Quadratic Cohomology

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Systems of inequalities.

Let M be a smooth compact manifold, $\varphi : M \to \mathbb{R}^{k+1}$ a C^1-vector function and $K \subset \mathbb{R}^{k+1}$ a closed convex cone. A system of inequalities is a relation $\varphi(x) \in K,\ x \in M$. We say that the system of inequalities is *regular* if $\mathrm{im} D_x\varphi + K = \mathbb{R}^{k+1},\ \forall\, x \in \varphi^{-1}(K)$.

We take the dual cone $K^\circ = \{p \in \mathbb{R}^{k+1} : \langle p, y\rangle \leq 0,\ \forall\, y \in K\}$ and consider the "manifold with a convex border" $(K^\circ \cap S^k) \times M$.

We say that a subset V of a smooth manifold is a *manifold with a convex border* if V is covered by coordinate neighborhoods whose intersections with V are diffeomorphic to closed convex sets.

Let $f : V \to \mathbb{R}$ be a C^1 function. We say that $v \in V$ is a critical point of f if $\langle d_v f, \xi \rangle \leq 0,\ \forall \xi \in T_v V$.

Now consider the function $\varphi^* : S^k \times M \to \mathbb{R}$ defined by the formula $\varphi^*(p, x) = \langle p, \varphi(x) \rangle$, where $p \in K^\circ \cap S^k$.

Lemma 1. *If the system of inequalities $\varphi(x) \in K$ is regular, then 0 is not a critical point of φ^*.*

A function $f \in C^1(V)$ is *regular* if 0 is not a critical value of f. A homotopy $f_t,\ t \in [0, 1]$, such that all f_t are regular is a *regular homotopy*.

Proposition. *Assume that* $f_t : (K^\circ \cap S^k) \times M \to \mathbb{R},\ t \in [0,1]$, *is a regular homotopy and* $f_0 = \varphi_0^*\big|_{(K^\circ \cap S^k)\times M}$, $f_1 = \varphi_1^*\big|_{(K^\circ \cap S^k)\times M}$. *Then* $M \setminus \varphi_0^{-1}(K)$ *is homotopy equivalent to* $M \setminus \varphi_1^{-1}(K)$.

Proof. We set

$$B_t = \Big\{(p,x) \in (K^\circ \cap S^k) \times M : f_t(p,x) > 0\Big\}.$$

Note that the projections $(p,x) \mapsto x$ restricted to B_0 and B_1 are fiber bundles over $M \setminus \varphi_0^{-1}(K)$ and $M \setminus \varphi_1^{-1}(K)$ whose fibers are hemispheres. In particular, B_0 is homotopy equivalent to $M \setminus \varphi_0^{-1}(K)$ and B_1 is homotopy equivalent to $M \setminus \varphi_1^{-1}(K)$.

Lemma 2. *There exists a smooth family of diffeomorphisms* $F_t : S^k \times M \to S^k \times M$ *such that* $F_0 = id, F_t(B_0) \subset B_t,\ \forall\, t \in [0,1]$.

Proposition. *Assume that* $f_t : (K^\circ \cap S^k) \times M \to \mathbb{R},\ t \in [0,1]$, *is a regular homotopy and* $f_0 = \varphi_0^*\big|_{(K^\circ \cap S^k)\times M}$, $f_1 = \varphi_1^*\big|_{(K^\circ \cap S^k)\times M}$. *Then the homology groups of* $\varphi_0^{-1}(K)$ *and* $\varphi_1^{-1}(K)$ *with coefficients in a field are isomorphic.*

Proof. Assume that $K \neq -K$, i. e. not all inequalities are equations (otherwise we add the tautological inequality $1 \geq 0$). Then $K^\circ \cap S^k$ is contractible and we have the following series of homotopy equivalences of the pairs:

$$\Big(M, M \setminus \varphi_0^{-1}(K)\Big) \sim \Big((K^\circ \cap S^k) \times M, B_0\Big) \sim$$

$$\Big((K^\circ \cap S^k) \times M, B_1\Big) \sim \Big(M, M \setminus \varphi_1^{-1}(K)\Big),$$

where $B_t = \Big\{(p, x) \in (K^\circ \cap S^k) \times M : f_t(p, x) > 0\Big\}$. Hence $H^*(M, M \setminus \varphi_0^{-1}(K)) \cong H^*(M, M \setminus \varphi_1^{-1}(K))$. The Alexander duality completes the proof.

Localization.

Let V be a manifold with a convex border and $f : V \times M \to \mathbb{R}$ a C^1-function. Assume that M is a real-analytic manifold and $f_v \doteq f(v, \cdot)$ is a subanalytic function, $\forall\, v \in V$.

Proposition. *Assume that the family f_v, $v \in V$, is regular at $v_0 \in V$. Then v_0 has a compact neighborhood O_{v_0} and centered at v_0 local coordinates Φ such that $U_0 \doteq \Phi(O_{v_0})$ is convex and the function $(f \circ \Phi + t)\big|_{\varepsilon U_0 \times M}$ is regular for any sufficiently small nonnegative constants t, ε one of which is strictly positive.*

A cohomology theory.

Let $\mathcal{A} \subset C^1(M)$ be a set of subanalytic functions, $W \subset V$ a pair of manifolds with convex borders, and $f : V \times M \to \mathbb{R}$ a regular function such that $f_v \in \mathcal{A},\ \forall\, v \in V$, and $f\Big|_{W \times M}$ is also regular.

We set $B_f = \{(v,x) : v \in V,\ f(v,x) > 0\}$ and define

$$H^{\cdot}_{\mathcal{A}}(f_V, f_W) \doteq H^{\cdot}\left(V \times M, (W \times M) \cup B_f\right), \qquad H^{\cdot}_{\mathcal{A}}(f) \doteq H^{\cdot}_{\mathcal{A}}(f_V . f_\emptyset).$$

The pairs of regular functions (f_V, f_W) form a category $\mathfrak{F}_{\mathcal{A}}$ with morphisms $\varphi^* : (f^0_{V_0}, f^0_{W_0}) \mapsto (f^1_{V_1}, f^1_{W_1})$, where $\varphi : V_1 \to V_0$ is a C^1-map such that $\varphi(W_1) \subset W_0$ and $f^1_v = f^0_{\varphi(v)},\ \forall\, v \in V_1$.

Then $H^{\cdot}_{\mathcal{A}}$ is a functor from this category to the category of commutative groups.

This is a kind of cohomology functor which satisfies natural modifications of the Steenrod–Eilenberg axioms. The exactness and excision are obvious. Homotopy axiom deals with $f : [0,1] \times V \times M \to \mathbb{R}$ such that $f_{\{t\}\times V} \in \mathfrak{F}_{\mathcal{A}}, \forall\, t \in [0,1]$, and claims that the inclusions $\{t\} \times V \hookrightarrow [0,1] \times V,\ t \in [0,1]$, induce the isomorphisms of cohomology groups:

$$H^{\cdot}_{\mathcal{A}}\left(f_{[0,1]\times V}, f_{[0,1]\times W}\right) \cong H^{\cdot}_{\mathcal{A}}\left(f_{\{t\}\times V}, f_{\{t\}\times W})\right).$$

The cohomology of a "point": if $V = \{v\}$ then

$$H^i_{\mathcal{A}}\left(f_{\{v\}}\right) = H^i\left(M, \{x \in M : f_v(x) > 0\}\right) = H_{n-i}\left(\{x \in M : f_v(x) \le 0\}\right),$$

$0 \le i \le n$, where $n = \dim M$.

The "points" for us are regular elements of $\mathcal{A}$ and different points may have different cohomology.

Now assume that $\mathcal{A} + t \subset \mathcal{A}$ for any nonnegative constant t. The localization implies:

$$H^{\cdot}_{\mathcal{A}}\left(f_{U_v}\right) \cong H^{\cdot}_{\mathcal{A}}\left(f_{\{v\}} + \varepsilon\right)$$

for small $U_v \ni v$, $\varepsilon > 0$. In other words, cohomology of a "small neighborhood" is equal to the cohomology of a "point".

Let $M = \mathbb{R}P^N = \{(x, -x) : x \in S^k\}$ and $\mathcal{Q}$ the space of real quadratic forms on $\mathbb{R}^{N+1}$ treated as functions on $\mathbb{R}P^N$.

Let $\lambda_1(q) \geq \cdots \geq \lambda_{N+1}(q)$ be the eigenvalues of the symmetric operator associated to the quadratic form $q \in \mathcal{Q}$. We set

$$\Lambda_{j,m} = \{q \in \mathcal{Q} : \lambda_{j-1}(q) \neq \lambda_j(q) = \lambda_{j+m-1}(q) \neq \lambda_{j+m}(q)\};$$

then $\Lambda_{i,m}$ is a smooth submanifold of codimension $\frac{m(m+1)}{2} - 1$ in $\mathcal{Q}$.

We say that the pair $(f_V, f_W) \in \mathfrak{F}_{\mathcal{Q}}$ is in the general position if the borders $\partial V, \partial W$ are smooth and the map $v \mapsto f_v,\ v \in V$, as well as the restrictions of this map to W, ∂V, ∂W are transversal to $\Lambda_{j,m}$.

We set:

$$V_f^j = \{v \in V : \lambda_j(f_v) > 0\}, \quad C^n(f) = \bigoplus_{i+j=n} C^i(V, V_f^{j+1}),$$

where C^i is the group of singular cochains of dimension i.

Besides the standard coboundary $\delta : C^i \to C^{i+1}$ we consider a differential:

$$\Delta : C^i(V, V_f^{j+1}) \to C^{i+2}(V, V_f^j), \quad \Delta(\xi) = \xi \smile \ell_f^j,$$

where ℓ_f^j is the Poincaré dual cocycle to the cycle $\{v \in V : f_v \in \Lambda_{j,2}\}$. Then

$$\ldots \to C^{n-1}(f) \xrightarrow{\delta+\Delta} C^n(f) \xrightarrow{\delta+\Delta} C^{n+1}(f) \to \ldots$$

is a cochain complex.

Relative cochain groups $C^n(f_V, f_W) = \bigoplus_{i+j=n} C^i(V, V_f^{j+1}) \cap C^i(V, W)$ equipped with the same differential $\delta + \Delta$ define a cohomology $\widehat{H}_{\mathcal{Q}}(f_V, f_W)$ that satisfy all the axioms.

Consider a filtration of the complex $\bigoplus_{n \geq 0} C^n(f)$ by subcomplexes $\bigoplus_{i \geq \alpha, j \geq 0} C^i_{j+1}(f)$, $\alpha = 0, 1, \ldots$ and the spectral sequence $E^r_{i,j}$ of this filtration converging to $\widehat{H}_{\mathcal{Q}}(f)$. We have:

$$E^2_{ij} = H^i(V, V_f^{j+1}), \quad d_2 : \xi \mapsto \xi \smile \overline{\ell}^j_f.$$

Moreover, $d_r : \bar{\xi} \mapsto \langle \bar{\xi}, \overline{\ell}^j_f, \ldots, \overline{\ell}^{j-r+2}_f \rangle$ are Massey operations for $r \geq 3$.

Assume that $\dim V = 3$, $H_1(V;\mathbb{Z}_2) = 0$ and ∂V is connected or empty. We have:

$$d_3 : H^0(V, V_f^{j+1}) \longrightarrow H^3(V, V_f^{j-1}).$$

Moreover, ranks of $H^0(V, V_f^{j+1})$ and $H^3(V, V_f^{j-1})$ are either one or zero.

If both ranks are equal to one, then d_3 sends the generator of $H^0(V, V_f^{j+1})$ to the generator of $H^3(V, V_f^{j-1})$ multiplied by the linking number of 1-dimensional submanifolds $f^{-1}(\Lambda_{j,2})$ and $f^{-1}(\Lambda_{j-1,2})$