Criterion of hyperbolicity in hyperelasticity

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(joint work with N. Favrie and S. Ndanou)

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Hyper or hypo?

Hyperelasticity : formulation of the governing equations in terms of deformations (Godunov, Kulikovskii, Miller and Colella, …)

Hypoelasticity : formulation of the governing equations for the deviatoric part of the stress tensor (Wilkins)

Hypoelasticity has serious problems with thermodynamics. However, it is very popular in the engineering community (CTH code of Los Alamos, ABACUS software…)

Plan

I. Hyperelastic model of isotropic solids

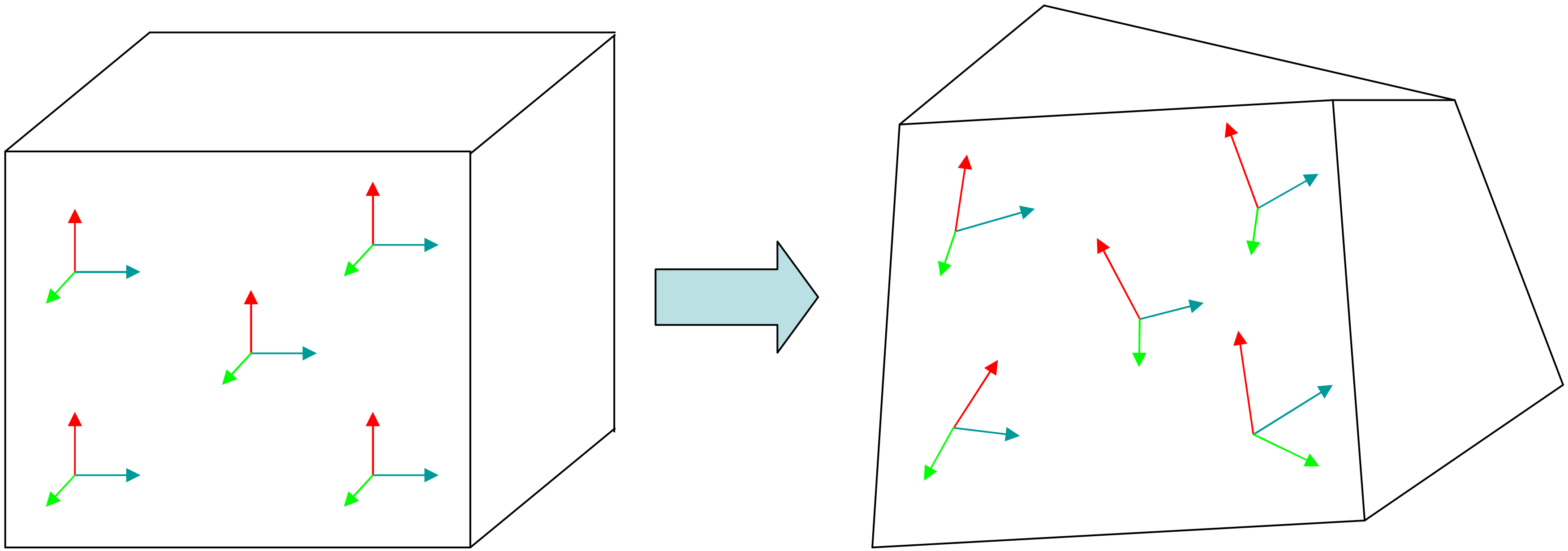
II. Hyperbolicity problem

III. Criterion of hyperbolicity

I. Hyperelastic model (modified)

A material is *hyperelastic* if the stress tensor is defined in terms of a stored energy function.

Basic idea



- Use a local cobasis at each point of the solid.
- Determine evolution equations for the cobasis vectors.
- Link the stress tensor to the cobasis vectors.

Definitions

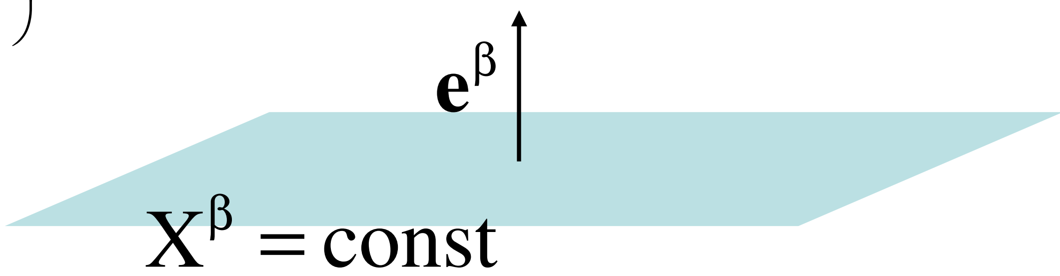
Deformation gradient $\mathbf{F} = \dfrac{\partial \mathbf{x}}{\partial \mathbf{X}}$

$\mathbf{X}$ Lagrangian coordinates

$\mathbf{x}$ Eulerian coordinates

Cobasis $\left(\mathbf{F}^{-1}\right)^{\mathrm{T}} = \left(\mathbf{e}^1 \quad \mathbf{e}^2 \quad \mathbf{e}^3\right) = \left(\nabla \mathrm{X}^1, \nabla \mathrm{X}^2, \nabla \mathrm{X}^3\right)$

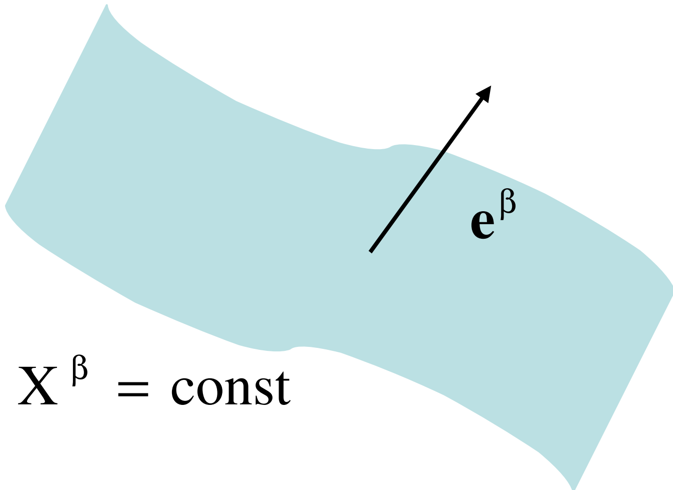
$$\mathbf{X} = \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$



Evolution of the local cobasis

Evolution of the Lagrangian coordinates $$\frac{DX^{\beta}}{Dt} = 0, \ \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla,$$

Evolution of the local cobasis $$\nabla\left(\frac{DX^{\beta}}{Dt}\right) = 0 \Longrightarrow \frac{\partial \mathbf{e}^{\beta}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{e}^{\beta}) = 0, \quad \mathrm{rot}\,\mathbf{e}^{\beta} = 0$$



Definitions

Finger tensor (inverse of the left Cauchy - Green tensor) $\mathbf{G} = \left(\mathbf{F}^{\mathrm{T}}\right)^{-1}\mathbf{F}^{-1} = \sum_{\beta=1}^{3} \mathbf{e}^{\beta} \otimes \mathbf{e}^{\beta}$

Evolution equation for the Finger tensor $\frac{\mathrm{D}\,\mathbf{G}}{\mathrm{Dt}} + \left(\frac{\partial\,\mathbf{v}}{\partial\,\mathbf{x}}\right)^{\mathrm{T}}\mathbf{G} + \mathbf{G}\,\frac{\partial\,\mathbf{v}}{\partial\,\mathbf{x}} = 0$

Alternative equations $\frac{\partial \mathbf{e}^{\beta}}{\partial t} + \nabla(\mathbf{v}\cdot\mathbf{e}^{\beta}) = 0, \quad \mathrm{rot}\mathbf{e}^{\beta} = 0, \quad \beta = 1,2,3.$

Energy of isotropic solids

$$e(\mathbf{G},\eta) = e(J_1, J_2, J_3, \eta), \quad J_i = tr(\mathbf{G}^i)$$

Stress tensor

$$\boldsymbol{\sigma} = -2\rho \frac{\partial e}{\partial \mathbf{G}} \mathbf{G} = -2\rho \left(\frac{\partial e}{\partial J_1} \mathbf{I} + 2 \frac{\partial e}{\partial J_2} \mathbf{G} + 3 \frac{\partial e}{\partial J_3} \mathbf{G}^2 \right) \mathbf{G}$$

Conservative model of elasticity (derived by Hamilton's principle)

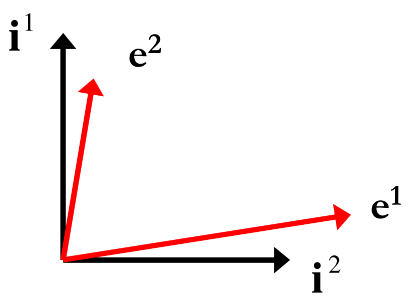
Mass equation $$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0$$

Momentum equation $$\frac{\partial \rho \mathbf{v}}{\partial t} + \mathrm{div}(\rho \mathbf{v} \otimes \mathbf{v} - \boldsymbol{\sigma}) = 0$$

Energy equation $$\frac{\partial \rho E}{\partial t} + \mathrm{div}(\rho E \mathbf{v} - \boldsymbol{\sigma} \cdot \mathbf{v}) = 0, \quad E = e + \frac{|\mathbf{v}|^2}{2}$$

Cobasis $$\frac{\partial \mathbf{e}^\beta}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{e}^\beta) = 0 \quad \text{and} \quad \mathrm{rot}(\mathbf{e}^\beta) = 0$$

with $\mathbf{e}^\beta\big|_{t=0} = \mathbf{i}^\beta$ where $\mathbf{i}^\beta$ are the vector of the Cartesian base



II. Hyperbolicity

- **Definition** : The energy is *rank one convex* function of **F** if it is convex function of t obtained by replacing

$$\mathbf{F} \mapsto \mathbf{F} + t\mathbf{m} \otimes \mathbf{n}, \quad \det(\mathbf{F} + t\mathbf{m} \otimes \mathbf{n}) > 0.$$

Theorem : Equations are hyperbolic if the energy is rank one convex (see, for example, Dafermos).

This problem is solved only mathematically : the equation of state should be rank one convex. However it is difficult and almost impossible to check it in practice. A new simple criterion is nedeed.

Symmetric *t*-hyperbolic systems

$$A\mathbf{U}_t + B\mathbf{U}_x + C\mathbf{U}_y + D\mathbf{U}_z = 0,$$

$$A = A^T > 0, \quad B = B^T, \quad C = C^T, \quad D = D^T$$

Godunov S.K. (1961), Godunov (1978), Godunov and Romenskii (2003)

Non-conservative form

Mass equation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0$$

Momentum equation

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \mathrm{div}(\rho \mathbf{v} \otimes \mathbf{v} - \boldsymbol{\sigma}) = 0$$

Energy equation

$$\frac{\partial \rho E}{\partial t} + \mathrm{div}(\rho E \mathbf{v} - \boldsymbol{\sigma} \cdot \mathbf{v}) = 0, \quad E = e + \frac{|\mathbf{v}|^2}{2}$$

Cobasis

$$\frac{\partial \mathbf{e}^{\beta}}{\partial t} + \frac{\partial \mathbf{e}^{\beta}}{\partial \mathbf{x}} \mathbf{v} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^{T} \mathbf{e}^{\beta} = 0$$

This system contains the hyperelasticity in a particular case where the cobasis vector are gradients of the Lagrangian coordinates. Hence, the hyperbolicity of such a system will imply the hyperbolicity of hyperelasticity.

Equation of state in separable form

Let $\mathbf{g} = \frac{\mathbf{G}}{|\mathbf{G}|^{1/3}}$

$$\mathrm{e} = \underbrace{\mathrm{e}^{\mathrm{h}}(\rho, \eta)}_{\text{Hydrodynamic}} + \underbrace{\mathrm{e}^{\mathrm{e}}(\mathbf{g})}_{\text{Shear}}$$

$$\mathrm{e}^{\mathrm{h}}(\rho, \mathrm{p}) = \frac{\mathrm{p} + \gamma \mathrm{p}_{\infty}}{\rho(\gamma - 1)} \qquad e^{e}(\mathbf{g}) = \frac{\mu}{4\rho_0}((2-4a)\ j_1 + a j_2 + 3(3a-2)), \quad j_i = tr(\mathbf{g}^i)$$

The stress tensor can be written as:

$$\boldsymbol{\sigma} = -2\rho \frac{\partial e}{\partial \mathbf{G}} \mathbf{G} = -p\mathbf{I} + \mathbf{S} = -\rho^2 \frac{\partial e^h}{\partial \rho} \mathbf{I} - 2\rho \left(\frac{\partial e^e}{\partial j_1} \left(\mathbf{g} - \frac{j_1}{3} \mathbf{I} \right) + 2 \frac{\partial e^e}{\partial j_2} \left(\mathbf{g}^2 - \frac{j_2}{3} \mathbf{I} \right) \right)$$

Invariance under group SO(3)

Mass equation

$$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{v}) = 0$$

Momentum equation

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\mathbf{v} - \frac{div(\boldsymbol{\sigma})}{\rho} = 0,$$

Energy equation

$$\frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \eta = 0$$

Cobasis

$$\frac{\partial \mathbf{e}^{\beta}}{\partial t} + \frac{\partial \mathbf{e}^{\beta}}{\partial \mathbf{x}}\mathbf{v} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^{T} \mathbf{e}^{\beta} = 0$$

Transformation

$$\mathbf{x}' = O\mathbf{x}, \quad \mathbf{v}' = O\mathbf{v}, \quad \mathbf{e}^{\beta}{}' = O\mathbf{e}^{\beta}$$

Hyperbolic system

3D case $$\mathbf{U}_t + A\mathbf{U}_x + B\mathbf{U}_y + C\mathbf{U}_z = 0$$

Equations are invariant under group SO(3).

Hence, it is sufficient to consider 1D case.

1D case $$\mathbf{U}_t + A\mathbf{U}_x = 0, \quad \mathbf{U} \in R^{14}$$

Let $$F^{-1} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$$

Then $$\mathbf{U} = (\rho, u, v, w, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, \eta)^T$$

Hyperbolic systems

For the energy in separable form $e = e^h(\rho, \eta) + e^e(j_1, j_2)$

$$j_1 = tr\left(\frac{G}{|G|^{1/3}}\right) = \frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2}{|\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})|^{2/3}},$$

$$j_2 = tr\left(\frac{G^2}{|G|^{1/3}}\right) = \frac{|\mathbf{a}|^4 + |\mathbf{b}|^4 + |\mathbf{c}|^4 + 2\left((\mathbf{a}\cdot\mathbf{b})^2 + (\mathbf{c}\cdot\mathbf{b})^2 + (\mathbf{c}\cdot\mathbf{a})^2\right)}{|\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})|^{4/3}}$$

the hyperbolicity problem is reduced to the study of the positive definiteness of the symmetric matrix $\mathbf{K}$

Hyperbolic systems

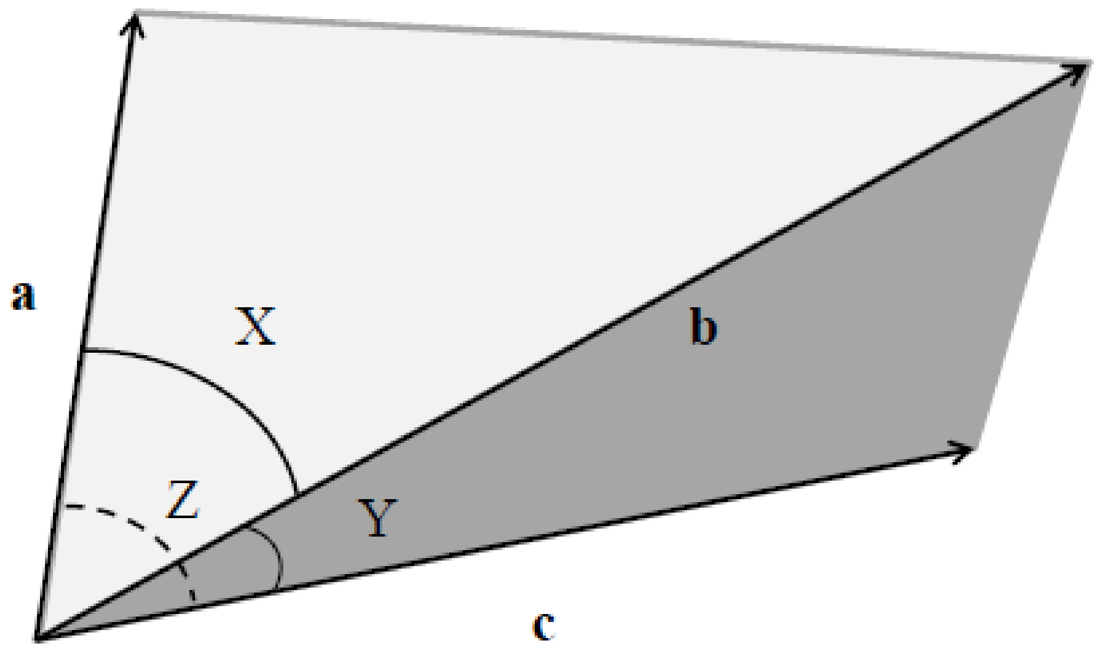
$$\mathbf{K} = \begin{pmatrix} c^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathbf{M}, \qquad \mathbf{M} = \frac{1}{\Delta}\begin{pmatrix} \mathbf{a}^T E''\mathbf{a} & \mathbf{a}^T E''\mathbf{b} & \mathbf{a}^T E''\mathbf{c} \\ \mathbf{a}^T E''\mathbf{b} & \mathbf{b}^T E''\mathbf{b} & \mathbf{b}^T E''\mathbf{c} \\ \mathbf{a}^T E''\mathbf{c} & \mathbf{b}^T E''\mathbf{c} & \mathbf{c}^T E''\mathbf{c} \end{pmatrix}$$

Here

$$\Delta = \mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) > 0, \quad E = \Delta e^e\left(j_1, j_2\right), \quad E'' = \frac{\partial^2 E}{\partial \mathbf{a}^2}$$

Property : the matrix M is homogeneous function of degree zero with respect to **a**, **b**, **c**. Hence, we can always suppose that the determinant of the deformation gradient is one.

Angles between vectors



Unit determinant deformation gradients

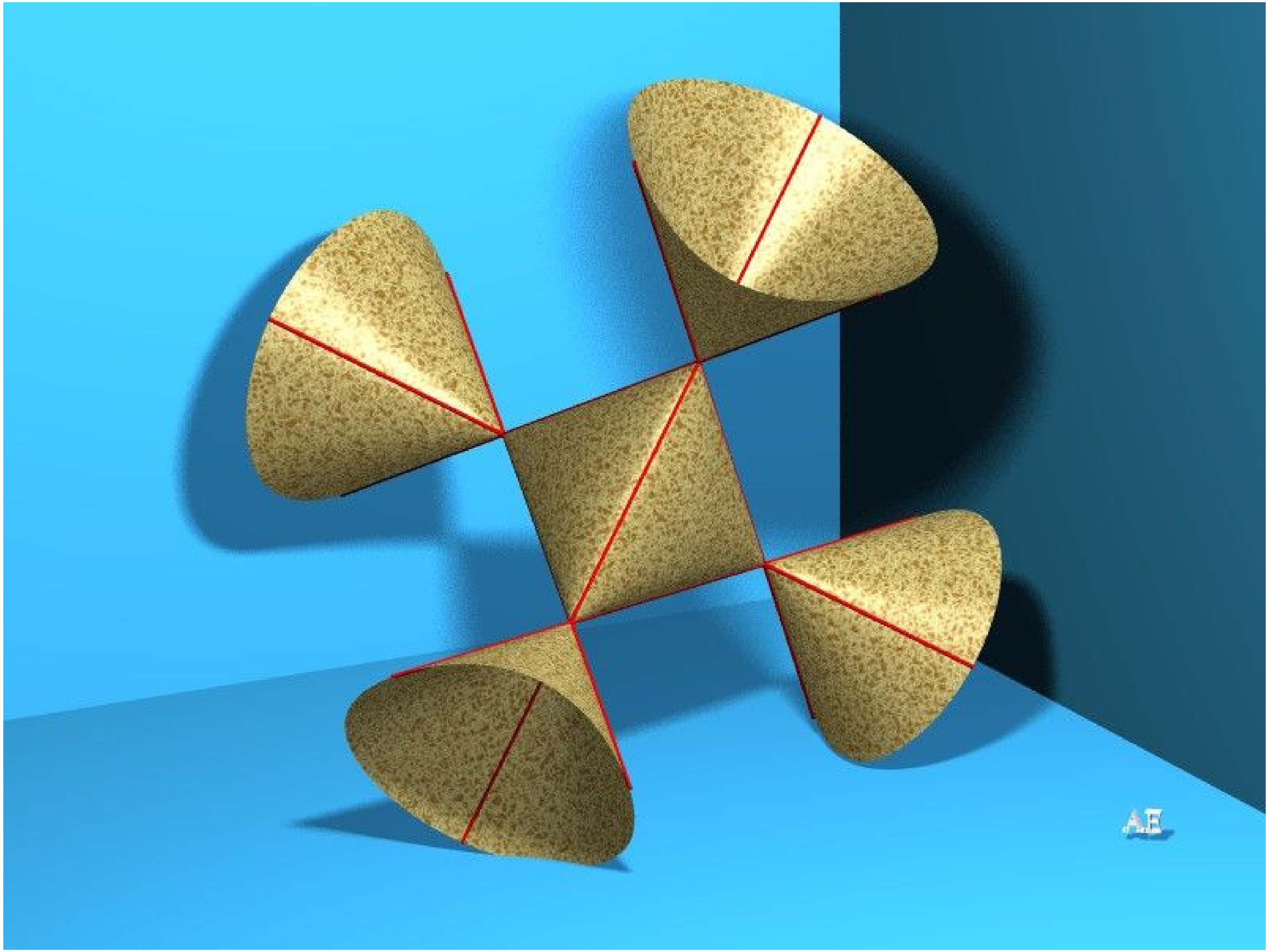
Introducing the angles X,Y,Z between vectors **a,b,c** one can find that the unit determinant can be expressed as

$$\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})=1 \Leftrightarrow X^2+Y^2+Z^2-2XYZ=1-\frac{1}{|\mathbf{a}|^2|\mathbf{b}|^2|\mathbf{c}|^2}=\alpha<1$$

Theorem The equations are hyperbolic for any deformations and shears if the matrix M is positive definite at any surface S_α given by

$$X^2+Y^2+Z^2-2XYZ=\alpha<1$$

The case $\alpha=1$ corresponds to the Cayley surface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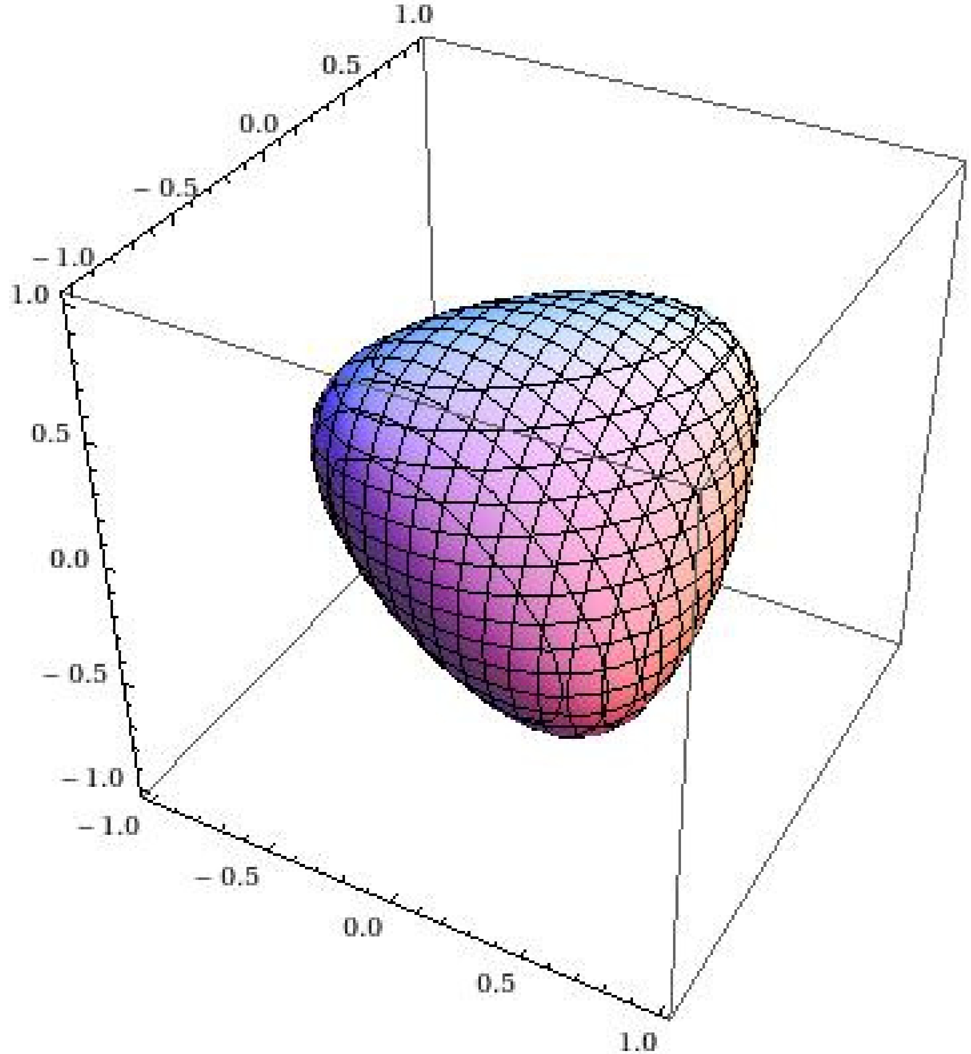
Surface $X^2 + Y^2 + Z^2 - 2XYZ = \alpha < 1$

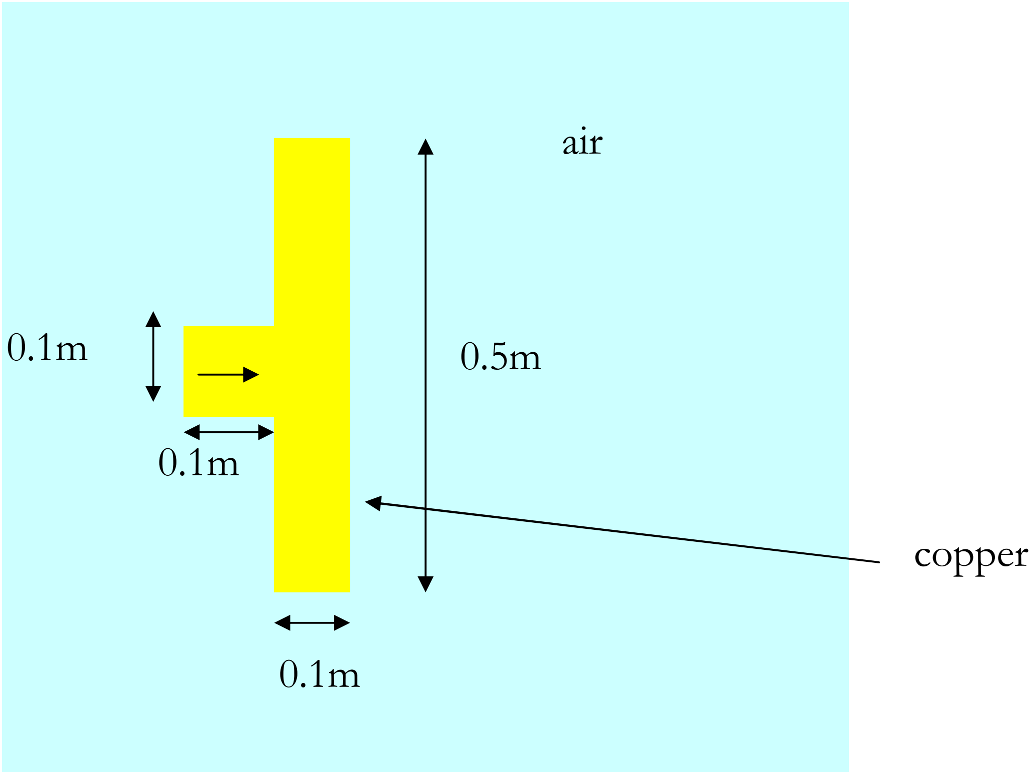


Examples

Let $e^e = j_2$

Then the equations are hyperbolic.

2D Impact



$\gamma_s = 4.22$

$p_{\infty s} = 34.2\ \text{GPa}$

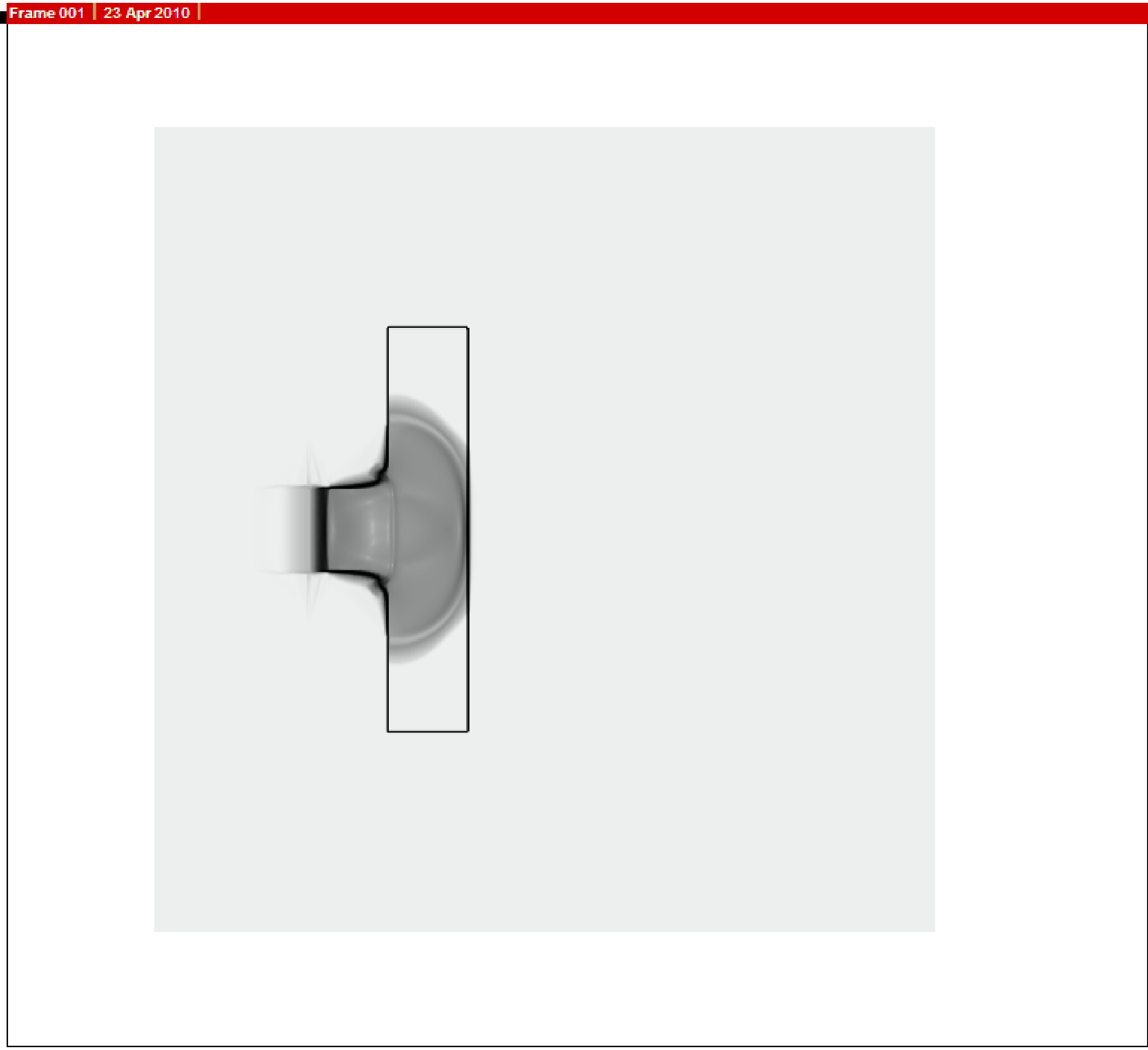
$\rho_s = 8900\ \text{kg/m}^3$

$\mu_s = 92\ \text{GPa}$

$Y_s = 2.49\ \text{GPa}$

A copper projectile impacts a copper plate at 800m/s

Impact problem



Conclusion

- A new criterion of hyperbolicity for hyperelasticity is formulated
- Equations of state are given satisfying this criterion

End