Foam Hurwitz operators realizing open-closed strings

Sergey M. Natanzon
(join with A.Mironov, A.Morosov)

Algebra of open-closed topological strings theory by G.Moor and C.Lazoroiu

1° **Associative, commutative algebra A with unit** and

a) **splitting** on finite-dimensional subspaces $A = \prod\limits_{\alpha} A_\alpha$, such that $A_{\alpha_1}A_{\alpha_2} \subset \delta_{\alpha_1,\alpha_2}A_{\alpha_1}$.

b)**linear forms** $l_\alpha : A \to \mathbb{C}$, such that $l_\beta(A \setminus A_\alpha) = 0$ and the bilinear form $< a_1, a_2 >_\alpha = l_\alpha(a_1a_2)$ is non-degenerated on A_α.

2° **Associative algebra B with unit** and

a) **splitting** on finite-dimensional subspaces $B = \prod\limits_{\beta,\gamma} B_{\beta,\gamma}$ such that $B_{\beta_1,\gamma_1}B_{\beta_2,\gamma_2} \subset \delta_{\gamma_1,\beta_2}B_{\beta_1,\gamma_2}$;

b)**linear forms** $l_{\beta,\gamma} : B \to \mathbb{C}$, such that $l_{\beta,\gamma}(B \setminus B_{\beta,\gamma}) = 0$, the coupling $< b_1, b_2 >_{\beta,\gamma} = l_{\beta,\gamma}(b_1b_2)$ is non-degenerated on $B_{\beta,\gamma} \times B_{\gamma,\beta}$ and $< b_1, b_2 >_{\beta,\gamma} = < b_2, b_1 >_{\gamma,\beta}$;

3° **Homomorphism of algebras $\Phi : A \to B$ and homomorphism of vector spaces** $\Phi^* : B \to A$ such that

a) $< \Phi^*(b), a >_\alpha = < b, \Phi(a) >_{\beta,\gamma}$ for $a \in A_\alpha$, $b \in B_{\beta,\gamma}$;

b) $\sum\limits_{\alpha} < \Phi^*(b_1), \Phi^*(b_2) >_\alpha = \mathsf{tr}\, W_{b_1,b_2}$, where the operator $W_{b_1,b_2} : B \to B$ is $W_{b_1,b_2}(b) = b_1bb_2$ and $b_1 \in B_{\beta,\beta}$, $b_2 \in B_{\gamma,\gamma}$

Operators from variables $X_{ij}\{i,j=1,2,\dots\}$

Put $D =: D_{a_1b_1}\dots D_{a_nb_n} := \sum_{e_1,\dots,e_n=1}^{\infty} X_{a_1e_1}\dots X_{a_ne_n}\frac{\partial}{\partial X_{a_1e_1}}\dots\frac{\partial}{\partial X_{a_ne_n}}$

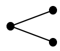
Corresponding bipartite graph $\Gamma(D)=(L,E,R)$, where

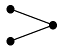
Left vertexes $L=L(D)$ is the set of different number, among $\{a_1,\dots,a_n\}$

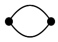
Right vertexes $R=R(D)$ is the set of different number, among $\{b_1,\dots,b_n\}$

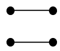
Set of edges $E=E(D)$.

Examples for $a_1\neq a_2$, $b_1\neq b_2$

$\Gamma(: D_{ab_1}D_{ab_2} :) =$ 

$\Gamma(: D_{a_1b}D_{a_2b} :) =$ 

$\Gamma(: D_{ab}D_{ab} :) =$ 

$\Gamma(: D_{a_1b_1}D_{a_2b_2} :) =$ 

Put: $V[\Gamma]=\{D|\Gamma(D)=\Gamma\}$

Graph (L,E,R) is called **balanced** if $|L|=|R|$

$\mathbb{G}_n$ **is set** of balanced graphs of degree $|E|=n$.

Algebra B

Consider vector space B_n generated by $\{V[\Gamma]|\Gamma\in\mathbb{G}_n\}$

Lemma *Formal infinite sums $b_1+b_2+\dots$, where $b_n\in B_n$, generate an associative algebra B with unit.*

Algebra A

$\Delta = [\mu_1, \dots, \mu_k]$ is the **Young diagram** with length row $\mu_1, \dots, \mu_k$
Put $m_j = m_j(\Delta) = |\{i | m_i = j\}|$.
Put $\kappa(\Delta) = |\mathsf{Aut}(\Delta)|^{-1} = \frac{1}{\prod_j m^j ! j^{m_j}}$.

Consider the infinite matrix $\mathrm{D} = \{D_{a,b}\}$

Consider the vector space A_n, generated by $W(\Delta) = \kappa(\Delta) : \prod_j (\mathsf{tr}(\mathrm{D}^j))^{m_j} :$,

where $|\Delta| = n$.

Lemma *Formal infinite sums $a_1 + a_2 + \dots$, where $a_i \in A_n$, generate an associative, commutative algebra A.*

Any $\Gamma(W(\Delta))$ is a balancent graph and moreover $W(\Delta) = \kappa(\Delta) \sum_{c_1, \dots, c_1 \in \mathbb{N}} : D_{c_1 c_{g(1)}} \dots D_{c_n c_{g(n)}} :$, were $g \in S_n$ with the cyclic type Δ. Thus there exists a **natural homomorphism** $\Phi : A \to B$.

Theorem There exists splittings $A = \prod_\alpha A_\alpha$ and $B = \prod_{\beta,\gamma} B_{\beta,\gamma}$, where α, β, γ — Young diagrams and linear functions l_α, $l_{\beta,\gamma}$, converting $\Phi : A \to B$ to algebra of open-closed topological strings theory.

For the prove we consider the finite-dimensional algebras $\phi_n : A_n \to B_n$ of degree n Hurwitz numbers of foam coverings (A.Alekseevskii and S.Nat. 2007), endow it a structure of algebra of open-closed topological strings theory and constrict an isomorphism between $\Phi : A \to B$ and $\prod_{n=1}^{\infty} (\phi_n : A_n \to B_n)$.

1. Classical Hurwitz numbers.
2. Algebra of classical Hurwitz numbers.
3. Foam surfaces
4. Disk Hurwitz numbers.
5. Algebra of disk Hurwitz numbers.
6. Algebra of foam Hurwitz numbers.
7. Spiling.