Trigonometric Sums in Number Theory and Analysis

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International Conference
devoted 75 Anniversary of V.I.Arnol'd,
Moscow, December 21th, 2012

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Introduction I

By trigonometric sum we mean a sum of the form

$$S = \sum_{x \in \Omega} e^{2\pi i f(x)},$$

where $f(x)$ is a real function of one or several variables defined on a discrete set Ω. When a real function is a polynomial of one variable such sums are called *Weyl sums*. The history of Weyl sums, the methods available for studying and some results are presented in I.M.Vinogradov's monograph [2] and in L.-K.Hua's review [6]. The general formulation of the problem of estimating Weyl sums due to I.M.Vinogradov. Such sums has the form

$$S = \sum_{x=1}^{P} e^{2\pi i f(x)},$$

where $f(x) = \alpha_1 x + \cdots + \alpha_n x^n$ is a polynomial with real coefficients.

Introduction II

It is easy to see that S is a periodic function with period 1 to each coefficient of $f(x)$. Hence it suffices to study S on the n-dimensional unit cube $0 \leq \alpha_1, \ldots, \alpha_n < 1$, which we denote by E. All points of E are divided into two sets E_1 and E_2.
At each point of E_1 I.M.Vinogradov obtains estimates for sum S, and in many cases these estimates are best possible. The set E_1 itself consists of intervals and its measure is small:

$$\mathrm{meas}\, E_1 \ll P^{-\frac{n(n+1)}{2}+1+\frac{2}{n}}.$$

At each point of E_2 there is an estimate for S which is uniform in the domain E_2 and has the form

$$S \ll P^{1-\rho}, \rho = \rho(n) = \frac{c}{n^2 \ln n}, c > 0;$$

the set E_2 has measure $1 - \mathrm{meas}\, E_2$.

I.M.Vinogradov's problems in the theory of numbers I

The first I.M.Vinogradov problem in the theory of trigonometric sums is that of obtaining an upper bound for the modulus of the sum $\sum\limits_{x\in\Omega} e^{2\pi i f(x)}$.

Closely related with this bound is the problem of the distribution of values of the fractional part of a real function $f(x) = f(x_1, \ldots, x_r)$ on a discrete set Ω.

At the last, it is the problem of the distribution of values of a function $f(x) = f(x_1, \ldots, x_r)$ taking integer values on Ω.

We note that, in general, these problems, formulated by Vinogradov (see [2],Introduction) are of interest only in the case where $f(x)$ and Ω have some arithmetic properties. Choosing the function $f(x)$ and the domain Ω approximately, we arrive at the problems Goldbach, Waring, Goldbach – Waring – Vinogradov, Hilbert – Kamke, the problem of estimating of Weyl's sums, etc.

I.M.Vinogradov's problems in the theory of numbers II

Problems related to multiple trigonometric sums over prime numbers belong to the first class of Vinogradov's problems. More precisely, if we take for Ω the set of points $(p_1, \ldots, p_r)$ with coordinates $p_s, 1 \leq s \leq r$, running independently over the set consecutive primes, and for $f(p_1, \ldots, p_r)$ a polynomial with arbitrary real coefficients, we arrive at the problem of obtaining an upper bound for the modules of a multiple trigonometric sum with prime numbers, i.e., of a sum of the form

$$S = S(A) = \sum_{p_1 \leq P_1} \cdots \sum_{p_r l, x_r eq P_r} e^{2\pi i f(p_1, \ldots, p_r)},$$

where A is a point with real coordinates $\alpha(t_1, \ldots, t_r)$ in the m-dimensional space, $m = (n_1 + 1) \ldots (n_r + 1), P_1, \ldots, P_r \geq 1$,

$$f(\bar{x}) = f(x_1, \ldots, x_r) = \sum_{t_1=0}^{n_1} \cdots \sum_{t_r=0}^{n_r} \alpha(t_1, \ldots, t_r) x_1^{t_1} \ldots x_r^{t_r}.$$

I.M.Vinogradov's problems in the theory of numbers III

For $r = 1$ the sum S becomes a simple trigonometric sum with prime numbers. We note that getting estimates for such sums even in the case of a linear polynomial $f(p) = \alpha p$ is connected with great difficulties. For the first time such estimates were Vinogradov in 1937 in [3], this allowed him to solve the Goldbach problem. The problem of finding bounds for sums with arbitrary polynomial $f(p)$ of higher degree is substantially more complicated. Its complete solution due to I.M. Vinogradov. Combination of Vinogradov's method for estimating sums with prime numbers and the theory of multiple trigonometric sums, created by G.I. Arkhipov, A.A. Karatsuba and V.N. Chubarikov [1], has made it possible to obtain estimates of multiple trigonometric sums with primes [5]. The basis of our theory of multiple trigonometric sums is the mean value theorem, i.e., the theorem that estimates the integral

$$J = J(\bar{P}; \bar{n}, k, r) = \int_0^1 \dots \int_0^1 \left| \sum_{x_1=1}^{P_1} \cdots \sum_{x_r=1}^{P_r} e^{2\pi i f_A(x_1,\dots,x_r)} \right|^{2k} dA.$$

I.M.Vinogradov's problems in the theory of numbers IV

The first theorem in multiple case got G.I. Arkhipov [4]. We give the simple version the theorem on the mean value of the $2k$th power of the modulus of an r-fold trigonometric sum of the form

$$S(A) = \sum_{x_1=1}^{P} \cdots \sum_{x_r=1}^{P} e^{2\pi i f_A(x_1,\ldots,x_r)}.$$

Theorem 1.1. *Suppose that $\tau \geq 0$ is an integer, $k \geq m\tau$, and $P \geq 1$. Then the following estimate holds for $J = J(P; n, k, \tau)$,*

$$J = \int_0^1 \ldots \int_0^1 |S(A)|^{2k}\, dA \leq k^{2m\tau} 4^{mr^2 n\tau} (nr)^{2nr\Delta(\tau)} P^{2rk-0.5rmn+\delta(\tau)},$$

where
$m = (n+1)^r, \delta(\tau) = 0.5rnm(1-1/(rn))^\tau, \Delta(\tau) = 0.5rnm - \delta(\tau).$

Singular series and integrals in Tarry's problem I

The history of Tarry's problem is discussed sufficiently complete in Hua's review [6]. This problem is connected with the number of solutions $J = J(P; k, n)$ of the following system of equations

$$\begin{cases} x_1 + \cdots + x_k = y_1 + \cdots + y_k, \\ x_1^2 + \cdots + x_k^2 = y_1^2 + \cdots + y_k^2, \\ \ldots \qquad \ldots \qquad \ldots \\ x_1^n + \cdots + x_k^n = y_1^n + \cdots + y_k^n, \end{cases}$$

where the unknown variables $x_1, \ldots, x_k, y_1, \ldots, y_k$ are integers ranging from 1 to $P > 1$.

Hua L.-K. derived the following asymptotic formula for J as $P \to \infty$:

$$J = \sigma\theta P^{2k-0.5(n^2+n)} + O\left(P^{2k-0.5(n^2+n)-\varepsilon}\right),$$

where $\varepsilon = \varepsilon(n, k) > 0$, k is of order $n^2 \ln n$,

Singular series and integrals in Tarry's problem II

a singular series is

$$\sigma = \sum_{q_n=1}^{\infty} \cdots \sum_{q_1=1}^{\infty} \sum_{\substack{a_n=1\\(a_n,q_n)=1}}^{q_n} \cdots \sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1} |S(\bar{a}/\bar{q})|^{2k},$$

$$S(\bar{a}/\bar{q}) = \frac{1}{q}\sum_{x=1}^{q} \exp 2\pi i \left(\frac{a_n}{q_n}x^n + \cdots + \frac{a_1}{q_1}x\right), q = q_n \dots q_1;$$

a singular integral is

$$\theta = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \left| \int_0^1 \exp 2\pi i(\alpha_n x^n + \cdots + \alpha_1 x) \right|^{2k} d\alpha_n \dots d\alpha_1.$$

Singular series and integrals in Tarry's problem III

Hua L.-K. found the convergence exponent of the singular series σ.
Theorem 2.1. *The singular series σ converges for* $2k > 0.5n(n+1)+2$ *and diverges for* $2k \leq 0.5n(n+1)+2$. He proved that the singular integral converges for $2k > 0.5n^2+n$.
G.I. Arkhipov, A.A. Karatsuba and V.N.Chubarikov found the convergence exponent of the singular integral θ.
Theorem 2.2. *The singular integral θ converges for* $2k > 0.5n(n+1)+1$ *and diverges for* $2k \leq 0.5n(n+1)+1$.

Singular series and integrals in Tarry's problem IV

The proof of the last theorem is based on following statement.
Theorem 2.3. *Suppose that $n \geq 1, \alpha_1, \dots, \alpha_n$ are a real numbers*

$$f(x) = \alpha_n x^n + \dots + \alpha_1 x, \quad \beta_r(x) = f^{(r)}(x)/r!, \quad r = 1, \dots, n,$$

$$H = H(\alpha_n, \dots, \alpha_1) = \min_{a \leq x \leq b} \sum_{r=1}^{n} |\beta_r(x)|^{1/r}.$$

Then the integral

$$I = \int_a^b e^{2\pi i f(x)}\, dx$$

satisfies the inequality

$$|I| \leq \min\left(b - a, 6en^3 H^{-1}\right).$$

Singular series and integrals in Tarry's problem V

We let $F_A(\bar{x}) = F_A(x_1, \ldots, x_r)$ denote a polynomial of the form

$$F_A(\bar{x}) = \sum_{t_1=0}^{n_1} \cdots \sum_{t_r=0}^{n_r} \alpha(t_1, \ldots, t_r) x_1^{t_1} \ldots x_r^{t_r},$$

where $\alpha(t_1, \ldots, t_r)$ are real numbers, the monomials $\alpha(t_1, \ldots, t_r) x_1^{t_1} \ldots x_r^{t_r}$ are arranged in ascending order of numbers $t_1 + (n_1 + 1)t_2 + \cdots + (n_1 + 1) \ldots (n_{r-1} + 1)t_r$, and A is the vector in m-dimensional Euclidean space whose coordinates are the coefficients of the the polynomial $F_A(\bar{x}), m = (n_1 + 1) \ldots (n_r + 1)$.

Singular series and integrals in Tarry's problem VI

Let

$$S(A) = \sum_{1 \le x_1 \le P_1} \cdots \sum_{1 \le x_r \le P_r} \exp 2\pi i F_A(\bar{x}), 1 < P_1 = \min(P_1, \ldots, P_r),$$

and E denote the unit m-dimensional cube of the form $0 \le \alpha(t_1, \ldots, t_r) < 1, 0 \le t_1 \le n_1, \ldots, 0 \le t_r \le n_r$.
Then the mean value $J = J(\bar{P}; n, k, r)$ of the $2k$th power of the modulus of a multiple trigonometric sum $S(A)$ is equal to the number of solutions of the system of Diophantine equations of the form

$$\sum_{j=1}^{2k} (-1)^j x_{1j}^{t_1} \ldots x_{rj}^{t_r} = 0, 0 \le t_1 \le n_1, \ldots, 0 \le t_r \le n_r,$$

where the unknowns vary within the limits

$$1 \le x_{1j} \le P_1, \ldots, 1 \le x_{rj} \le P_r, j = 1, \ldots, 2k.$$

Singular series and integrals in Tarry's problem VII

Let $\nu_1, \ldots, \nu_r$ are natural numbers such that

$$-1 < \frac{\ln P_s}{\ln P_1} - \nu_s \leq 0, s = 1, \ldots, r,$$

$$\varkappa = n_1\nu_1 + \ldots n_r\nu_r, \gamma\varkappa = 1, P = (P_1^{n_1} \ldots P_r^{n_r})^{\gamma}.$$

Theorem 2.4. *Let $k \geq 4m\varkappa \ln 16m\varkappa$. Then the following asymptotic formula holds:*

$$J = \sigma_r\theta_r(P_1 \ldots P_r)^{2k} P^{-0.5m\varkappa} + O\left(e^a(P_1 \ldots P_r)^{2k} P^{-0.5m\varkappa-\rho}\right),$$

where
$\rho = (32\varkappa \ln m\varkappa)^{-1}$, $\quad a = 64m\varkappa^2 \ln 16m\varkappa + 32m\varkappa \ln^2 16m\varkappa$, *and the constant in the sign O depends only on $\bar{n}$ and r.*
The singular integral of this asymptotical formula is

$$\theta_r = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \left| \int_0^1 \ldots \int_0^1 e^{2\pi i F_A(\bar{x})}\, dx_1 \ldots dx_r \right|^{2k} dA,$$

Singular series and integrals in Tarry's problem VIII

the singular series is

$$\sigma_r = \sum_{q(0,\ldots,1)=1}^{+\infty} \cdots \sum_{q(n_1,\ldots,n_r)=1}^{+\infty} 1 \times$$

$$\times \sum_{\substack{a(0,\ldots,1)=1 \\ (a(0,\ldots,1),q(0,\ldots,1))=1}}^{q(0,\ldots,1)} \cdots \sum_{\substack{a(n_1,\ldots,n_r)=1 \\ (a(n_1,\ldots,n_r),q(n_1,\ldots,n_r))=1}}^{q(n_1,\ldots,n_r)} |U(\bar{a},\bar{q})|^{2k},$$

$$U(\bar{a},\bar{q}) = q^{-r} \sum_{x_1=1}^{q} \cdots \sum_{x_r=1}^{q} e^{2\pi i \Phi(\bar{x})}, q = q(0,\ldots,1)\ldots q(n_1,\ldots,n_r),$$

$$\Phi(\bar{x}) = \sum_{t_1=0}^{n_1} \cdots \sum_{t_r=0}^{n_r} \frac{a(t_1,\ldots,t_r)}{q(t_1,\ldots,t_r)} x_1^{t_1} \ldots x_r^{t_r}.$$

Singular series and integrals in Tarry's problem IX

We put $n_1 = \cdots = n_r = n$. In 2006 we prove [7]
Theorem 2.5. *The singular integral* θ_r *converges for* $2k > n(n+1)^r$ *and diverges for* $2kr \leq rT + 1$, *where*

$$T = \frac{1}{r+1} \sum_{s=1}^{r+1} \binom{r+1}{s} (n+1)^s B_{r+1-s} \sim \frac{(n+1)^{r+1}}{r+1},$$

$B_s, s \geq 0$ *are Bernoulli's numbers.*

Hilbert – Kamke problem and its generalizations I

Let J is the number of solutions of the system of Diophantine equations

$$\sum_{j=1}^{k} x_{1j}^{t_1} \dots x_{rj}^{t_r} = N(t_1, \dots, t_r)$$

$$(0 \leq t_1 \leq n_1, \dots, 0 \leq t_r \leq n_r, t_1 + \dots + t_r \geq 1),$$

$$1 \leq x_{1j} \leq P_1, \dots, 1 \leq x_{rj} \leq P_r, j = 1, \dots, k; P_1 = \min(P_1, \dots, P_r).$$

We obtained for $k \geq 8m\varkappa \ln 16m\varkappa$ the asymptotic formula for J as $P_1 \to +\infty$. It has the form as $\frac{\ln P_s}{\ln P_1} \ll 1$ for $s = 1, \dots, r$:

$$J = \sigma_r' \theta_r' (P_1 \dots P_r)^k (P_1^{n_1} \dots P_r^{n_r})^{-m/2} +$$

$$+O((P_1 \dots P_r)^k (P_1^{n_1} \dots P_r^{n_r})^{-m/2} P_1^{-0.1}),$$

Hilbert – Kamke problem and its generalizations II

where θ'_r is the singular integral

$$\theta'_r = \int\limits_{-\infty}^{+\infty} \dots \int\limits_{-\infty}^{+\infty} \left(\int\limits_{0}^{1} \dots \int\limits_{0}^{1} e^{2\pi i F_A(\bar{x})} \, dx_1 \dots dx_r \right)^k e^{-2\pi i (A \times M)} \, dA,$$

$$(A \times M) = \sum_{t_1=0}^{n_1} \cdots \sum_{t_r=0}^{n_r} \alpha(t_1, \dots, t_r) M(t_1, \dots, t_r),$$

$$M(t_1, \dots, t_r) = N(t_1, \dots, t_r) P_1^{-t_1} \dots P_r^{-t_r};$$

Hilbert – Kamke problem and its generalizations III

and σ'_r is the singular series

$$\sigma'_r = \sum_{q(0,\dots,1)=1}^{+\infty} \cdots \sum_{q(n_1,\dots,n_r)=1}^{+\infty} 1\times$$

$$\times \sum_{\substack{a(0,\dots,1)=1 \\ (a(0,\dots,1),q(0,\dots,1))=1}}^{q(0,\dots,1)} \cdots \sum_{\substack{a(n_1,\dots,n_r)=1 \\ (a(n_1,\dots,n_r),q(n_1,\dots,n_r))=1}}^{q(n_1,\dots,n_r)} U^k(\bar{a},\bar{q})e^{-2\pi i\left(\frac{a}{q}\times N\right)},$$

$$U(\bar{a},\bar{q}) = q^{-r}\sum_{x_1=1}^{q}\cdots\sum_{x_r=1}^{q} e^{2\pi i\Phi(\bar{x})}, q = q(0,\dots,1)\dots q(n_1,\dots,n_r),$$

$$\Phi(\bar{x}) = \sum_{t_1=0}^{n_1}\cdots\sum_{t_r=0}^{n_r}\frac{a(t_1,\dots,t_r)}{q(t_1,\dots,t_r)}x_1^{t_1}\dots x_r^{t_r},$$

$$\left(\frac{a}{q}\times N\right) = \sum_{t_1=0}^{n_1}\cdots\sum_{t_r=0}^{n_r}\frac{a(t_1,\dots,t_r)}{q(t_1,\dots,t_r)}N(t_1,\dots,t_r).$$

Hilbert – Kamke problem and its generalizations IV

There are known that $\theta'_r \geq 0, \sigma'_r \geq 0$. For a non-triviality of the asymptotic formula for J we need to prove that the singular series and the singular integral are positive. Hence we have two types of conditions for the solvability the system of Diophantine equations. It is arithmetic conditions related to the fact that the singular series is positive, and ordering conditions related to the fact that the singular integral is positive. Moreover, the arithmetic conditions are equivalent to the solvability conditions for the system of congruences of the form

$$\sum_{j=1}^{k} x_{1j}^{t_1} \ldots x_{rj}^{t_r} \equiv N(t_1, \ldots, t_r) \pmod q$$

$$(0 \leq t_1 \leq n_1, \ldots, 0 \leq t_r \leq n_r, t_1 + \cdots + t_r \geq 1),$$

for all moduli q that do not exceed $T = T(n_1, \ldots, n_r)$.

Hilbert – Kamke problem and its generalizations V

We note that the ordering conditions are the conditions that exists a solution of the system of equations in real numbers such that the Jacobi matrix corresponding to this solution has maximal rank. Consider the system of equations

$$F_{t_1,\dots,t_r}(\bar{x}) = \sum_{j=1}^{k} x_{1j}^{t_1} \dots x_{rj}^{t_r} = \beta(t_1, \dots, t_r)$$

$$(0 \le t_1 \le n_1, \dots, 0 \le t_r \le n_r, t_1 + \dots + t_r \ge 1),$$

where $\bar{x}$ is the aggregate $x_{11}, \dots, x_{r1}, \dots, x_{1k}, \dots, x_{rk}$ of real numbers.

Hilbert – Kamke problem and its generalizations VI

The Jacobi matrix of solution $\bar{x}$ of this system is the matrix

$$\left(\frac{\partial}{\partial x_{sj}} F_{t_1,\dots,t_r}(\bar{x}) \right),$$

the rows of which are indexed by
$(t_1, \dots, t_r), 0 \leq t_1 \leq n_1, \dots, 0 \leq t_r \leq n_r, t_1 + \dots + t_r \geq 1$, ordered in some fashion, and the column are indexed as follows:
$s + r(j-1), 1 \leq s \leq r, 1 \leq j \leq k$.

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