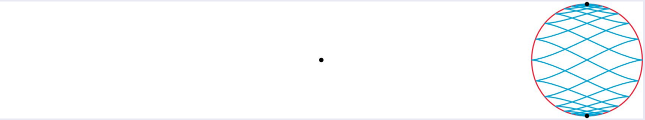

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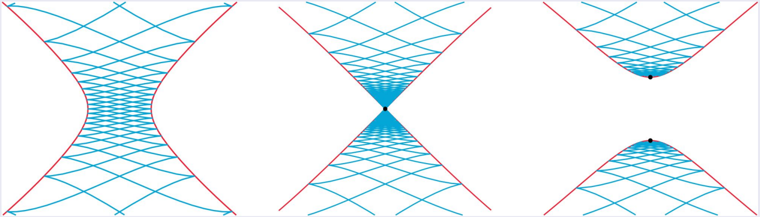

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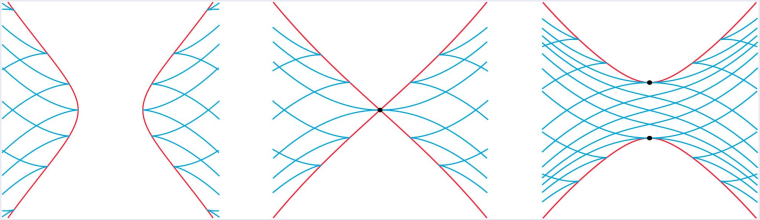

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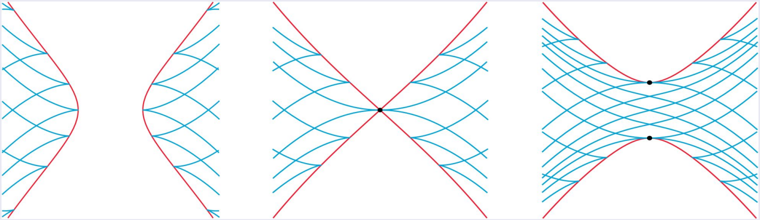

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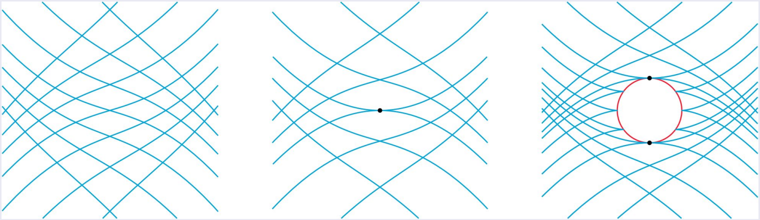

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Theorem (V. I. Arnold, Math. Notes, 1988, 44:1, 489–497, Remark 3)

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be a typical formal surface depending on a parameter $\varepsilon\in\mathbb{R}$ *such that* $\partial_p F(0)=\partial_x F(0)=\partial_y F(0)=0$ *and the quadratic part of* $F(p,x,z,0)$ *is a non-degenerate indefinite form of the variables* p, x, z.

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Contact $=$ *preserves* $\Delta=\left\{\frac{1}{2}p\,dx-\frac{1}{2}x\,dp-dz\right\}$.

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The case of sphere:

If the quadratic part of $F(p, x, z, 0)$ is non-degenerate definite form of the variables p, x, z then the family $F(p, x, z, \varepsilon) = 0$ is reduced to the following normal form:

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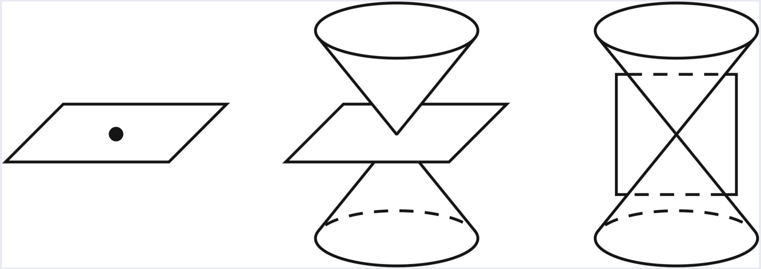
3 Arnold's cases:



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$$\mathcal{E}_{p,x,y} = \mathcal{E}_{x,y} + p\mathcal{E}_{x,y} + I_\Lambda$$

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- $(p^2) \subseteq I_\Lambda|_{\pi^{-1}(0)}$ is a criterion in one-dimensional case.

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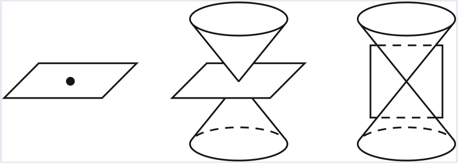
Application to an implicit ODE:

If π_S is a fold then π is S-stable. The contact structure can define any direction field on S! (As degenerate as we want.)

Proof of the main result:



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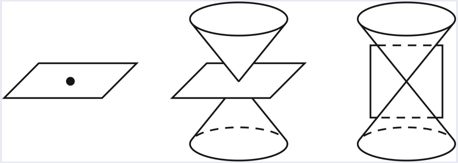

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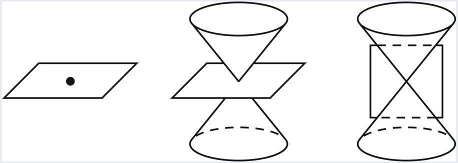

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2. In the second case the germs of all Legendre fibrations are stable and form one connected component. So all germs are equivalent.
3. In the third case the stable germs form two connected components. All germs from each connected component are equivalent.