Universal Algorithm for Online Trading Based on the Method of Calibration

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Stock Market with one stock:

- $S_1, S_2, \ldots, S_{i-1}$ – rescalled prices of the stock; $0 \le S_i \le 1$.
- side information $\mathbf{x}_i \in [0,1]$.
- *Trader M* uses a randomized strategy $\tilde{M}_i$.
- *Trader D* uses a stationary strategy $D(\mathbf{x}_i)$, where $D \in \mathscr{F}$. $\mathscr{F}$ is a function space.

FOR $i = 1, 2 \ldots$
Stock Market announces a signal $\mathbf{x}_i \in [0,1]$.
Trader M bets by buying or selling the random number $\tilde{M}_i$ of shares of the stock by S_{i-1} each.
Trader D bets by buying or selling a number $D(x_i)$ of shares of the stock by S_{i-1} each.
Stock Market reveals a price S_i of the stock.
Trader M updates his total gain (or loss):
$\mathcal{K}_i^M = \mathcal{K}_{i-1}^M + \tilde{M}_i(S_i - S_{i-1})$. We get $\mathcal{K}_0^M = 0$.
Trader D updates his total gain (or loss)
$\mathcal{K}_i^D = \mathcal{K}_{i-1}^D + D(\mathbf{x}_i)(S_i - S_{i-1})$. We get $\mathcal{K}_0^D = 0$.
ENDFOR

Figure: Protocol of the trading game

Players can borrow money.

Information available to players

At any round *i* of the game

- *Trader M* can use all past information available before his move: past prices $S_1, S_2, \ldots, S_{i-1}$, moves of both players, side information;
- *Trader D* uses only a value of the fixed function D from side information $\mathbf{x}_i \in [0,1]$: $\mathbf{x}_i$ can contain in an encoded form not only past prices but also, for example, the future price S_i of the stock.

Universal trading strategy

Compute a forecast p_i of a future price S_i.
Randomize it to $\tilde{p}_i$.
Randomize the past price S_{i-1} of the stock to $\tilde{S}_{i-1}$.
Universal trading strategy:

$$\tilde{M}_i = \begin{cases} 1 \text{ if } \tilde{p}_i > \tilde{S}_{i-1}, \\ -1 \text{ otherwise.} \end{cases}$$

The core of universal strategy is the algorithm for computing the well-calibrated forecasts $\tilde{p}_i$.

Main result

$\|D\|_\infty = \sup_{0 \le x \le 1} |D(x)|$, where D is a continuous function.

$S_1, S_2, \dots \in [0,1]$ and $\mathbf{x}_1, \mathbf{x}_2, \dots \in [0,1]$ be given online according to the protocol.

Theorem

An algorithm for computing forecasts and a sequential method of randomization can be constructed such that for any continuous nonzero function D

$$\liminf_{n \to \infty} \frac{1}{n} \left(\mathscr{K}_n^M - \|D\|_\infty^{-1} \mathscr{K}_n^D \right) \ge 0$$

holds almost surely with respect to a probability distribution generated by the corresponding sequential randomization.

There is some analogy with Cover's universal portfolio.

Forecasting task

Nature: outputs $\omega_1, \omega_2, \ldots, \omega_{n-1}, \ldots$ – reals from $[0, 1]$ (or binary 0 and 1);

Forecaster: computes $p_1, p_2, \ldots, p_n, \ldots$ – forecasts (p_n – expected value of a future outcome ω_n).

Example for binary case: p_n is a probability that it will rain ($\omega_n = 1$ means rain).

A problem: how to evaluate the forecaster?

Predictive-sequential framework

- we observe only individual sequence $\omega_1, \omega_2, \ldots, \omega_{n-1}$ of outcomes;
- we can not define a probability distribution on the whole space.

Method of calibration: informal setting

Informally: a forecaster is said to be well-calibrated if it rains as often as he leads us to expect. It should rain about 80% of the days for which $p_n = 0.8$, and so on.

A sequence of forecasts $p_1, p_2, \ldots$ is “calibrated” for an infinite binary sequence $\omega_1, \omega_2 \ldots$ if for any p^*

$$\frac{\sum_{p_i \approx p^*} \omega_i}{\sum_{p_i \approx p^*} 1} \approx p^*, \; n \to \infty$$

as the denominator of this relation tends to infinity.

A sequence of forecasts $p_1, p_2, \ldots$ is well-calibrated for an infinite sequence $\omega_1, \omega_2 \ldots$ if for the characteristic function $I(p)$ of any subinterval of $[0, 1]$ (checking rule) the calibration error tends to zero, i.e.,

$$\frac{\sum_{i=1}^{n} I(p_i)(\omega_i - p_i)}{\sum_{i=1}^{n} I(p_i)} \longrightarrow 0,\ n \to \infty$$

as the denominator of this relation tends to infinity.

A more weak condition:

$$\frac{1}{n}\sum_{i=1}^{n} I(p_i)(\omega_i - p_i) \to 0 \text{ as } n \to \infty$$

Adversarial Nature (Dawid and Oakes (1985):

Any total deterministic forecasting algorithm f

$$p_n = f(\omega_1, \omega_2, \ldots, \omega_{n-1})$$

is not calibrated for the sequence $\omega_1, \omega_2 \ldots$, where

$$\omega_i = \begin{cases} 1 \text{ if } p_i < 0.5 \\ 0 \text{ otherwise} \end{cases}$$

and $p_i = f(\omega_1, \ldots, \omega_{i-1})$, $i = 1, 2, \ldots$. The condition of calibration fails for this ω, where $I = [0, 0.5)$ or $I = [0.5, 1]$.

Probability forecasting game

FOR $i = 1, 2, \dots n$

Forecaster computes a random forecast $\tilde{p}_i \in [0, 1]$.
In other words,
Forecaster outputs a probability distribution P_i on $p_i \in [0, 1]$.

Nature reveals an outcome $\omega_i \in [0, 1]$

ENDFOR

The sequence P_n, $n = 1, 2, \dots$ of defines an overal probability distribution Pr on infinite trajectories $p_1, p_2, \dots$ of forecasts.

Foster and Vohra (1994) – first result

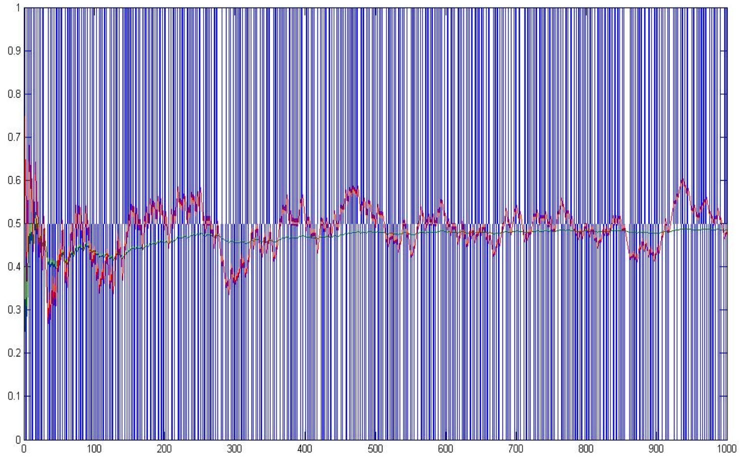
For any $\Delta > 0$, Kakade and Foster's (2004) algorithm given binary $\omega_1 \dots \omega_{i-1}$ computes a deterministic forecasts p_i and randomly rounds it up to Δ to $\tilde{p}_i$ such that:

For any infinite sequence $\omega_1, \omega_2 \dots$ and for the characteristic function $I(p)$ of any subinterval of $[0,1]$

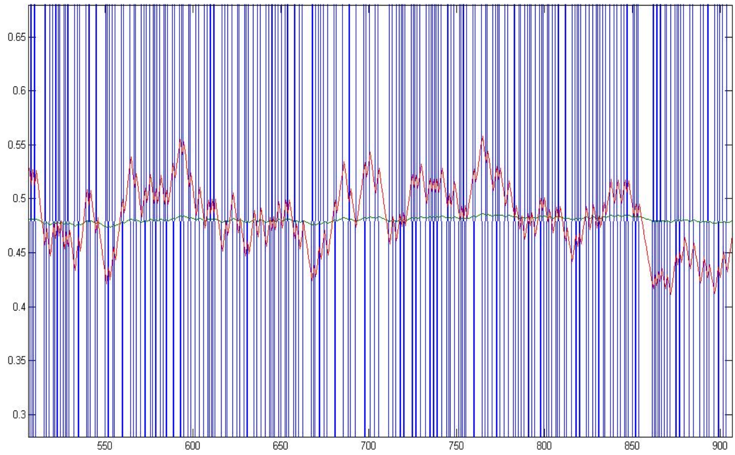
$$\limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^{n} I(\tilde{p}_i)(\omega_i - \tilde{p}_i) \right| \le \Delta$$

holds with the overall probability Pr one, where Pr is generated by these randomizations.
This result also holds for real outcomes $\omega_i = S_i \in [0,1]$.

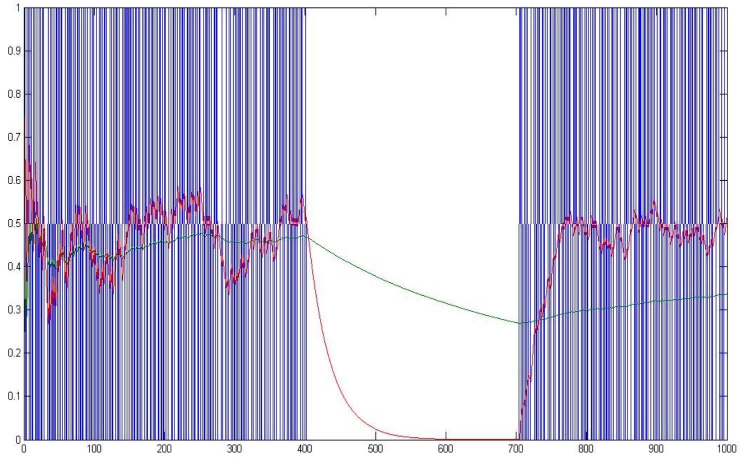
Numerical experiment



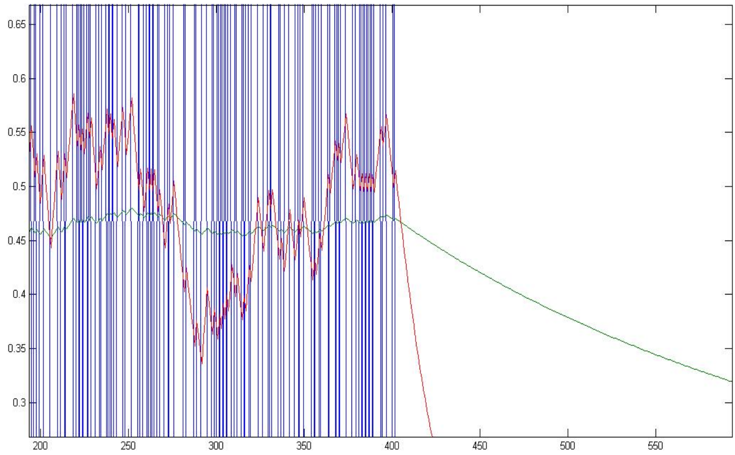
Numerical experiment



Numerical experiment



Numerical experiment



Proof. Random rounding:
Divide [0,1] on subintervals: $v_i = i\Delta$, where $i = 0, 1, \dots, K$.
$V = \{v_0, \dots, v_K\}$. For any $p \in [0,1]$
$p = \sum_{v \in V} w_v(p) v = w_{v_{i-1}}(p) v_{i-1} + w_{v_i}(p) v_i$, where $p \in [v_{i-1}, v_i]$.
We will define a deterministic forecast p and randomize it:

$$\tilde{p} = \begin{cases} v_{i-1} \text{ with probability } w_{v_{i-1}}(p) \\ v_i \text{ with probability } w_{v_i}(p) \end{cases}$$

$\bar{w}(p) = (w_v(p) : v \in V)$ – vector of probabilities of rounding.
$p_i = E_{\bar{w}}(\tilde{p}_i)$.

In general, $\omega_i = S_i \in [0,1]$ – real outcomes,
$\bar{x}_i$ is *information* vector of dimension $k \geq 1$: $\bar{x}_i \in [0,1]^k$.
The information vector contains all information used by checking rules, besides the forecast
Examples: $\bar{x}_i = S_{i-1}$ or $\bar{x}_i = (\mathbf{x}_i, S_{i-1})$.
New checking rule is a subset $\mathscr{S} \subseteq [0,1]^{k+1}$

$$I_{\mathscr{S}}(p,\bar{x}) = \begin{cases} 1 \text{ if } (p,\bar{x}) \in \mathscr{S}, \\ 0 \text{ otherwise,} \end{cases}$$

where $\bar{x}$ is an k-dimensional vector. Example: $k = 1$,

$$I_{\mathscr{S}}(p,x) = \begin{cases} 1 \text{ if } p > x, \\ 0 \text{ otherwise,} \end{cases}$$

- RKHS is a Hilbert space $\mathscr{F}$ of real-valued functions on a compact metric space X such that the evaluation functional $f \to f(x)$ is continuous for each $x \in X$.
- $f(x) = (f \cdot \Phi(x))$.
- $K(x, y) = (\Phi(x) \cdot \Phi(y))$ – kernel.
- $\| \cdot \|_{\mathscr{F}}$ be a norm in $\mathscr{F}$.
- $c_{\mathscr{F}}(x) = \sup_{\|f\|_{\mathscr{F}} \leq 1} |f(x)|$.
- The embedding constant of $\mathscr{F}$: $c_{\mathscr{F}} = \sup_x c_{\mathscr{F}}(x) = \|\Phi(\bar{x})\|_{\mathscr{F}}$.
- We consider RKHS $\mathscr{F}$ with $c_{\mathscr{F}} < \infty$.

Theorem

Given $\varepsilon > 0$ we can compute forecasts $p_1, p_2, \ldots$ and a sequential method of randomization such that:

- *for any subset $\mathscr{S} \subseteq [0.1]^{k+1}$, n, and for any $\delta > 0$, with probability at least $1 - \delta$,*

$$\left| \sum_{i=1}^{n} I_{\mathscr{S}}(\tilde{p}_i, \tilde{x}_i)(S_i - \tilde{p}_i) \right| \leq$$

$$18 \left(\frac{k+1}{2} \right)^{\frac{2}{k+3}} (c_{\mathscr{F}}^2 + 1)^{\frac{1}{k+3}} n^{1 - \frac{1}{k+3} + \varepsilon} + \sqrt{\frac{n}{2} \ln \frac{2}{\delta}}.$$

- *for any $D \in \mathscr{F}$,*

$$\left| \sum_{i=1}^{n} D(\mathbf{x}_i)(y_i - p_i) \right| \leq \|D\|_{\mathscr{F}} \sqrt{(c_{\mathscr{F}}^2 + 1)n}$$

Universal strategy is a randomized decision rule – it takes only two values:

$$\tilde{M}_i = \begin{cases} 1 \text{ if } \tilde{p}_i > \tilde{S}_{i-1}, \\ -1 \text{ otherwise.} \end{cases}$$

Assume that prices $S_1, S_2, \dots \in [0,1]$ and signals $\mathbf{x}_1, \mathbf{x}_2, \dots \in [0,1]$ be given online according to the protocol

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Theorem

Given $\varepsilon > 0$ an algorithm for computing forecasts p_i and a sequential method of randomization can be constructed such that for any n and $\delta > 0$, with probability at least $1-\delta$, for all nontrivial $D \in \mathscr{F}$ (RKHS)

$$\sum_{i=1}^{n} \tilde{M}_i \Delta S_i \ge \|D\|_\infty^{-1} \sum_{i=1}^{n} D(\mathbf{x}_i)\Delta S_i -$$
$$-38(c_{\mathscr{F}}^2+1)^{\frac{1}{4}} n^{\frac{3}{4}+\varepsilon} - \|D\|_\infty^{-1}\|D\|_{\mathscr{F}}\sqrt{(c_{\mathscr{F}}^2+1)n} -$$
$$-\sqrt{\frac{n}{2}\ln\frac{2}{\delta}}$$

for all n, where $\Delta S_i = S_i - S_{i-1}$.

Information (vector) $x_i = S_{i-1}$

Calibration theorem for $k = 1$ and $\mathscr{S} = \{(p,x) : p > x\}$ (and $\mathscr{S} = \{(p,x) : p \leq x\}$):

Given $\varepsilon > 0$ we can compute forecasts $p_1, p_2, \ldots$ and a sequential method of randomization such that:

- for any $\delta > 0$, with probability at least $1 - \delta$,

$$\left| \sum_{i=1}^{n} I(\tilde{p}_i > \tilde{S}_{i-1})(S_i - \tilde{p}_i) \right| \leq$$

$$18(c_{\mathscr{F}}^2 + 1)^{\frac{1}{4}} n^{\frac{3}{4}+\varepsilon} + \sqrt{\frac{n}{2} \ln \frac{2}{\delta}}.$$

- for any $D \in \mathscr{F}$,

$$\left| \sum_{i=1}^{n} D(\mathbf{x}_i)(y_i - p_i) \right| \leq \|D\|_{\mathscr{F}} \sqrt{(c_{\mathscr{F}}^2 + 1)n}$$

for all n.

Sketch of the proof of the main result

$$\sum_{i=1}^{n} \tilde{M}_i \Delta S_i = \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (S_i - S_{i-1}) =$$

$$= \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (S_i - \tilde{p}_i) + \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (\tilde{p}_i - \tilde{S}_{i-1}) + \sum_{\tilde{p}_i > \tilde{S}_{i-1}} (\tilde{S}_{i-1} - S_{i-1}) \approx$$

$$\sum_{\tilde{p}_i > \tilde{S}_{i-1}} (\tilde{p}_i - \tilde{S}_{i-1}) \geq \|D\|_\infty^{-1} \sum_{i=1}^{n} D(x_i)(\tilde{p}_i - \tilde{S}_{i-1}) =$$

$$= \|D\|_\infty^{-1} \sum_{i=1}^{n} D(x_i) \Big((p_i - S_{i-1}) + (\tilde{p}_i - p_i) - (\tilde{S}_{i-1} - S_{i-1}) \Big) \geq$$

$$\|D\|_\infty^{-1} \sum_{i=1}^{n} D(x_i)(p_i - S_{i-1}) =$$

$$\|D\|_\infty^{-1} \sum_{i=1}^{n} D(x_i)(S_i - S_{i-1}) - \|D\|_\infty^{-1} \sum_{i=1}^{n} D(x_i)(S_i - p_i) \approx$$

$$\|D\|_\infty^{-1} \sum^{n} D(x_i)(S_i - S_{i-1})$$

Universal algorithmic trading: competing with continuous trading strategies

Theorem

An algorithm for computing forecasts p_i and a sequential method of randomization can be constructed such that for any nontrivial continuous function D,

$$\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{M}_i^1\Delta S_i-\frac{1}{n}\|D\|_\infty^{-1}\sum_{i=1}^{n}D(\mathbf{x}_i)\Delta S_i\right)\geq 0 \qquad (1)$$

holds almost surely with respect to a probability distribution generated by the corresponding sequential randomization.

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An RKHS $\mathscr{F}$ on a compact metric space X is universal if for any continuous function f, for each $\varepsilon > 0$, a function $D \in \mathscr{F}$ exists such that

$$\sup_{x \in X} |f(x) - D(x)| \leq \varepsilon$$

(cf. Steinwart (2001), Vovk (2005)).
The Sobolev space $\mathscr{F} = H^1([0,1])$, which consists of absolutely continuous functions $f : [0,1] \to \mathscr{R}$ with $\|f\|_{\mathscr{F}} \leq 1$, where

$$\|f\|_{\mathscr{F}} = \sqrt{\int_0^1 (f(t))^2 dt + \int_0^1 (f'(t))^2 dt},$$

is **universal** RKHS.
For this space, $c_{\mathscr{F}} = \sqrt{\coth 1}$ (cf. Vovk (2005)).
The existence of the universal RKHS on $[0,1]$ implies the theorem.

Competing with discontinuous trading strategies
Deterministic signals $\mathbf{x}_i$: counterexample

Theorem

Let $\tilde{M}_i$ be an arbitrary sequence of independent random variables (randomized trading strategy) such that $|\tilde{M}_i| \le 1$ for all i.
Consider the protocol of trading game with two players and with signals $\mathbf{x}_i = P\{\tilde{M}_i > 0\}$.
Then a binary decision rule $D(\mathbf{x})$ and a sequence $S_1, S_2, \dots$ of prices can be defined such that with probability one

$$\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{M}_i\Delta S_i - \frac{1}{2}\frac{1}{n}\sum_{i=1}^{n}D(\mathbf{x}_i)\Delta S_i\right) \le 0. \qquad (2)$$

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$\mathbf{x}_i = P\{\tilde{M}_i > 0\}$ – signals
Define a sequence of stock prices: $S_0 = 1/2$ and for $1 \leq i \leq 1$

$$S_i = \begin{cases} S_{i-1} - 2^{-(i+1)} \text{ if } \mathbf{x}_i > \frac{1}{2} \\ S_{i-1} + 2^{-(i+1)} \text{ otherwise.} \end{cases}$$

By definition $S_i > 0$ for all i.
Define the decision rule D:

$$D(\mathbf{x}_i) = \begin{cases} -1 \text{ if } \mathbf{x}_i > \frac{1}{2} \\ 1 \text{ otherwise.} \end{cases}$$

Competing with discontinuous trading strategies
Randomized signals $\mathbf{x}_i$: positive result

Theorem

An algorithm for computing forecasts and a sequential method of randomization of forecasts $\tilde{p}_i$, past prices $\tilde{S}_{i-1}$, and signals $\tilde{\mathbf{x}}_i$ can be constructed such that for any nontrivial decision rule D for any $\delta > 0$, with probability at least $1-\delta$,

$$\sum_{i=1}^{n} \tilde{M}_i \Delta S_i \geq \|D\|_{\infty}^{-1} \sum_{i=1}^{n} D(\tilde{\mathbf{x}}_i) \Delta S_i - O\left(n^{\frac{4}{5}+\varepsilon} + \sqrt{\frac{n}{2} \ln \frac{2m}{\delta}} \right).$$

$\mathbf{x}_i$ can be perturbed by noise.

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Let $I_{\mathscr{S}}$ be the characteristic function of the set $\mathscr{S} \subseteq [0,1]^{k+3}$.
Information vector $x_i = (S_{i-1}, \mathbf{x}_i)$.
Calibration theorem for $k = 2$:
Given $\varepsilon > 0$ we can compute forecasts $p_1, p_2, \dots$ and a sequential method of randomization such that for any $\delta > 0$, with probability at least $1 - \delta$,

$$\left| \sum_{i=1}^{n} I_{\mathscr{S}}(\tilde{p}_i, \tilde{S}_{i-1}, \tilde{\mathbf{x}}_i)(S_i - \tilde{p}_i) \right| \leq 18 n^{\frac{4}{5}+\varepsilon} + \sqrt{\frac{n}{2} \ln \frac{2}{\delta}}.$$

for all n.
We use $\mathscr{S} = \{(p,s,x) : p > s\}$ and $\mathscr{S} = \{(p,s,x) : D(x) = d\}$.

Randomized signals: asymptotic result

Theorem

An algorithm for computing forecasts and a sequential method of randomization of forecasts $\tilde{p}_i$, *past prices* $\tilde{S}_{i-1}$, *and signals* $\tilde{\mathbf{x}}_i$ *can be constructed such that for any nontrivial decision rule* D

$$\liminf_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\tilde{M}_i\Delta S_i - \frac{1}{n}\|D\|_\infty^{-1}\sum_{i=1}^{n}D(\tilde{\mathbf{x}}_i)\Delta S_i\right)\geq 0$$

holds almost surely with respect to a probability distribution generated by the corresponding sequential randomization.

Numerical experiments (with V.G.Trunov)

Data has been downloaded from FINAM site: www.finam.ru
Number of trading points in each game is 88000–116000 min.
(From March 26 2010 to March 25 2011).
Arbitrarily chosen 11 US stocks, and 6 Russian stocks, TEST
We buy and sell 5 shares of each stock.
It was found that $\mathscr{K}_i > 0$ for $i = 1, 2, \ldots, 17$, i.e., we never incur debt in our experiments (with an exception of TEST stock).

Ticker	Buy& Hold Profit %	UN for a rise Profit %	UN for a fall Profit %	ARMA for a rise Profit %	ARM for Pro
TEST	6.85	-1.39	-8.19	9.88	3.08
AT-T	7.71	137.40	129.70	30.73	23.0
CTGR	15.04	1594.34	1579.34	1167.22	115
KOCO	16.55	62.66	46.15	2.90	-13.
GOOG	10.25	114.85	104.62	12.85	2.62
InBM	24.28	85.38	61.09	29.31	5.02
INTL	4.29	111.70	107.50	25.86	21.6
MSD	10.71	58.32	47.60	18.66	7.95
US1.AMT	22.01	22.74	0.77	28.46	6.49
US1.IP	2.40	19.83	17.47	9.36	7.00
US2.BRCM	25.30	53.62	28.28	20.06	-5.2
US2.FSLR	40.15	143.92	103.61	-9.86	-50.
SIBN	-6.54	732.87	739.33	357.74	364
GAZP	22.75	101.20	78.45	31.75	9.00
LKOH	19.39	261.84	242.45	87.08	67.6

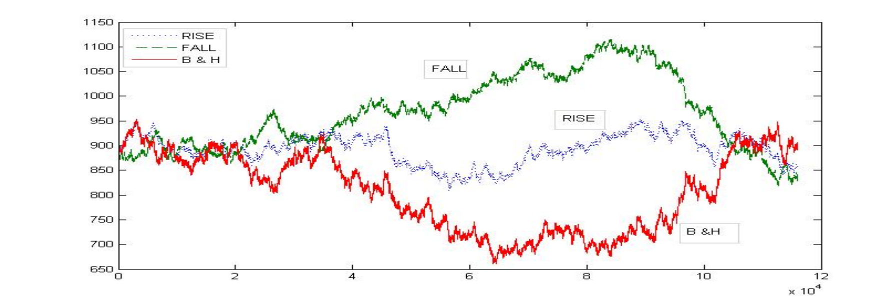

Figure: Evolution of capitals of the three trading strategies: Buy and Hold – solid (red) line, dealing for a rise – dotted (blue) line, dealing for a fall – dashed (green) line. One run of trading is performed with the simulated stock TEST.

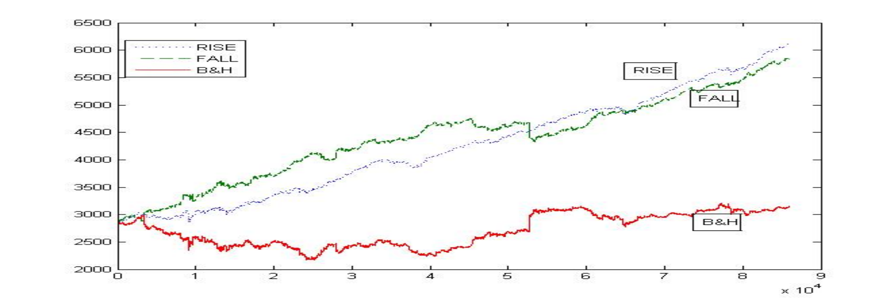

Figure: Evolution of capitals of the three trading strategies: Buy and Hold – solid (red) line, dealing for a rise – dotted (blue) line, dealing for a fall – dashed (green) line. One run of trading is performed with the stock GOOG).

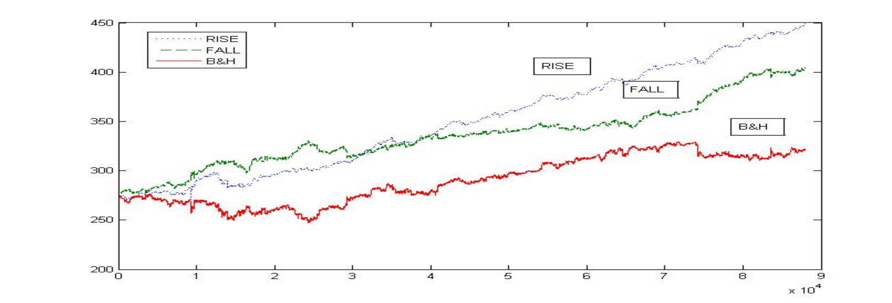

Figure: Evolution of capitals of the three trading strategies: Buy and Hold – solid (red) line, dealing for a rise – dotted (blue) line, dealing for a fall – dashed (green) line. One run of trading is performed with the stock KOCO).

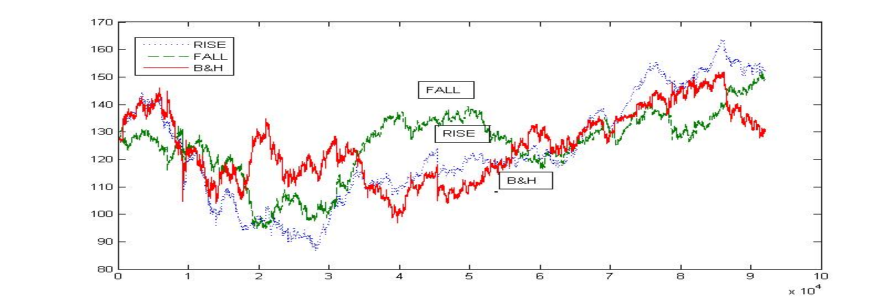

Figure: Evolution of capitals of the three trading strategies: Buy and Hold – solid (red) line, dealing for a rise – dotted (blue) line, dealing for a fall – dashed (green) line. One run of trading is performed with the stock US1.IP).

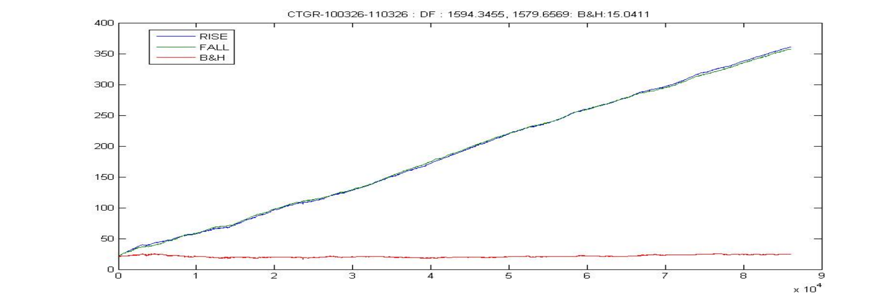

Figure: Evolution of capitals of the three trading strategies: Buy and Hold – solid (red) line, dealing for a rise – dotted (blue) line, dealing for a fall – dashed (green) line. One run of trading is performed with the stock CTGR).

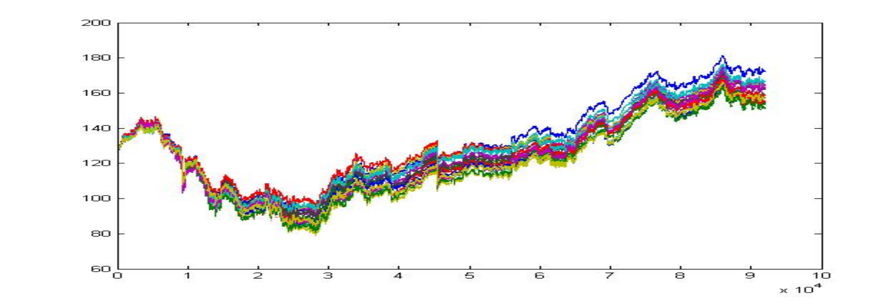

Figure: Evolution of capitals for 10 runs of UN strategy (dealing for a rise) using wll-calibrated forecasts of the stock US1.IP at the period 26.03.10–25.03.11. The size of grid for randomization is 0.2σ.

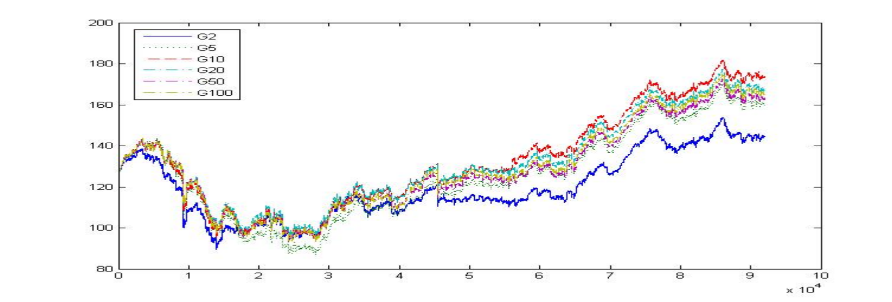

Figure: Means of capitals curves of UN trading (dealing for a rise) with stock US1.IP for randomization grid: 0.5σ ($G2$), 0.2σ ($G5$), 0.1σ ($G10$), 0.05σ ($G20$), 0.02σ ($G50$), 0.01σ ($G100$)