Right-angled billiards, pillowcase covers, and volumes of the moduli spaces

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(joint work with Jayadev Athreya and Alex Eskin)

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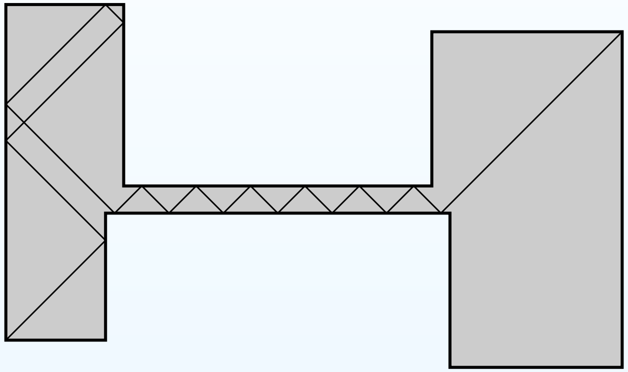
Billiards in right-angled polygons

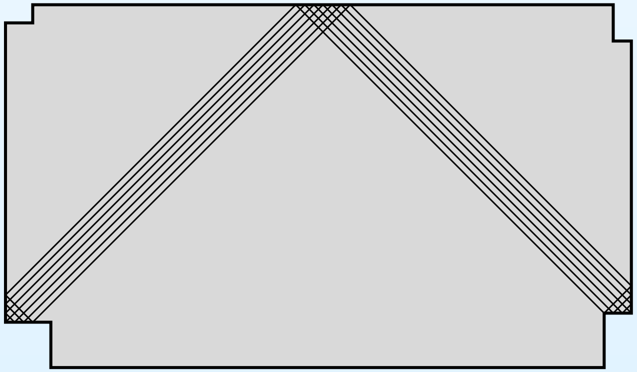
Moon Duchin playing a right-angled billiard.

Closed trajectories and generalized diagonals

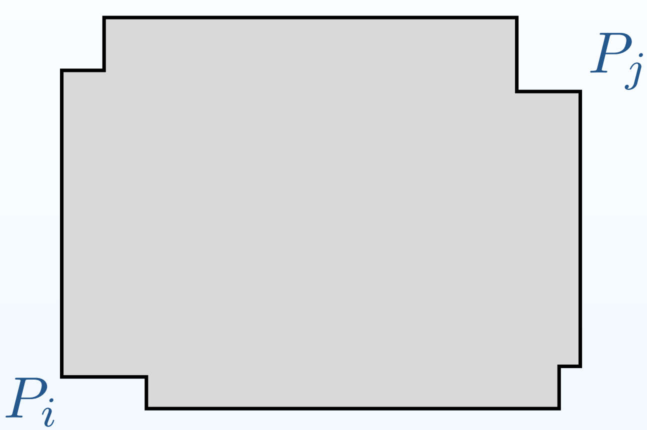
We count the asymptotic number of trajectories of bounded length joining a given pair of corners as the bound L tends to infinity.

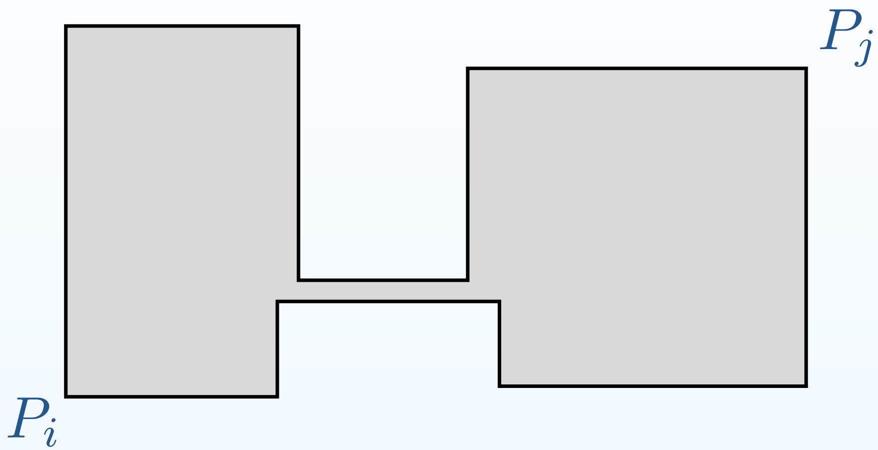


We also want to count the number of periodic trajectories of length at most L, or rather the number of *bands* of periodic trajectories. We might also count the bands with the weight representing the “thickness” of the band.



Number of generalized diagonals



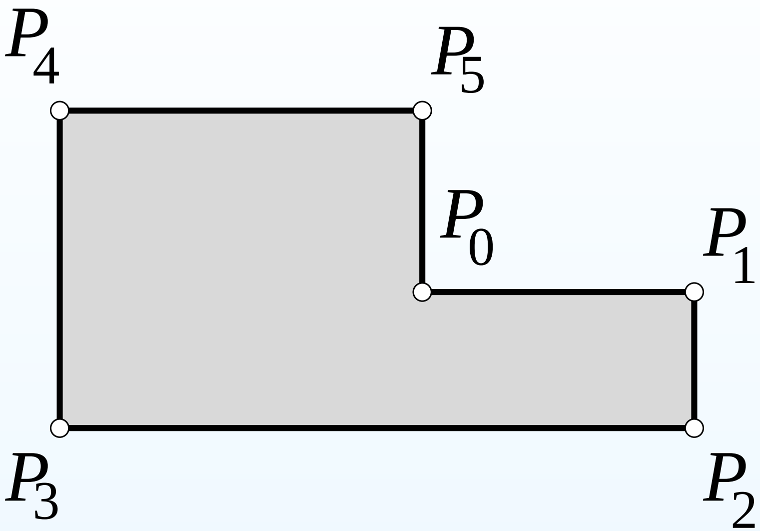


Example of a Theorem. *For almost any right-angled polygon the number of trajectories joining any two fixed corners with true right angles* $\pi/2$ *is "approximately" the same as for a rectangle:*

$$\frac{1}{2\pi}\cdot\frac{(\text{bound for the length})^2}{\text{area of the table}}$$

and does not depend on the shape of the polygon.

Naive intuition does not help...



However, say, for a typical L-shaped polygon the number of trajectories joining the corner with the angle $3\pi/2$ to some other corner is "approximately"

$$\frac{2}{\pi} \cdot \frac{(\text{bound for the length})^2}{\text{area of the table}}$$

which is 4 times (and not 3) times bigger than the number of trajectories joining a fixed pair of right corners...

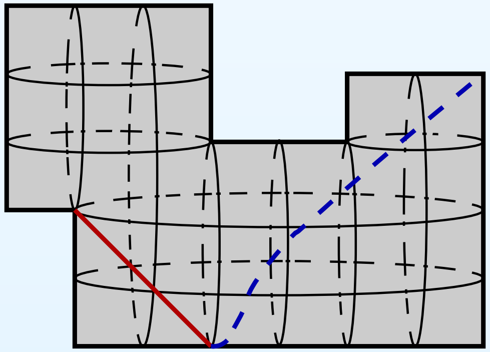
Billiard in a right-angled polygon: general answer

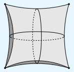
More generally, we explicitly computed the coefficients in the (approximate) quadratic asymptotics for the number of generalized diagonals and the number of closed trajectories in almost any table in any family of right-angled billiards.

Pillowcase covers and volumes of the moduli spaces

Billiards in rectangular polygons versus quadratic differentials on $\mathbb{CP}^1$

The topological sphere obtained by gluing two copies of the billiard table by the boundary is naturally endowed with a flat metric. This metric has conical singularities at the points coming from vertices of the polygon, otherwise it is nonsingular. In the case of a "rectangular polygon" the flat metric has holonomy in $\mathbb{Z}/(2\mathbb{Z})$. Hence it corresponds to a meromorphic quadratic differential with at most simple poles on $\mathbb{CP}^1$. Moreover, geodesics on this flat sphere project to billiard trajectories! Thus, to count billiard trajectories we may count geodesics on flat spheres!





Counting the number of general pillowcase covers (i.e. "integer" flat spheres with with any fixed collection of corner angles) is equivalent to counting the volumes of the corresponding moduli spaces of quadratic differentials.

Billiards in rectangular polygons versus quadratic differentials on $\mathbb{C}\mathrm{P}^1$

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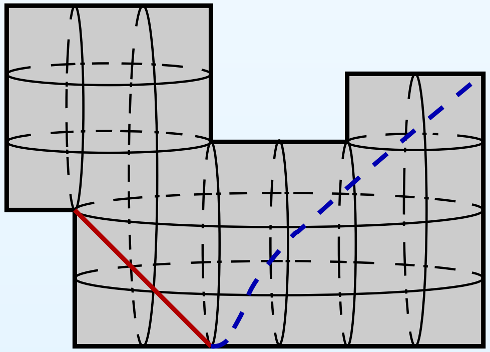

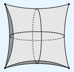

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Volumes of strata in genus zero

Let $$v(n) := \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \geq -1 \text{ is odd} \\ 2 & \text{when } n \geq 0 \text{ is even} \end{cases}$$

By convention we set $(-1)!! := 0!! := 1$, so $v(-1) = 1$ and $v(0) = 2$.

Theorem. *The volume of any stratum $\mathcal{Q}_1(d_1, \dots, d_k)$ of meromorphic quadratic differentials with at most simple poles on $\mathbb{C}\mathrm{P}^1$ (i.e. $d_i \in \{-1\,;\,0\} \cup \mathbb{N}$ for $i = 1, \dots, k$, and $\sum_{i=1}^{k} d_i = -4$) is equal to*

$$\operatorname{Vol} \mathcal{Q}_1(d_1, \dots, d_k) = 2\pi \cdot \prod_{i=1}^{k} v(d_i)$$

Corollary. *The number of pillowcase covers of degree at most N with ramification pattern corresponding to $\mathcal{Q}(d_1, \dots, d_k)$ has the following leading term in the asymptotics as $N \to \infty$*

$$\text{Number of pillowcase covers} \sim \frac{\pi}{k-2} \prod_{i=1}^{k} v(d_i) \cdot N^{k-2} \,.$$

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Siegel–Veech constants and ideas of the proofs

Siegel—Veech constant

Closed regular geodesics on flat surfaces appear in families of parallel closed geodesics sharing the same length. Every such family fills a *maximal cylinder* having conical points on each of the boundary components.

Denote by $N_{area}(S, L)$ the sum of areas of all cylinders spanned by geodesics of length at most L.

Theorem [after W. Veech] *For every stratum of meromorphic quadratic differentials $\mathcal{Q}(d_1, \dots, d_k)$ the following ratio is constant (i.e. does not depend on the value of a positive parameter L):*

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The constant $c_{area}(d_1, \dots, d_k)$ is called the *Siegel–Veech constant*.

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Two expressions for Siegel–Veech constants

Theorem (A. Eskin, H. Masur, A. Z.) For any stratum $\mathcal{Q}(d_1,\dots,d_k)$ in genus zero one has

$$c_{area}(d_1,\dots,d_k) =$$

$$= (\text{explicit combinatorial factor}) \cdot \frac{\prod \mathrm{Vol}(\text{adjacent simpler strata})}{\mathrm{Vol}\,\mathcal{Q}_1(d_1,\dots,d_k)}.$$

Theorem (A. Eskin, M. Kontsevich, A. Z.) For any stratum $\mathcal{Q}(d_1,\dots,d_k)$ in genus zero one has

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Combining two expressions we get series of combinatorial identities recursively defining volumes of all strata. It remains to verify that the guessed answer satisfy these identities.

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Back to billiards in polygons