Trading network as Boltzmann mechanics of communicating vessels

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January 10, 2013

Motivation

Standard financial mathematics considers games of one or small number of players against the chance (random market). Recently, a new approach (called multi-agent models) appeared which considers the games of many players against each other.
Here some **local** models are considered where there are many players and many financial or trading instruments. The model resembles communication and transportation networks - the main difference is that the nodes have special dynamical values (moving boundaries, or real prices). The clients have also their own subjective prices and their interaction (transaction) with the nodes depend on these prices. This model does not describe any real situation.

Free one-phase Boltzmann dynamics

Consider the phase space $\mathbf{S} = I \times I_0$, where $I \subset R$ is an infinite interval and $I_0 = [-V_0, V_0], 0 < V_0 < \infty$. On $\mathbf{S}$ at any time $t \geq 0$ a random locally finite configuration $\{(x_i(t), v_i(t))\}$ of particles is given with coordinates $x_i \in I$ and velocities $v_i \in I_0$. Assume that this configuration at any time t has distribution P_t with one-particle correlation function $f(x, v, t)$ defined so that for any subset $A \subset \mathbf{S}$ of the phase space

$$E\#\{i : (x_i, v_i) \in A\} = \int_A f(x, v, t) dx dv$$

One can have in mind Poisson measure P_0 at time $t = 0$.

Free one-phase Boltzmann dynamics

Any particle moves always with its initial velocity, independently of other particles. Also there is Poisson income flow of particles from exterior with rate $\lambda(x,v,t)$, that is during time interval $[t,t+dt]$ the mean number of incoming particles to the cell $[x,x+dx]\times[v.v+dv]$ of the phase space is $\lambda(x,v,t)dxdvdt$. Assume moreover that each particle can die (disappear) with exponential distribution having rate $\mu(x,v,t)$. This means that during time dt $\mu(x,v,t)dxdvdt$ particles leave the cell $dxdv$. Assume boundedness of velocities, that is

$$f(x,v,t)=\lambda(x,v,t)=\mu(x,v,t)=0,|v|\geq V_0$$

Free one-phase Boltzmann dynamics

Lemma

For any $x \in I$ *and* $t < \frac{d(x,\partial I)}{V_0}$*, where* $d(x,\partial I)$ *is the distance of the point* x *from the boundary of* I*, the standard linear Boltzmann equation holds*

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = -\mu(x,v,t)f(x,v,t) + \lambda(x,v,t) \tag{1}$$

This is trivial for $\mu = \lambda = 0$. In fact, for small $\delta > 0$ we have

$$f(x,v,t+\delta) = f(x-v\delta,v,t) \tag{2}$$

if x is not on the boundary of I and δ is sufficiently small. Subtracting $f(x,v,t)$ from both parts of this equality, dividing by δ and taking the limit $\delta \to 0$, we have

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0 \tag{3}$$

Free one-phase Boltzmann dynamics

The unique solution of the Cauchy problem for (3) is

$$f(x,v,t) = f(x - vt, v, 0)$$

If $\lambda \neq 0, \mu = \mu(x,v) \neq 0$ then it is also easy to see that the equation (1) holds. Note that if $\lambda = 0$ and μ does not depend on t, there is also explicit solution

$$f(x,v,t) = f(x - vt, v, 0)\exp(-\int_0^t \mu(x - vs, v)ds)$$

Two phases - particle dynamics

We shall define two types of dynamics - particle dynamics and continuum media dynamics.
In the particle dynamics $(\pm)$-phases consist of $(\pm)$-particles so that each $(-)$-particle is to the left of any $(+)$-particle. Denote $b(t) \in R$ (boundary between phases) the coordinate of the leftmost $(+)$-particle. Then for $x \geq b(t)$ there is $(+)$-phase and for $x < b(t)$ there is $(-)$-phase. Particles move, as above, with their own velocities until a $(-)$-minus particle reaches the point $b(t)$, then it disappears together with the $(+)$-particle at $b(t)$ and the point $b(t)$ jumps to the coordinate of the new leftmost $(+)$-particle. After this, the process proceeds similarly.
Random configurations of particles are defined by the correlation functions $f_{\pm}(x, v, t)$ correspondingly. Assume that also the functions $\lambda_{\pm}(r, v, t), \mu_{\pm}(r, v, t), r \geq 0$, are defined, smooth on R_+ and zero if $r \geq R_0$ for some $0 < R_0 < \infty$.

Two phases - particle dynamics

The dynamics of one point correlation functions $f_{\pm}(x, v, t)$ for $x \neq b(t)$, that is on $(b(t), \infty)$ and $(-\infty, b(t))$ correspondingly, is given by the equations (already non-linear as $b(t)$ is unknown)

$$\frac{\partial f_{\pm}}{\partial t} + v\frac{\partial f_{\pm}}{\partial x} = -\mu_{\pm}(x - b(t), v, t)f_{\pm}(x, v, t) + \lambda_{\pm}(x - b(t), v, t) \tag{4}$$

This means that we assume that arrivals and departures depend only on the distance $r = |x - b(t)|$.

Two phases - particle dynamics

Thus two phases add reactions between particles of different phases. The following interpretation is useful. We consider one instrument (**stocks, futures, houses or other real estate** etc.). There are two types of traders - $(+)$-particles correspond to **sellers** and $(-)$-particles to **buyers,** x_i are subjective prices comfortable for the trader i. Collision between particles corresponds to transaction, after this both leave the market. In more general cases it will be possible that they do nor leave the market. We consider here a particular case when for some constant velocities $v_{\pm}$ and for any t

$$f_{\pm}(x, v, t) = \rho_{\pm}(x, t)\delta(v - v_{\pm})$$

For this to hold at any time t it is sufficient to demand that this holds for $t = 0$. *Initial conditions* are defined by the initial densities $\rho_{\pm}(r, 0)$. The velocities $v_{\pm}$ can be interpreted as averaged velocities for sellers and buyers correspondingly.

Two phases - fluid dynamics

Under some scaling the defined particle dynamics tends to some kind of **continuous (fluid) picture**, but we consider continuous densities of $(+)$-masses and $(-)$-masses and shall define their dynamics directly. We assume that at each time t there exists point $b(t)$ - boundary between phases. There are two phases with initial densities $\rho_+(r,0), \rho_-(r,0)$ where

$$r = r(t) = |x - b(t)| = \pm(x - b(t))$$

correspondingly. Phases move with velocities $v_\pm$correspondingly. Collision of plus and minus masses (at the point $b(t)$) leads to their cancellation in equal amount.
We obtain equations for the triple $(b(t), \rho_+(r,t), \rho_-(r,t))$ similarly to the way how the equations of continuum mechanuics are derived, that is using conservation laws. Here there is only one - mass conservation law.

Two phases - fluid dynamics

Obtain the equation for the boundary. Assume $b(t)$ smooth and put $\beta(t) = \frac{db(t)}{dt}$. Then for time dt the amount of positive mass, reaching the boundary will be

$$\begin{aligned} M_+(\beta, t)dt = & \int_{v-\beta<0} \int_{r<(-v+\beta)dt} f_+(r, v, t)dv + o(dt) \\ = & dt \int_{v-\beta<0} f_+(0, v, t)(-v+\beta)dv + o(dt) \end{aligned}$$

In fact, income and outcome give the contribution $o(dt)$. Similarly for negative mass

$$\begin{aligned} M_-(\beta, t)dt = & \int_{v-\beta>0} \int_{r<(v-\beta)dt} f_-(r, v, t)dv + o(dt) \\ = & dt \int_{v-\beta>0} f_-(0, v, t)(v-\beta)dv + o(dt) \end{aligned}$$

Lemma

For any t there exists unique $\beta = \beta(t)$ such that

$$M_+(\beta, t) = M_-(\beta, t) \tag{5}$$

In fact, consider the equation with respect to β

$$\int_{v-\beta<0} f_+(0, v, t)(-v+\beta)dv = \int_{v-\beta>0} f_-(0, v, t)(v-\beta)dv$$

Then if β increases, then the right-hand side decreases and the left-hand side increases.

Equation for boundary

We can rewrite the equation (5) in our case

$$\rho_+(0,t)(-v_+ + \beta(t)) = \rho_-(0,t)(v_- - \beta(t)) \tag{6}$$

from where we can get $\beta(t)$

$$\beta(t) = \frac{\rho_+(0,t)v_+ + \rho_-(0,t)v_-}{\rho_+(0,t) + \rho_-(0,t)} \tag{7}$$

Equations for densities

Now we should write the equations for the densities. For $\rho_+(r,t)$ we get

$$\begin{aligned}\rho_+(r,t+\Delta t) = &\rho_+(r-(v_+-\beta(t))\Delta t, t) - \mu_+(r,t)\rho_+(r,t)\Delta t \\ &+\lambda_+(r,t)\Delta t + o(\Delta t) \\ =&\rho_+(r,t) - (v_+-\beta(t))\frac{\partial \rho_+(r,t)}{\partial r}\Delta t \\ &-\mu_+(r,t)\rho_+(r,t)\Delta t + \lambda_+(r,t)\Delta t + o(\Delta t)\end{aligned}$$

In the limit $\Delta t \to 0$

$$\frac{\partial \rho_+(r,t)}{\partial t} = -(v_+-\beta(t))\frac{\partial \rho_+(r,t)}{\partial r} - \mu_+(r,t)\rho_+(r,t) + \lambda_+(r,t) \tag{8}$$

Similarly for $\rho_-(r,t)$:

$$\frac{\partial \rho_-(r,t)}{\partial t} = (v_- -\beta(t))\frac{\partial \rho_-(r,t)}{\partial r} - \mu_-(r,t)\rho_-(r,t) + \lambda_-(r,t) \tag{9}$$

System

We come to the following system from three equations

$$\beta(t) = \frac{\rho_+(0,t)v_+ + \rho_-(0,t)v_-}{\rho_+(0,t) + \rho_-(0,t)}$$

$$\frac{\partial \rho_+(r,t)}{\partial t} = -(v_+ - \beta(t))\frac{\partial \rho_+(r,t)}{\partial r} - \mu_+(r,t)\rho_+(r,t) + \lambda_+(r,t)$$

$$\frac{\partial \rho_-(r,t)}{\partial t} = (v_- - \beta(t))\frac{\partial \rho_-(r,t)}{\partial r} - \mu_-(r,t)\rho_-(r,t) + \lambda_-(r,t)$$

Networks with many markets

Let us call the previous model an elementary market. A network is a set V of elementary markets with similar parameters and variables indexed by $m \in V$

$$v_{\pm,m}, \lambda_{\pm,m}(r,t), \mu_{\pm,m}(r,t), \rho_{\pm,m}(r,t), b_m(t), \beta_m(t)$$

Denote by $|V|$ the cardinality of the set V.
There are also other parameters interconnecting the markets.
Denote $v_{+,m}(t)$ ($v_{-,m}(t)$) the total annihilation flow of ($\pm$)-particles from the market m. As they are equal we denote $v_m(t) = v_{+,m}(t) = v_{-,m}(t)$.

Networks with many markets

Let

$$v_{k,m}(+,+,r,t), v_{k,m}(+,-,r,t), v_{k,m}(-,+,r,t), v_{k,m}(-,+,r,t)$$

be the parts of these annihilation flows of $(\pm)$-particles, that after the transaction on the market m, become $(\mp)$-particles on the market k with the coordinate r. Denote

$$p_{km}(\pm,\pm,r,t) = \frac{v_{k,m}(\pm,\pm,r,t)}{v_m(t)}$$

We mean that $p_{km}(+,+,r,t) = p_{km}(-,-,r,t) \equiv 0$. Then for any k and t the conditions

$$\sum_{m \in V} \int_0^\infty p_{km}(+,-,r,t)dr \leq 1, \sum_{m \in V} \int_0^\infty (p_{km}(-,+,r,t)dr \leq 1$$

should hold.

Networks with many markets

We have then the following system of $3|V|$ equations:

$$
\begin{aligned}
\beta_m(t) &= \frac{v_{-,m}\rho_{-,m}(0,t) + v_{+,m}\rho_{+,m}(0,t)}{\rho_{+,m}(0,t) + \rho_{+,m}(0,t)} \\
\frac{\partial \rho_{+,m}(r,t)}{\partial t} &= -(v_{+,m} - \beta_m(t))\frac{\partial \rho_{+,m}(r,t)}{\partial r} - \mu_{+,m}(r,t)\rho_{+,m}(r,t) \\
&+ \lambda_{+,m}(r,t) + \sum_{k\in V}(v_{-,k} - \beta_k(t))\rho_{-,k}(0,t)p_{km}(-,+,r,t) \\
\frac{\partial \rho_{-,m}(r,t)}{\partial t} &= (v_{-,m} - \beta_m(t))\frac{\partial \rho_{-,m}(r,t)}{\partial r} - \mu_{-,m}(r,t)\rho_{-,m}(r,t) \\
&+ \lambda_{-,m}(r,t) - \sum_{k\in V}(v_{+,k} - \beta_k(t))\rho_{+,k}(0,t)p_{km}(+,-,r,t)
\end{aligned}
$$

Fixed points

Assume $\lambda_{\pm,m}(r,t), \mu_{\pm,m}(r,t), p_{km}(\pm,\pm,r,t)$ do not depend on t and have a compact support. Put

$$F_{+}^{(m)}(x) = -v_{+,m}^{-1}\int_0^x \mu_{+,m}(y)dy,\ F_{-}^{(m)}(x) = v_{-,m}^{-1}\int_0^x \mu_{-,m}(y)dy$$

$$\hat{\lambda}_{+,m} = \int_0^\infty \lambda_{+,m}(x)\exp\left(-F_{+}^{(m)}(x)\right)dx$$

$$\hat{\lambda}_{-,m} = \int_0^\infty \lambda_{-,m}(x)\exp\left(-F_{-}^{(m)}(x)\right)dx$$

$$\alpha_{km}(-,+) = \int_0^\infty p_{km}(-,+,x)\exp\left(-F_{+}^{(m)}(x)\right)dx$$

$$\alpha_{km}(+,-) = \int_0^\infty p_{km}(+,-,x)\exp\left(-F_{-}^{(m)}(x)\right)dx$$

for $k,m \in V$.

Fixed points

Define matrices A_{-+}, A_{+-} with elements $\alpha_{km}(-,+)$ $\alpha_{km}(+,-)$, where $k,m \in V$, and assume, that they have the following property:

$$\forall\, k \sum_{m \in V} \alpha_{km}(\pm,\pm) \leq 1,\ \exists\, k_0 \sum_{m \in V} \alpha_{km}(\pm,\pm) < 1 \tag{10}$$

Consider the following system of inequalities with respect $\bar{s}$

$$\bar{s}(E - A_{-+}) \geq \overline{\lambda}_+,\ \bar{s}(E - A_{+-}) \geq \overline{\lambda}_- \tag{11}$$

where E is the identity matrix and $\overline{\lambda}_\pm$ are vectors with coordinates $\hat{\lambda}_{\pm,m}$. We say that this system has a positive solution if there is vector $\bar{s}$ with positive coordinates satisfying both inequalities in (11). Generally, this system may not have a positive solution.

Theorem

Each solution $\bar{s} = (s_m,\, m \in V) > 0$ of the system (11) uniquely defines the fixed point as follows:

$$\rho_{+,m}(r) = -v_{+,m}^{-1} \exp\left(F_{+}^{(m)}(r)\right) \times \left(s_m - \int_0^r \left(\lambda_{+,m}(x) + \sum_{k \in V} s_k p_{km}(-,+,x)\right) \exp\left(-F_{+}^{(m)}(x)\right) dx\right)$$

$$\rho_{-,m}(r) = v_{-,m}^{-1} \exp\left(F_{-}^{(m)}(r)\right) \times \left(s_m - \int_0^r \left(\lambda_{-,m}(x) + \sum_{k \in V} s_k p_{km}(+,-,x)\right) \exp\left(-F_{-}^{(m)}(x)\right) dx\right)$$

If the set of positive solutions of system (11) is empty there is no any fixed point.

Fixed points and stationary points

Assume that the functions $\lambda_\pm(r) = \lambda_\pm(r,t)$ and $\mu_\pm(r) = \mu_\pm(r,t)$ do not depend on t (remind that they were assumed to have compact support). Denote

$$\gamma_{cr}^{(+)} = -v_+^{-1} \int_0^\infty \lambda_+(x) \exp\left(\frac{1}{v_+} \int_0^x \mu_+(y) dy \right) dx$$

$$\gamma_{cr}^{(-)} = v_-^{-1} \int_0^\infty \lambda_-(x) \exp\left(-\frac{1}{v_-} \int_0^x \mu_-(y) dy \right) dx$$

and

$$\gamma_{cr} = \max\left(\gamma_{cr}^{(+)}, \frac{v_- \gamma_{cr}^{(-)}}{-v_+} \right)$$

Fixed points and stationary points

We define the fixed point of our dynamics by the conditions: $\beta(t) = 0$ and $\rho_\pm(r,t)$ do not depend on time. Alternatively the fixed points are defined as any solutions of the stationary version

$$\rho_+(0)v_+ + \rho_-(0)v_- = 0 \tag{12}$$

$$-v_+\frac{\partial \rho_+(r)}{\partial r} - \mu_+(r)\rho_+(r) + \lambda_+(r) = 0 \tag{13}$$

$$v_-\frac{\partial \rho_-(r)}{\partial r} - \mu_-(r)\rho_-(r) + \lambda_-(r) = 0 \tag{14}$$

of the system (6,8,9). We call stationary point any solution of the system of equations (6,8,9), where $\beta = \beta(t)$ and the densities do not depend on t. We say that a fixed (or stationary) point has finite mass if

$$\int_0^\infty \rho_\pm(r)dr < \infty$$

Fixed points and stationary points

Theorem

Let the parameters $\lambda_{\pm}(r), \mu_{\pm}(r)$ and $v_{\pm}$be fixed. Then

1. *For any value of the parameter $\gamma_+ = \rho_+(0)$ there is at most one fixed point. For $\gamma_+ < \gamma_{cr}$ there is no any fixed point. For $\gamma_+ \geq \gamma_{cr}$ there exists exactly one fixed point*
2. *The fixed point has finite mass if $\gamma_{cr}^{(+)} = \gamma_{cr}^{(-)}$*
3. *For any γ_+, γ_-such that*

   $$\gamma_+ = \rho_+(0) \geq \gamma_{cr}^{(+)}, \ \gamma_- = \rho_-(0) \geq \gamma_{cr}^{(-)}$$

   there is exactly one stationary point and the boundary velocity is

   $$\beta = \frac{\rho_+(0)v_+ + \rho_-(0)v_-}{\rho_+(0) + \rho_-(0)}$$

4. *Stationary point has finite mass iff $\gamma_+ = \gamma_{cr}^{(+)}, \ \gamma_- = \gamma_{cr}^{(-)}$.*

where stationary densities are defined by

$$\rho_+(r) = \exp\left(-\frac{1}{v_+}\int_0^r \mu_+(x)dx\right) \times \left(\rho_+(0) + v_+^{-1}\int_0^r \lambda_+(x)\exp\left(\frac{1}{v_+}\int_0^x \mu_+(y)dy\right)dx\right)$$

$$\rho_-(r) = v_-^{-1}\exp\left(\frac{1}{v_-}\int_0^r \mu_-(x)dx\right) \times \left(-v_+\rho_+(0) - \int_0^r \lambda_-(x)\exp\left(-\frac{1}{v_-}\int_0^x \mu_-(y)dy\right)dx\right)$$