# Complex cobordisms and toric topology

**V. M. Buchstaber**

Steklov Mathematical Institute, Russian Academy of Sciences

buchstab@mi.ras.ru

"Novikov Day"

dedicated to the 75th anniversary of S. P. Novikov

MIAN 2013

S. P. Novikov about *complex cobordism theory*

"In the process of the work the author ran into a whole series
of new and tempting algebraic and topological situations,
analogues to which in the classical case
are either completely lacking or strongly degenerate;
many of them have not been considered in depth."
1967

"The store of algebraic ideas originating from complex
cobordism theory is still far from being exhausted."
2003

A cycle in a space $X$ can be defined as a map $f\colon M \to X$ of a smooth closed manifold $M$ to $X$.

This concept ascends to the work of H. Poincaré (1895).

A cycle is homologous to zero if the map $f$ can be extended to a map $W \to X$ where $W$ is a smooth manifold with boundary $\partial W = M$.

The appropriate notion of cocycle uses the *Pontryagin–Thom construction*.

Given an embedding of smooth manifolds $i\colon M_1 \to M_2$ such that $\partial M_1 = \varnothing$ and $i(M_1) \cap \partial M_2 = \varnothing$, the Pontryagin–Thom construction produces a map

$$M_2/\partial M_2 \to T\nu,$$

where $\nu$ is the normal bundle of $i$ and $T\nu = D\nu/\partial D\nu$ is its Thom space (here $D\nu \subset \nu$ denotes the subbundle of unit balls).

These geometric constructions can be used to define
a generalised theory of homology and cohomology,
the theory of bordism and cobordism.
The terminology is due to M. Atiyah (1961).

The definition of cobordism relies heavily on the fundamental notion of transverse regularity for maps between manifolds.

The theory of complex bordism $U_*(\cdot)$ and cobordism $U^*(\cdot)$ deals with stably complex manifolds ($U$-manifolds), that is, smooth manifolds $M^n$ which admit embeddings $M^n \subset \mathbb{R}^{n+2k}$ with a complex structure in the normal bundle.

The *disjoint union* of stably complex manifolds determines the addition in the group $U_{2n}(pt) = U^{-2n}(pt)$.
The *direct product* of manifolds determines the multiplication in the *graded* ring

$$\Omega_U = U^*(pt) = \sum_{n \geqslant 0} \Omega_U^{-2n}.$$

The group $\Omega_U$ was calculated independently by J. Milnor and S. P. Novikov (1960).

The proof of the *ring* isomorphism

$$\Omega_U \simeq \mathbb{Z}[a_1, \ldots, a_n, \ldots],$$

$\deg a_n = -2n$, known as the *Milnor–Novikov Theorem*, was first published in 1962 by S. P. Novikov.

S. P. Novikov in his talk
at the 1966 International Congress of Mathematicians in Moscow
outlined a programme for applications of complex cobordism
to problems of algebraic topology.

A detailed exposition of this programme
was given by S. P. Novikov in the fundamental paper
"*The methods of algebraic topology
from the viewpoint of cobordism theory*", 1967.

In this paper S. P. Novikov described the algebra $\mathcal{A}_U = \Omega_U S$ of cohomology operations in complex cobordism.

This algebra contains a subalgebra $S$ arising from the *Chern–Conner–Floyd characteristeic classes* $cf_k(\xi)$, $k = 1, 2, \dots$, of complex vector bundles $\xi$.

The algebra $S$ has a structure of a connected *graded Hopf algebra* $S = \sum_{q \geqslant 0} S_{2q}$ with $S_0 = \mathbb{Z}$.
It has been introduced independently by S. P. Novikov and P. Landweber and has since become known as the *Landweber–Novikov algebra*.

S. P. Novikov came to the problem of describing the first Chern–Conner–Floyd class

$$cf_1(\xi_1 \otimes \xi_2) \in U^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

as a special power series

$$f(u, v) \in \Omega_U[[u, v]]$$

in $u = cf_1(\xi_1)$ and $v = cf_1(\xi_2)$.

As it follows from the properties of the tensor product of complex line bundles, the series

$$cf_1(\xi_1 \otimes \xi_2) = f(u, v) = u + v + \sum a_{i,j} u^i v^j$$

defines a formal group law over the ring $\Omega_U$, which was called the *formal group law of geometric cobordisms*.

If the ring $A$ is a module over the rationals $\mathbb{Q}$,
then any formal group law $F(u,v)$ over $A$ has the form

$$F(u,v) = g^{-1}(g(u) + g(v)).$$

The series $g(u)$ is referred to as the *logarithm*
of the formal group law,
and the series $g^{-1}(t)$ satisfying $g^{-1}(g(u)) = u$ is its *exponential*.

A. S. Mishchenko using the Landweber–Novikov algebra
found an explicit formula for the *logarithm*
of the formal group law of geometric cobordisms:

$$g(u) = u + \sum_{n\geqslant 1} [\mathbb{C}P^n]\frac{u^{n+1}}{n+1}\,.$$

Here $\mathbb{C}P^n$ is an $n$-dimensional complex projective space.

The formal group law $f(u,v)$ and its logarithm were crucial for the construction of cobordism analogues $\Psi^k_U$ of the famous Adams operations $\Psi^k$ in $K$-theory, and for introducing the Chern character

$$\sigma_h\colon K^*(\cdot) \to U^*(\cdot) \otimes \mathbb{Q}.$$

The operations $\Psi^k_U$, now known as the Adams–Novikov operations, play a key role in Novikov's approach to the calculation of the group

$$E_2^{1,*} = \mathrm{Ext}_S^{1,*}(\mathbb{Z}, \Omega_U)$$

in the *Adams–Novikov spectral sequence*, and also in the deduction of important relations on fixed points of *group actions* on stably complex manifolds.

In spring 1967 S. P. Novikov visited the USA,
where he presented his programme, including
*applications of the formal group law of geometric cobordisms*.

After his Harward seminar talk Novikov was approached
by J. Tate, who said that he never thought
that *algebraic topology* was so close to his field of interest,
the *theory of formal groups*.

S. P. Novikov remarked that D. Quillen was in the audience.

In 1954 Lazard showed that the ring $\mathscr{A}$ of coefficients of the *universal* formal group law is $\mathbb{Z}[a_1, \ldots, a_n, \ldots]$.

In 1969 Quillen proved that the ring homomorphism

$$\mathscr{A} \to \Omega_U$$

*classifying* the formal group law of geometric cobordisms is an isomorphism.

He obtained important topological applications of this isomorphism using deep algebraic results, such as Cartier theory.

Note that the constuction of Morava K-theory, which is a powerful tool in stable homotopy theory, is based on this isomorphism.

Consider the ring $\Omega_U(\mathbb{Z}) = \sum_{n \geqslant 0} \Omega_U^{-2n}(\mathbb{Z})$ with

$$\Omega_U^{-2n}(\mathbb{Z}) = \{\sigma \in \Omega_U^{-2n} \otimes \mathbb{Q} \colon s\sigma \in \Omega_U^0 = \mathbb{Z} \text{ for } s \in S_{2n}\},$$

where $S = \sum_{n \geqslant 0} S_{2n}$ is the Landweber–Novikov algebra.

The subring $\Omega_U(\mathbb{Z}) \subset \Omega_U \otimes \mathbb{Q}$ is invariant
with respect to the action of the algebra $\mathcal{A}_U$.

There is an isomorphism of $S$-modules $\Omega_U(\mathbb{Z}) \cong S^*$ where
the $S$-module structure on $S^*$ is defined by
the canonical left action $l$ of the Landweber–Novikov
Hopf algebra $S$ on the dual Hopf algebra $S^*$.

The classical Grothendieck–Hirzebruch–Riemann–Roch theorem is formulated in terms of classical Chern character

$$ch : K^*(\cdot) \to H^*(\cdot; \mathbb{Q}).$$

In 1970 V. M. Buchstaber,
motivated by the problem of realization of cycles
and the problem of obtaining an analogue
of the Grothendieck–Hirzebruch–Riemann–Roch theorem
in complex cobordisms,
has constructed the theory of Chern–Dold character in cobordism

$$ch_U : U^*(X) \xrightarrow{\widehat{ch}_U} H^*(X; \Omega_U(\mathbb{Z})) \longrightarrow H^*(X; \Omega_U \otimes \mathbb{Q}),$$

where $\widehat{ch}_U$ is the multiplicative transformation of cohomology theories.

The transformation $\widehat{ch}_U$ is a homomorphism of $\mathcal{A}_U$-modules for any $X$ and induces the isomorphism

$$U^*(X) \otimes_{\Omega_U} \Omega_U(\mathbb{Z}) \to H^*(X; \Omega_U(\mathbb{Z}))$$

in the case when $H^*(X;\mathbb{Z})$ is torsion-free.

A spectral sequence associated with the filtration

$$\Omega_U = N_0 \subset N_1 \subset \ldots \subset N_\infty = \Omega_U(\mathbb{Z}) = S^*,$$

$$N_k = \{\sigma \in \Omega_U(\mathbb{Z}) : s_\omega \sigma \in N_{k-1}, |\omega| > 0\}$$

was introduced.

The Chern–Dold character is uniquely defined by the formula

$$ch_U g(u) = g(\psi(t)) = t,$$

where

$$\psi(t) = \widehat{ch}_U(cf_1(\xi)) \in H^*(\mathbb{C}P^\infty; \Omega_U(\mathbb{Z})) = \Omega_U(\mathbb{Z})[[t]],$$

thus the series $\psi(t)$ is the *exponential*
of the formal group law of geometric cobordisms.

$$ch_U(u) = t + \sum_{n \geqslant 1} (-1)^n [M^{2n}] \frac{t^{n+1}}{(n+1)!},$$

where $[M^{2n}] \in \Omega_U$ and Td $M^{2n} = 1$.

*Toric geometry* is actively developing for the last 30 years.
It is a field in algebraic geometry.
Its main object of study is a toric variety.
First topological analogues of toric varieties were introduced by M. Davis and T. Januszkiewicz, and have become known as *quasitoric manifolds*.

Following the papers of V. Buchstaber, N. Ray and T. Panov we describe quasitoric manifolds
using a topological analogue of the quotient construction of nonsingular projective toric varieties.

A convex $n$-dimensional polytope $P \subset \mathbb{R}^n$ is called *simple* if at each vertex of $P$ exactly $n$ facets meet.

Let

$$P = \{x \in \mathbb{R}^n \colon \langle a_i, x\rangle + b_i \geqslant 0,\ 1 \leqslant i \leqslant m\},$$

where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.
We assume that none of the $m$ inequalities can be removed without changing $P$.

Let us form an $(m \times n)$-matrix $A_P$ whose rows are vectors $a_i$ written in the standard basis, and set $b = (b_1, \dots, b_m) \in \mathbb{R}^m$.

The image of $P$ under the map

$$L_P \colon \mathbb{R}^n \to \mathbb{R}^m \colon L_P(x) = A_P x + b$$

is the intersection of the $n$-dimensional plane

$$\{y \in \mathbb{R}^m \colon y = A_P x + b\}$$

with the positive cone

$$\mathbb{R}^m_{\geqslant} = \{y \in \mathbb{R}^m \colon y_i \geqslant 0,\ i = 1, \dots, m\}.$$

The canonical action of the standard torus $\mathbb{T}^m$ on $\mathbb{C}^m$ defines the projection

$$\rho\colon \mathbb{C}^m \to \mathbb{R}^m_{\geqslant}\colon \rho(z) = (|z_1|^2, \ldots, |z_m|^2).$$

We define the space $\mathcal{Z}_P$ with the canonical action of $\mathbb{T}^m$ from the commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \longrightarrow & \mathbb{C}^m \\ \Big\downarrow \rho_P & & \Big\downarrow \rho \\ P & \xrightarrow{L_P} & \mathbb{R}^m_{\geqslant} \end{array}.$$

The space $\mathcal{Z}_P$ has the structure of a smooth manifold.

In the case when $P$ is an $n$-dimensional simplex $\Delta^n$ the space $\mathcal{Z}_P$ is a sphere $S^{2n+1}$.

In the general case the space $\mathcal{Z}_P$ is a complete intersection of quadratic surfaces.

The manifolds $\mathcal{Z}_P/T^k$,
where $T^k \subset \mathbb{T}^m$ is a torus freely acting on $\mathcal{Z}_P$,
give examples of manifolds important in different areas.

The manifolds $M^{2n} = \mathcal{Z}_P/T^{m-n}$ with a canonical action of $T^n$ are particullary interesting.

Let $\{F_1, \ldots, F_m\}$ be the set of facets of a *simple* polytope $P^n$.

An integral $(n \times m)$-matrix $\Lambda$ defines a characteristic map

$$\ell \colon \{F_1, \ldots, F_m\} \to \mathbb{Z}^n,$$

if its columns $\lambda_{j_1}, \ldots, \lambda_{j_n}$ form a basis of $\mathbb{Z}^n$
for any vertex $v = F_{j_1} \cap \ldots \cap F_{j_n}$.

The matrix $\Lambda$ defines an epimorphism $\ell \colon \mathbb{T}^m \to \mathbb{T}^n$.
The group $K(\Lambda) = \ker \ell$ of rank $(m-n)$ acts freely on $\mathcal{Z}_P$.

The orbit space $M = \mathcal{Z}_P/K(\Lambda)$ is a $2n$-dimensional smooth manifold with an action of the $n$-torus $T^n = \mathbb{T}^m/K(\Lambda)$.
We denote this action by $\alpha$.
It satisfies the Davis–Januszkiewicz conditions:

1) $\alpha$ is locally isomorphic to the standard $\mathbb{T}^n$-action on $\mathbb{C}^n$;

2) there is a projection $\pi\colon M \to P$ such that $\pi^{-1}(x)$ is an orbit of the action $\alpha$ for any $x \in P$.

Note that $\alpha$ has only isolated fixed points.

We refer to $M = M(P, \Lambda)$ as the *quasitoric manifold* with combinatorial data $(A_P, \Lambda)$.

The combinatorial data $(A_P, \Lambda)$ defines a canonical $T^n$-invariant tangential stably complex structure on the manifold $M(P, \Lambda)$.

Consider the projection $\pi \colon M(P, \Lambda) \to P$.

The set of facets $\{F_1, \dots, F_m\}$ of a polytope $P$ defines the set of mutually transverse codimension-two submanifolds

$$M_j = \pi^{-1}(F_j), \quad j = 1, \dots, m, \quad \text{in} \quad M(P, \Lambda).$$

Let $v = F_{j_1} \cap \dots \cap F_{j_n}$ be a vertex of $P$.
Then $x = \pi^{-1}(v) = M_{j_1} \cap \dots \cap M_{j_n}$ is a fixed point for the $T^n$-action $\alpha$ on $M(P, \Lambda)$.

We have a one-to-one correspondence between the set of vertices $\{v_1, \ldots, v_N\}$ and the set of fixed points $\{x_1, \ldots, x_N\}$ of the action.

The $2n$-dimensional tangent space $\tau_x$ to $M = M(P, \Lambda)$ at a fixed point $x$ decomposes into a direct sum

$$\tau_x\Big(M(P, \Lambda)\Big) = \nu_{j_1}|_x \oplus \ldots \oplus \nu_{j_n}|_x,$$

where $\nu_j \to M_j$ is the normal bundle of the submanifold $M_j$ in $M$.

The combinatorial data is also used to turn each $\nu_j$ into a complex line bundle for any $j = 1, \dots, m$.

Let $x = M_{j_1} \cap \dots \cap M_{j_n}$ be a fixed point.
Then $\sigma(x) = 1$ if the orientation of the space $\tau_x(M)$ determined by the stably complex structure on $M$ coincides with the orientation of $\nu_{j_1}|_x \oplus \dots \oplus \nu_{j_n}|_x$ determined by the orientations of the bundles $\nu_{j_k}$, $k = 1, \dots, m$.
Otherwise, $\sigma(x) = -1$.

This can be expressed by the formula

$$\sigma(x) = \operatorname{sign}\Big(\det(\lambda_{j_1}, \dots, \lambda_{j_n}) \det(a_{j_1}, \dots, a_{j_n})\Big).$$

Given a fixed point $x = F_{j_1} \cap \ldots \cap F_{j_n}$ denote
$\Lambda_x$ the square submatrix of $\Lambda$ of column vectors $j_1, \ldots, j_n$,
$W_x$ the matrix determined by $W_x^t = \Lambda_x^{-1}$ .

The weight vectors $w_1(x)$, $\ldots w_n(x)$ are the columns of $W_x$.

Consider the power system $[k](u) \in \Omega_U[[u]]$, $k \in \mathbb{Z}$,
for the formal group law of geometric cobordisms $f(u, v)$.

We have $[0](u) = 0$ and by induction $[k](u) = f(u, [k-1](u))$.

For $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{Z}^n$ and $u = (u_1, \ldots, u_n)$ one defines $[\omega](u)$ inductively with $[\omega](u) = [\omega_1](u)$ for $n = 1$ and

$$[\omega](u) = f_{q=1}^{n}[\omega_q](u_q) = f\left(f_{q=1}^{k-1}[\omega_q](u_q), [\omega_n](u_n)\right)$$

for $n \geqslant 2$.

For $u = (u_1, \ldots, u_n)$ set

$$\Phi((P, \Lambda); u) = \sum_{Ver(P)} \sigma(x) \prod_{j=1}^{n} \frac{1}{[w_j(x)](u)}.$$

We have

$$\Phi((P, \Lambda); u) \in \Omega_U[[u]],$$

$$\Phi((P, \Lambda); 0) = [M^{2n}(P, \Lambda)].$$

According to a theorem of Milnor,
any cobordism class $[M^{2n}] \in \Omega_U^{-2n}$ can be represented by a nonsingular algebraic variety.
The resulting algebraic variety is not connected in general.

F. Hirzebruch proposed the following problem:
describe the cobordism classes $[M^{2n}] \in \Omega_U^{-2n}$ which contain an irreducible (that is, connected) *nonsingular* algebraic variety.
This problem is still unsolved.

It is proved by V. M. Buchstaber and N. Ray (1998) that for any $n > 1$ the group $\Omega_U^{-2n}$ has a basis consisting of complex cobordism classes of quasitoric manifolds.

The standard connected sum $M_1^{2n} \# M_2^{2n}$ of quasitoric manifolds at neighbourhoods of their fixed points $x_k \in M_k^{2n}$, $k = 1, 2$, gives a manifold $M^{2n}$ with a $T^n$-action.

By the result of V. M. Buchstaber, T. E. Panov and N. Ray (2007), this manifold has a structure of a quasitoric manifold compatible with the quasitoric structures on $M_1^{2n}$ and $M_2^{2n}$ if and only if $\sigma(x_1)\sigma(x_2) = -1$.

A new operation of the "box sum"

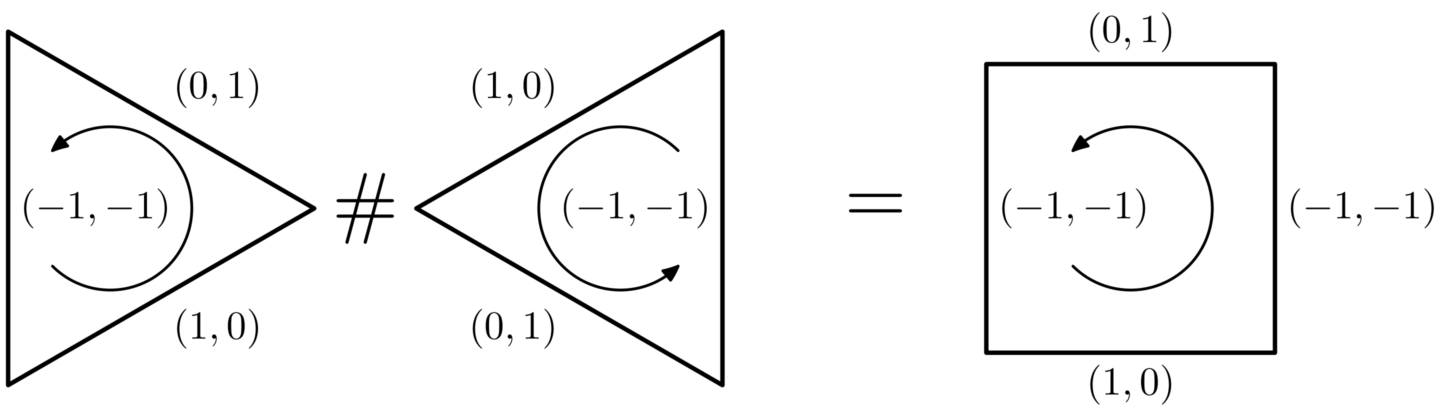
$$M_1^{2n} \diamondsuit M_2^{2n}$$

is introduced by V. M. Buchstaber, T. E. Panov and N. Ray, and it is proved that the following identity holds in the group $\Omega_U^{-2n}$:

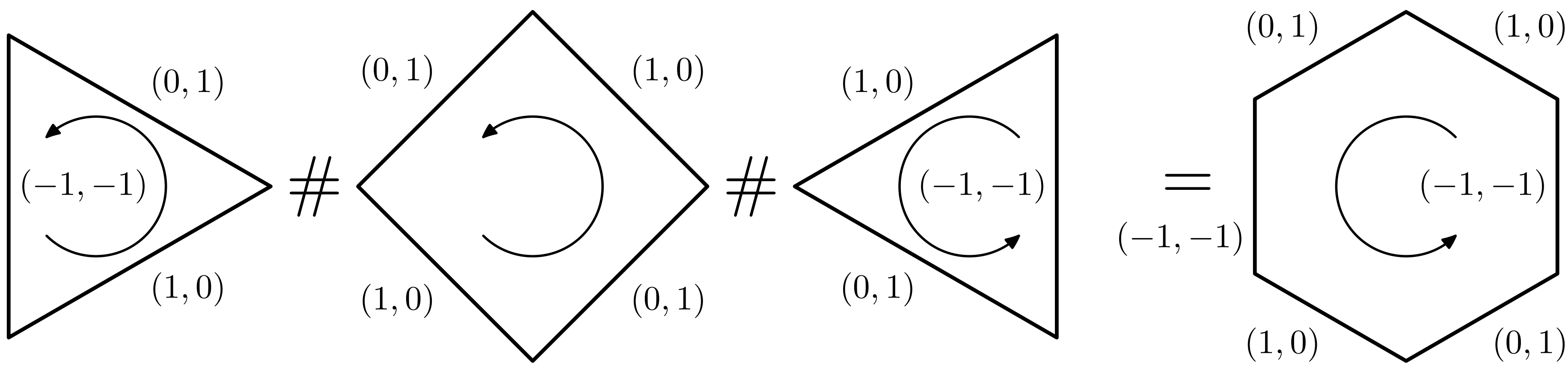
$$[M_1^{2n} \diamondsuit M_2^{2n}] = [M_1^{2n}] + [M_2^{2n}].$$

The operation $\diamondsuit$ allows us to connect-sum quasitoric manifolds at fixed points of the same sign.

In particular, the box sum $M_1^{2n} \diamondsuit M_2^{2n}$ of two toric manifolds $M_1$ and $M_2$ is defined; it is a quasitoric manifold.




The standard connected sum $\Delta^2 \# \Delta^2 \Rightarrow \mathbb{C}P^2 \# \mathbb{C}P^2$.




The box sum $\Delta^2 \Diamond \Delta^2 \Rightarrow \mathbb{C}P^2 \Diamond \mathbb{C}P^2$.

As a corollary of the result above, we obtain:

*Any complex cobordism class* $[M^{2n}] \in \Omega_U^{-2n}$*,* $n > 1$*, contains a quasitoric manifold.*

"Quasitoric" cannot be replaced by "toric" in the statement above.
This can be seen from the fact that any toric manifold has Todd genus 1.

A. A. Kustarev (2009) showed that if $\sigma(x) = 1$ for any fixed point $x$, then the canonical $T^n$-invariant stably complex structure on $M(P, \Lambda)$ is equivalent to a $T^n$-invariant almost complex structure.

Therefore, for a given combinatorial data $(A_P, \Lambda)$, the quasitoric manifold $M(P, \Lambda)$ admits a $T^n$-invariant *almost complex* structure if and only if $\sigma(x) = 1$ for any fixed point $x$.

**Ring of polytopes.**

Denote by $\mathfrak{P}^{2n}$ the free abelian group generated by all combinatorial $n$-polytopes.
The sum is induced by *disjoint union* of polytopes.

The group $\mathfrak{P}^{2n}$ is infinitely generated for $n > 1$, and it splits into a direct sum of finitely generated components as follows:

$$\mathfrak{P}^{2n} = \bigoplus_{m \geqslant n+1} \mathfrak{P}^{2n,2(m-n)},$$

where $\mathfrak{P}^{2n,2(m-n)}$ is the group generated by $n$-polytopes with $m$ facets.

The *direct product* of polytopes turns the direct sum

$$\mathfrak{P} = \bigoplus_{n \geqslant 0} \mathfrak{P}^{2n} = \mathfrak{P}^0 + \bigoplus_{m \geqslant 2} \bigoplus_{n=1}^{m-1} \mathfrak{P}^{2n, 2(m-n)}$$

into a *bigraded* commutative ring, the *ring of polytopes*.
The unit is $P^0$, a point.

*Simple polytopes* generate a *bigraded subring* of $\mathfrak{P}$,
which we denote by $\mathfrak{S}$.

A polytope is *indecomposable* if it cannot be represented as a product of two other polytopes of positive dimension.

$\mathfrak{P}$ is a bigraded *polynomial* ring generated by indecomposable combinatorial polytopes.

Given $P \in \mathfrak{P}^{2n}$, denote by $dP \in \mathfrak{P}^{2(n-1)}$ the *sum of all facets* of $P$ in the ring $\mathfrak{P}$.

$d\colon \mathfrak{P} \to \mathfrak{P}$ is a linear operator of degree $-2$ satisfying the identity

$$d(P_1P_2) = (dP_1)P_2 + P_1(dP_2).$$

Therefore, $\mathfrak{P}$ is a *differential ring*, and $\mathfrak{S}$ is its *differential subring*.

We have

$$dI^n = n(dI)I^{n-1} = 2nI^{n-1}, \quad \text{and} \quad d\Delta^n = (n+1)\Delta^{n-1}.$$

**Face-polynomials.**

Let $P$ be a convex $n$-polytope.
Denote by $f_i$ the number of $i$-dimensional faces of $P$.

The integer sequence $\mathbf{f}(P) = (f_0, f_1, \dots, f_n)$ is known as the *$f$-vector* (or the *face vector*) of $P$. Note that $f_n = 1$.
The homogeneous *$F$-polynomial* of $P$ is defined by

$$F(P)(s,t) = s^n + f_{n-1}s^{n-1}t + \ldots + f_1 s t^{n-1} + f_0 t^n.$$

The *$h$-vector* $\mathbf{h}(P) = (h_0, h_1, \dots, h_n)$ and the *$H$-polynomial* of $P$ are defined by

$$h_0 s^n + h_1 s^{n-1} t + \ldots + h_n t^n = (s-t)^n + f_{n-1}(s-t)^{n-1} t + \ldots + f_0 t^n,$$

$$H(P)(s,t) = h_0 s^n + h_1 s^{n-1} t + \ldots + h_{n-1} s t^{n-1} + h_n t^n = F(P)(s-t,t).$$

The $F$-polynomial and the $H$-polynomial define *ring homomorphisms*

$$F\colon \mathfrak{P} \longrightarrow \mathbb{Z}[s,t], \quad H\colon \mathfrak{P} \longrightarrow \mathbb{Z}[s,t],$$

which send $P \in \mathfrak{P}$ to $F(P)(s,t)$ and $H(P)(s,t)$ respectively.

For any *simple* polytope $P$ we have

$$F(dP) = \frac{\partial}{\partial t} F(P).$$

Let $P_1$ and $P_2$ be two simple $n$-polytopes, $n > 0$, such that $dP_1 = dP_2$ in $\mathfrak{G}$. Then $F(P_1) = F(P_2)$.

Let $\widetilde{F}\colon \mathfrak{S} \to \mathbb{Z}[s,t]$ be a linear map such that

$$\widetilde{F}(dP) = \frac{\partial}{\partial t}\widetilde{F}(P) \quad \text{and} \quad \widetilde{F}(P)|_{t=0} = s^n.$$

Then $\widetilde{F}(P) = F(P)$.

The following identity holds for any *simple* $n$-polytope $P$:

$$F(P)(s,t) = F(P)(-s, s+t).$$

The ring homomorphism $H\colon \mathfrak{P} \longrightarrow \mathbb{Z}[s,t]$ satisfies

$$H(P)\Big|_{t=0} = s^n.$$

The restriction of $H$ to the subring $\mathfrak{S}$ of simple polytopes satisfies the equation

$$H(dP) = \partial H(P)$$

where $\partial = \frac{\partial}{\partial s} + \frac{\partial}{\partial t}$.

The image of $\mathfrak{S}$ is the ring of *symmetric* polynomials generated by

$$H(\Delta^1) = s + t \quad \text{and} \quad H(\Delta^2) = s^2 + st + t^2.$$

We will call a 2-*truncation* the truncation of a face of codimension 2.
We will call 2-*truncated cubes* the polytopes obtained from a cube by series of 2-truncations.

One can check that a product of 2-truncated cubes is also a 2-truncated cube.

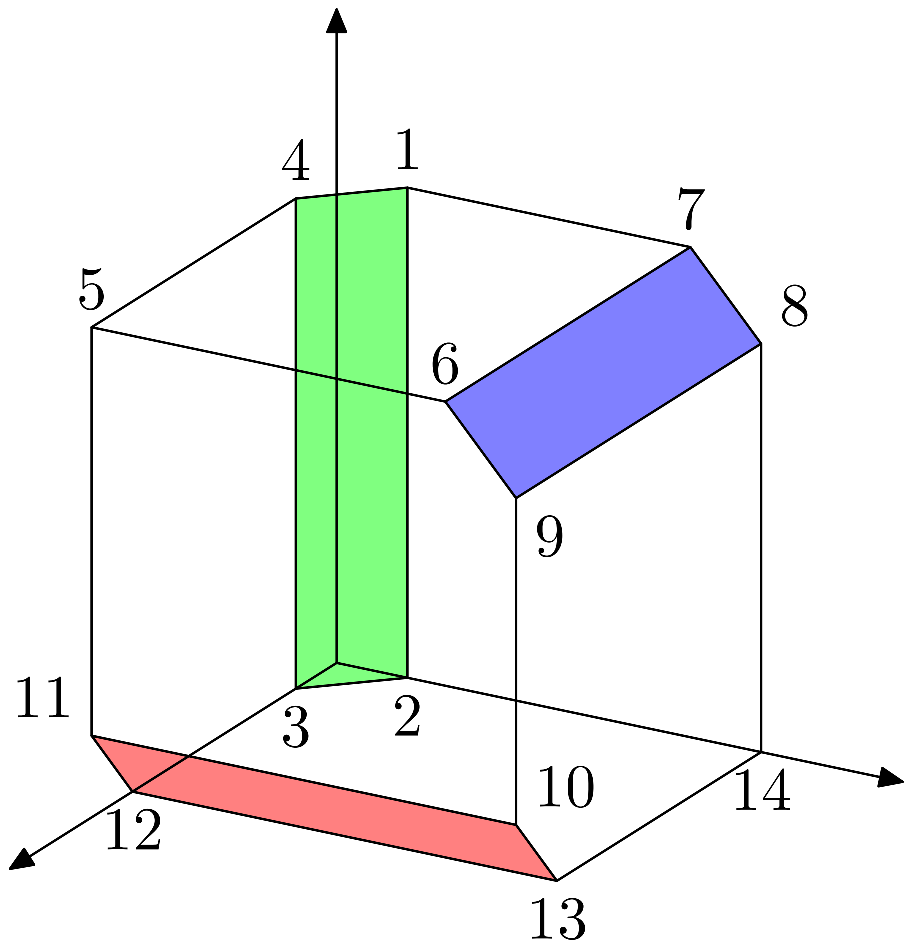
Thus we get the bigraded *ring of 2-truncated polytopes*.

An important class of simple polytopes is that of *nestohedra*. These polytopes arose in the work of C. De Concini and C. Procesi.
Nestohedra were constructed as *Minkowski sums* of special sets of simplices.

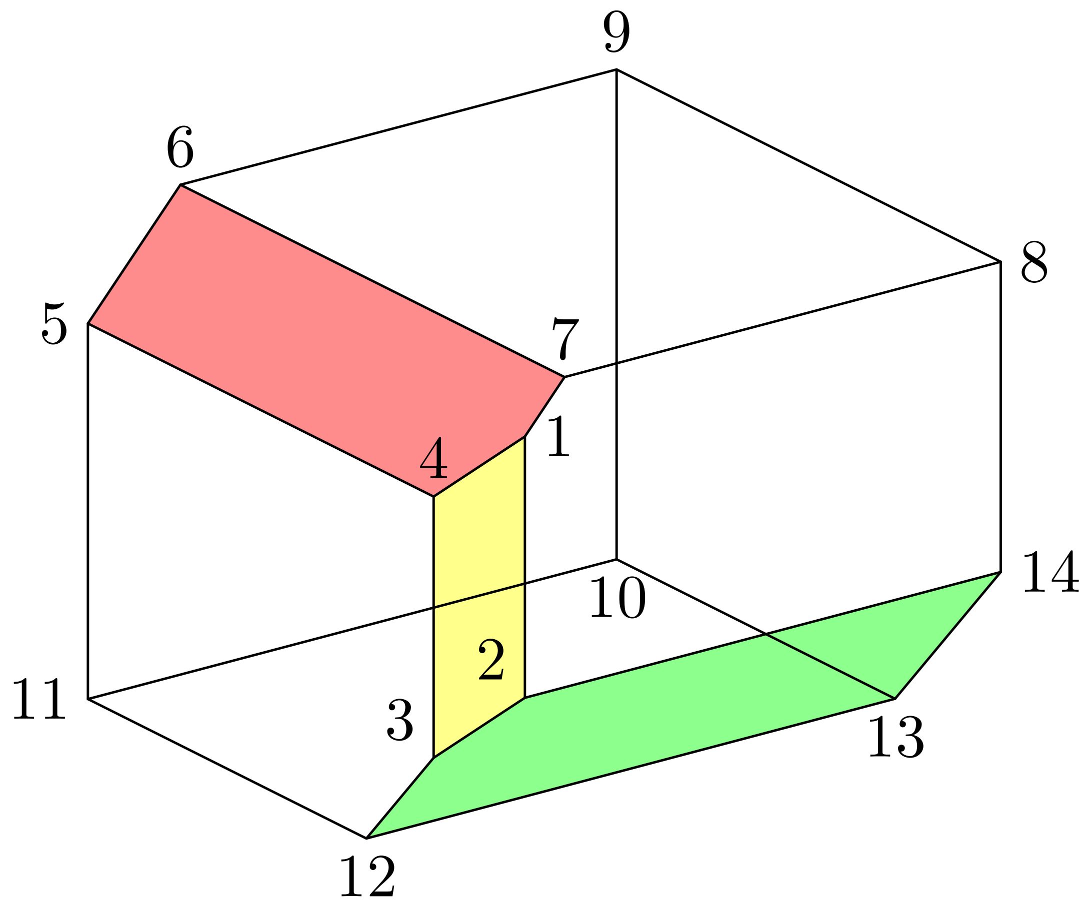
A simple polytope $P$ is called *flag* if every collection of its pairwise intersecting faces has a nonempty intersection.

V. M. Buchstaber and V. D. Volodin showed that a nestohedron is a 2-truncated cube if and only if it is a flag polytope.

A wide class of flag nestohedra, the graph-associahedra $P^n_\Gamma$, were introduced by M. Carr and S. Devadoss.
Any connected graph $\Gamma$ with $n+1$ vertex
gives a graph-associahedra $P^n_\Gamma$. Among them are
Stasheff polytopes (associahedra, $\Gamma$ is a path),
Bott–Taubes polytopes (cyclohedra, $\Gamma$ is a cycle),
permutohedra ($\Gamma$ is a complete graph),
stellohedra ($\Gamma$ is a star), . . .




3-dimensional associahedron (Stasheff polytope) with the Hamiltonian path $(1, 2, \ldots, 14)$.




Afinnely another 3-dimensional associahedron with the Hamiltonian path $(1, 2, \ldots, 14)$.

V. M. Buchstaber and V. D. Volodin (2011) described classes of nestohedra among which associahedra, stellohedra and permutohedra having extremal properties.

For any flag nestohedron $P^n$, we have

$$f_i(I^n) \leqslant f_i(P^n) \leqslant f_i(Pe^n).$$

Moreover, the lower bound is achieved iff $P^n \approx I^n$, the upper bound is achieved iff $P^n \approx Pe^n$.

For any connected graph $\Gamma_{n+1}$ on $[n+1]$, we have

$$f_i(As^n) \leqslant f_i(P_{\Gamma_{n+1}}) \leqslant f_i(Pe^n).$$

Moreover, the lower bound is achieved iff $\Gamma_{n+1}$ is a path, the upper bound is achieved iff $\Gamma_{n+1}$ is a complete graph.

For any Hamiltonian graph $\Gamma_{n+1}$ on $[n+1]$, we have

$$f_i(Cy^n) \leqslant f_i(P_{\Gamma_{n+1}}) \leqslant f_i(Pe^n).$$

Moreover, the lower bound is achieved iff $\Gamma_{n+1}$ is a cyclyc graph, the upper bound is achieved iff $\Gamma_{n+1}$ is a complete graph.

For any tree $\Gamma_{n+1}$ on $[n+1]$, we have

$$f_i(As^n) \leqslant f_i(P_{\Gamma_{n+1}}) \leqslant f_i(St^n).$$

Moreover, the lower bound is achieved iff $\Gamma_{n+1}$ is a path, the upper bound is achieved iff $\Gamma_{n+1}$ is a star graph.

The convex polytope $P \subset \mathbb{R}^n$ is called a *Delzant polytope* if at each vertex the normal vectors of the facets through the vertex can be chosen to form a $\mathbb{Z}$-basis for the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

**Delzant theorem.**
For every $n$-dimensional Delzant polytope $P^n$ there exists a *symplectic* $2n$-manifold $M^{2n}$ with *hamiltonian* torus action for which $P^n$ is the image of moment map.

Such a manifold can be constructed as quasi-toric manifold corresponding to geometric realization of the Delzant polytope.

It was proved that 2-truncated cubes are Delzant polytopes. Therefore each 2-truncated cube gives an *almost complex* manifold.

Denote by $\mathfrak{P}[q]$ the polynomial ring in an indeterminate $q$ with coefficients in $\mathfrak{P}$.

Let

$$Q\colon \mathfrak{P} \to \mathfrak{P}[q], \qquad P \mapsto Q(P;q)$$

be a linear map such that

$$Q(dP;q) = \frac{\partial}{\partial q} Q(P;q) \quad \text{and} \quad Q(P;0) = P$$

for any polytope $P$. Then

$$Q(P;q) = \sum_{k=0}^{n} (d^k P) \frac{q^k}{k!}.$$

Now assume given a sequence $\mathscr{P} = \{P^n,\ n \geqslant 0\}$ of polytopes, one in each dimension.

We define its *generating series* as the formal power series

$$\mathscr{P}(x) = \sum_{n \geqslant 0} \lambda_n P^n x^{n+n_0}$$

in $\mathfrak{P} \otimes \mathbb{Q}[[x]]$.

The parameter $n_0$ and the coefficients $\lambda_n$ will be chosen depending on a particular sequence $\mathscr{P}$.

Using the transformation $Q$ we may define the 2-parameter extension of the generating series:

$$\mathscr{P}(q,x) = \sum_{n \geqslant 0} \lambda_n Q(P^n; q) x^{n+n_0}.$$

We have $\mathscr{P}(0,x) = \mathscr{P}(x)$.

We consider the following generating series
of the six sequences of well-known polytopes:

$$\mathit{\Delta}(x) = \sum_{n\geqslant 0} \mathit{\Delta}^n \frac{x^{n+1}}{(n+1)!}\ ; \quad I(x) = \sum_{n\geqslant 0} I^n \frac{x^n}{n!}\ ;$$

$$As(x) = \sum_{n\geqslant 0} As^n x^{n+2}\ ; \qquad Pe(x) = \sum_{n\geqslant 0} Pe^n \frac{x^{n+1}}{(n+1)!}\ ;$$

$$Cy(x) = \sum_{n\geqslant 0} Cy^n \frac{x^{n+1}}{n+1}\ ; \qquad St(x) = \sum_{n\geqslant 0} St^n \frac{x^n}{n!}\ .$$

The differentials of the generating series above are given by

$$d\Delta(x) = x\Delta(x)\ ; \qquad dI(x) = 2xI(x)\ ;$$

$$d\,As(x) = As(x)\frac{d}{dx}As(x)\ ; \qquad d\,Pe(x) = Pe^2(x)\ ;$$

$$d\,Cy(x) = As(x)\frac{d}{dx}Cy(x)\ ; \qquad d\,St(x) = \Big(x + Pe(x)\Big)St(x)\ .$$

The two-parameter extensions
of the generating series mentioned above satisfy
the partial differential equations:

$$\frac{\partial}{\partial q}\Delta(q,x) = x\Delta(q,x)\ ; \qquad \frac{\partial}{\partial q}I(q,x) = 2xI(q,x)\ ;$$

$$\frac{\partial}{\partial q}As(q,x) = As(q,x)\frac{\partial}{\partial x}As(q,x)\ ; \quad \frac{\partial}{\partial q}Pe(q,x) = Pe^2(q,x)\ ;$$

$$\frac{\partial}{\partial q}Cy(q,x) = As(q,x)\frac{\partial}{\partial x}Cy(q,x)\ ; \quad \frac{\partial}{\partial q}St(q,x) = \Big(x + Pe(q,x)\Big)St(q,x)\ .$$

Consider series

$$Pe(x) = \sum_{n=0}^{\infty} Pe^n \frac{x^{n+1}}{(n+1)!}.$$

Applying $H$ we obtain series

$$H_{Pe}(x) = \sum_{n=0}^{\infty} H(Pe^n) \frac{x^{n+1}}{(n+1)!}.$$

This series satisfies the equation

$$\frac{dH_{Pe}(x)}{dx} = (1 + \alpha H_{Pe}(x))(1 + tH_{Pe}(x)),$$

where $H_{Pe}(x) = \sum_{n=0}^{\infty} H(Pe^n) \frac{x^{n+1}}{(n+1)!}$.

We obtain

$$H_{Pe}(x) = \frac{e^{\alpha x} - e^{tx}}{\alpha e^{tx} - t e^{\alpha x}} \; .$$

It is the exponential of the formal group law

$$F(u,v) = \frac{u + v + (\alpha + t)uv}{1 - \alpha t uv}.$$