Commuting ordinary differential operators of arbitrary genus and arbitrary rank with polynomial coefficients

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Theorem. *The operators L_{2r} and $M_{(2g+1)r}$ of orders $2r$ and $(2g+1)r$, respectively,*

$$L_{2r} = \left(aT_r\left(\frac{d}{dx}\right) - x^2\frac{d^2}{dx^2} - 3x\frac{d}{dx} + x^2 + b\right)^2 - ar^2g(g+1)T_r\left(\frac{d}{dx}\right)$$

$$M^2_{(2g+1)r} = L^{2g+1}_{2r} + a_{2g}L^{2g}_{2r} + \ldots + a_1L_{2r} + a_0,$$

where a_i are some constants, $T_r(x)$ is the Chebyshev polynomial of the first kind, of degree r, $r > 1$, the notation $T_r\left(\frac{d}{dx}\right)$ means the ordinary differential operator, which is the Chebyshev polynomial T_r of $\frac{d}{dx}$, a is an arbitrary nonzero constant, b is an arbitrary constant, are commuting operators of rank r, genus g, $[L_{2r}, M_{(2g+1)r}] = 0$, the coefficients of the operator $M_{(2g+1)r}$ are expressed polynomially in terms of the coefficients of the operator L_{2r} and their derivatives. For $r > 3$ the commuting operators L_{2r} and $M_{(2g+1)r}$ have the standard canonical form (for $a = 1/2^{r-1}$). For $r = 1$ this pair of operators is commuting one of rank 2 and genus g.

$$L = \sum_{i=0}^{n} u_i(x) \frac{d^i}{dx^i}, \quad M = \sum_{i=0}^{m} v_i(x) \frac{d^i}{dx^i},$$

$$[L, M] = 0.$$

Burchnall–Chaundy lemma. *There is a polynomial $Q(\lambda, \mu)$, in two variables, such that $Q(L, M) = 0$.*
The algebraic spectral curve Γ: $Q(\lambda, \mu) = 0$.
To a generic point (λ, μ) on the curve there corresponds at least one common eigenfunction $\psi = \psi(x, \lambda, \mu)$ of L and M:

$$L\psi = \lambda\psi, \quad M\psi = \mu\psi.$$

The dimension r of the vector bundle of common eigenfunctions $\psi(x, \lambda, \mu)$ at a generic point of the spectral curve $Q(\lambda, \mu) = 0$ is called *the rank of the commuting pair*.

The rank of any pair of commuting operators is a common divisor of the orders of these commuting operators.
For commuting operators of relatively prime orders $r = 1$.
In the case of rank $r = 1$ the commutation equations have been integrated by Burchnall, Chaundy, Baker, Krichever. In this case the common eigenfunctions and the coefficients of commuting operators are expressed explicitly in terms of the Θ-function of the spectral curve (Krichever).
The case $r > 1$ for spectral curves of nontrivial genus $g > 0$ is much more complicated.
The operators are usually assumed to be in the standard canonical form (Burchnall, Chaundy), i.e.

$$u_n(x) = v_m(x) = 1, \quad u_{n-1}(x) = 0.$$

For a pair of commuting operators this can always be achieved by a change of variables and a suitable conjugacy (Burchnall, Chaundy).

The first examples of commuting scalar ordinary differential operators of the nontrivial ranks 2 and 3 and the nontrivial genus $g = 1$ were constructed by Dixmier for the nonsingular elliptic spectral curve $\mu^2 = \lambda^3 - \alpha$, where α is an arbitrary nonzero constant:

$$L = \left(\left(\frac{d}{dx} \right)^2 + x^3 + \alpha \right)^2 + 2x,$$

$$M = \left(\left(\frac{d}{dx} \right)^2 + x^3 + \alpha \right)^3 + 3x \left(\frac{d}{dx} \right)^2 + 3\frac{d}{dx} + 3x(x^3 + \alpha),$$

where L and M is the commuting pair of the Dixmier operators of rank 2 and genus 1,

$$M^2 = L^3 - \alpha, \quad [L, M] = 0,$$

the orders of the commuting operators L and M are 4 and 6, rank 2;

$$L = \left(\left(\frac{d}{dx} \right)^3 + x^2 + \alpha \right)^2 + 2 \frac{d}{dx},$$

$$M = \left(\left(\frac{d}{dx} \right)^3 + x^2 + \alpha \right)^3 + 3 \left(\frac{d}{dx} \right)^4 + 3(x^2 + \alpha) \frac{d}{dx} + 3x,$$

where L and M is the commuting pair of the Dixmier operators of rank 3 and genus 1,

$$M^2 = L^3 - \alpha, \quad [L, M] = 0,$$

the orders of the commuting operators L and M are 6 and 9, rank 3.
These remarkable unusual examples of commuting scalar ordinary differential operators with polynomial coefficients were found by Dixmier as commutative subalgebras of the Weyl algebra W_1 by a quite nontrivial, purely algebraic way without any connection to the spectral theory of commuting operators.
Both the Dixmier examples are in the standard canonical form.

The general classification of commuting scalar ordinary differential operators of nontrivial ranks $r > 1$ was obtained by Krichever: a pair of commuting operators of rank r is determined by specifying the curve Γ, a point $P_0 \in \Gamma$, a local parameter $k^{-1}(P)$ in a neighbourhood of P_0, by specifying $r^2 g$ (g is the genus of the curve Γ) constants (α_{ij}, γ_i), $1 \leq i \leq rg$, $0 \leq j \leq r-2$, called the *Tyurin parameters,* and by specifying $r-1$ arbitrary functions $w_j(x)$.

The common eigenfunctions $\psi_j(x, P; x_0)$, $0 \leq j \leq r-1$, $P \in \Gamma$, normalized by the condition

$$\left(\frac{d^i}{dx^i} \psi_j(x, P; x_0) \right) \Big|_{x=x_0} = \delta_{ij}, \quad 0 \leq i, j \leq r-1,$$

have the following analytic properties (Krichever):

1. The common eigenfunctions $\psi_j(x, P; x_0)$, $0 \le j \le r-1$, are meromorphic on the spectral curve Γ outside P_0, and each one has rg simple poles $\gamma_i(x_0)$, with

$$\psi_j(x, z; x_0) \sim \frac{\psi_{ij}(x, x_0)}{z - \gamma_i(x_0)}, \quad 0 \le j \le r-1, \;\; 1 \le i \le rg,$$

in a neighbourhood of the pole $\gamma_i(x_0)$.
2. All the residues $\psi_{ij}(x, x_0)$ are proportional to one of them:

$$\psi_{ij}(x, x_0) = \alpha_{ij}(x_0)\psi_{ir-1}(x, x_0), \quad 0 \le j \le r-2.$$

3. If $k^{-1}(P)$ is a local parameter on Γ in a neighbourhood of P_0, then we have the asymptotics

$$\vec{\psi}(x, P; x_0) = \left(\sum_{s=0}^{\infty} \vec{\xi}_s(x) k^{-s} \right) \Phi_0(x, k; x_0),$$

where $\vec{\psi}(x, P; x_0) = (\psi_0(x, P; x_0), \ldots, \psi_{r-1}(x, P; x_0))$, $\vec{\xi}_0(x) = (1, 0, \ldots, 0)$, $\vec{\xi}_s(x_0) = \vec{0}$ for $s \geq 1$, and $\Phi_0(x, k; x_0) = (\Phi_0^{ij})$ is a solution of the matrix equation

$$\frac{d\Phi_0}{dx} = S\Phi_0, \quad \Phi_0^{ij}(x_0, k; x_0) = \delta^{ij}, \quad 0 \leq i, j \leq r-1,$$

and

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ k + w_0(x) & w_1(x) & w_2(x) & \cdots & w_{r-2}(x) & 0 \end{pmatrix}.$$

The analytic properties 1, 2 and 3, the arbitrary constants (γ_i, α_{ij}), $1 \leq i \leq rg$, $0 \leq j \leq r-2$, and the arbitrary functions $w_0(x), \ldots, w_{r-2}(x)$ determine a vector-valued function $\vec{\psi}(x, P; x_0)$ and a commuting pair L, M of rank r and genus g in general position (Krichever).

In order to find commuting operators Krichever and Novikov proposed the method of deforming the Tyurin parameters $(\gamma_i(x_0), \alpha_{ij}(x_0))$ that allowed to obtain the general form of commuting ordinary scalar differential operators of rank $r = 2$ for arbitrary elliptic spectral curve (genus $g = 1$).

Let us consider the Wronskian matrix

$$\widetilde{\psi}(x, P; x_0) = \begin{pmatrix} \psi_0 & \psi_1 & \cdots & \psi_{r-1} \\ \psi_0' & \psi_1' & \cdots & \psi_{r-1}' \\ \dots & \dots & \dots & \dots \\ \psi_0^{(r-1)} & \psi_1^{(r-1)} & \dots & \psi_{r-1}^{(r-1)} \end{pmatrix}$$

of the vector-valued function $\vec{\psi}(x, P; x_0)$, and

$$\widetilde{\psi}_x \widetilde{\psi}^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & 1 \\ \chi_0 & \chi_1 & \chi_2 & \cdots & \chi_{r-1} \end{pmatrix},$$

where $\chi_j(x, P)$ are meromorphic functions on Γ.

For $x = x_0$ the poles of $\chi_j(x_0, P)$ coincide with $\gamma_1(x_0), \ldots, \gamma_{rg}(x_0)$, and the ratios of the residues of the functions $\chi_j(x_0, P)$ at the points $\gamma_i(x_0)$ coincide with the parameters $\alpha_{ij}(x_0)$:

$$\alpha_{ij}(x_0) = \frac{\mathrm{res}_{\gamma_i(x_0)}\chi_j}{\mathrm{res}_{\gamma_i(x_0)}\chi_{r-1}}.$$

In a neighbourhood of P_0 on Γ the functions $\chi_j(x_0, P)$ have the form

$$\chi_0(x, P) = k + w_0(x) + O(k^{-1}),$$
$$\chi_s(x, P) = w_s(x) + O(k^{-1}), \quad 1 \le s \le r - 2,$$
$$\chi_{r-1}(x, P) = O(k^{-1}).$$

The expansion of χ_j in a neighbourhood of the pole $\gamma_i(x)$ has the form

$$\chi_j(x, k) = \frac{c_{ij}(x)}{k - \gamma_i(x)} + d_{ij}(x) + O(k - \gamma_i(x)),$$
$$c_{ij} = \alpha_{ij} c_{ir-1}, \quad 0 \le j \le r - 1, \ 1 \le i \le rg.$$

Krichever and Novikov proved that the functional parameters $\gamma_i(x)$ and $\alpha_{ij}(x)$, $1 \leq i \leq rg$, $0 \leq j \leq r-2$, satisfy the system (the Krichever–Novikov system of equations of deformation of the Tyurin parameters)

$$\gamma_i^{'} = -c_{ir-1},$$

$$\alpha_{i0}^{'} = \alpha_{i0}\alpha_{ir-2} + \alpha_{i0}d_{ir-1} - d_{i0},$$
$$\alpha_{ij}^{'} = \alpha_{ij}\alpha_{ir-2} - \alpha_{ij-1} + \alpha_{ij}d_{ir-1} - d_{ij}, \quad j \geq 1.$$

For arbitrary functions $w_i(x)$, $0 \leq i \leq r-2$, the solution of the Krichever–Novikov system uniquely determine an array of meromorphic functions $\chi_j(x_0, P)$ with the required analytic properties (Krichever, Novikov).
By the asymptotics of $\chi_j(x_0, P)$ in a neighbourhood of P_0 on Γ one can reconstruct the commuting operators.
Note that the coefficients of one from a pair of commuting operators are always expressed polynomially in terms of the coefficients of the second operator of the commuting pair and their derivatives.

The general form of commuting scalar ordinary differential operators (L_4, L_6) of rank 2 for an arbitrary elliptic spectral curve

$$\mu^2 = 4\lambda^3 - g_2\lambda - g_3$$

was found by Krichever and Novikov:

$$(L_6)^2 = 4\,(L_4)^3 - g_2 L_4 - g_3,$$

$$L_4 = \left(\frac{d^2}{dx^2} - v(x)\right)^2 - 2\varphi_1(x)\frac{d}{dx} - 2\varphi_1'(x) - 2\varphi_0(x),$$

$$\varphi_0(x) = \frac{1}{2}\left(\wp(\gamma_0 + c(x)) + \wp(\gamma_0 - c(x))\right) -$$

$$-\frac{1}{2}\left(c'(x)\left(\wp(\gamma_0 + c(x)) - \wp(\gamma_0 - c(x))\right)\right)_x,$$

$$\varphi_1(x) = c'(x)\left(\wp(\gamma_0 + c(x)) - \wp(\gamma_0 - c(x))\right),$$

$$v(x)=\frac{1}{4(c'(x))^2}-\frac{1}{4}\frac{(c''(x))^2}{(c'(x))^2}+\frac{1}{2}\frac{c'''(x)}{c'(x)}+$$

$$+2\Phi(x)c''(x)+\Phi^2(x)(c'(x))^2+\Phi_c(x)(c'(x))^2,$$

$$\Phi(x)=\frac{1}{2}\frac{\wp'(\gamma_0+c(x))+\wp'(\gamma_0-c(x))}{\wp(\gamma_0+c(x))-\wp(\gamma_0-c(x))},$$

where $\gamma_0=\text{ const}$, $c(x)$ is an arbitrary function,

$$L_6=2\left(\frac{d^2}{dx^2}-v(x)\right)^3-6\varphi_1(x)\frac{d^3}{dx^3}-6(\varphi_0(x)+2\varphi_1'(x))\frac{d^2}{dx^2}-$$

$$-2(3\varphi_3(x)+6\varphi_0'(x)+7\varphi_1''(x)-3v(x)\varphi_1(x))\frac{d}{dx}-$$

$$-8\varphi_1'''(x)-6\varphi_2(x)-12\varphi_3'(x)-14\varphi_0''(x)+6v(x)\varphi_0(x),$$

$$\varphi_2(x) = -\frac{c'(x)}{2}\left(\eta_1(x)\wp'(\gamma_0 + c(x)) - \eta_2(x)\wp'(\gamma_0 - c(x))\right),$$

$$\varphi_3(x) = -\frac{c'(x)}{2}\left(\wp'(\gamma_0 + c(x)) - \wp'(\gamma_0 - c(x))\right),$$

$$\eta_1(x) = \frac{1}{2c'(x)} - \frac{1}{2}\frac{c''(x)}{c'(x)} - \Phi(x)c'(x),$$

$$\eta_2(x) = -\frac{1}{2c'(x)} - \frac{1}{2}\frac{c''(x)}{c'(x)} - \Phi(x)c'(x).$$

Rational parametrization:

$$w(x) = \wp(c(x)).$$

$$\lambda(x) = \wp(\gamma_0) + \frac{1}{2}\frac{\wp''(\gamma_0)}{w(x) - \wp(\gamma_0)}$$

$$L_4 = \left(\frac{d^2}{dx^2} - v(x)\right)^2 - 2A\lambda'(x)\frac{d}{dx} -$$

$$-2\lambda(x) - A^2(\lambda(x) - \wp(\gamma_0))^2 - A\lambda''(x),$$

$$A = 2\frac{\wp'(\gamma_0)}{\wp''(\gamma_0)} = \text{ const},$$

$$v(x) = \frac{1}{2}\frac{\lambda'''(x)}{\lambda'(x)} - \frac{1}{4}\frac{(\lambda''(x))^2}{(\lambda'(x))^2} + \frac{4\lambda^3(x) - g_2\lambda(x) - g_3}{4(\lambda'(x))^2} +$$

$$+A^2\frac{(\lambda(x) - \wp(\gamma_0))^4 - (\wp''(\gamma_0)/2)^2}{4(\lambda'(x))^2}.$$

Self-adjoint commuting operators of rank 2 and genus 1: $A = 0$.

Self-adjoint commuting operators (L_4, L_6) of rank 2 and genus 1:

$$\mu^2 = 4\lambda^3 - g_2\lambda - g_3,$$

$$L_4 = \left(\frac{d^2}{dx^2} - v(x)\right)^2 - 2\lambda(x),$$

$$L_6 = 2\left(\frac{d^2}{dx^2} - v(x)\right)^3 - 6\lambda(x)\frac{d^2}{dx^2} - 6\lambda'(x)\frac{d}{dx} - 5\lambda''(x) + 6v(x)\lambda(x),$$

$$v(x) = \frac{1}{2}\frac{\lambda'''(x)}{\lambda'(x)} - \frac{1}{4}\frac{(\lambda''(x))^2}{(\lambda'(x))^2} + \frac{4\lambda^3(x) - g_2\lambda(x) - g_3}{4(\lambda'(x))^2}.$$

Generalization of the Dixmier examples to the general elliptic curve:

$$\lambda(x) = -x,$$

$$v(x) = -x^3 + \frac{g_2}{4}x - \frac{g_3}{4},$$

$$L_4 = \left(\frac{d^2}{dx^2} + x^3 - \frac{g_2}{4} x + \frac{g_3}{4} \right)^2 + 2x,$$

$$L_6 = 2 \left(\frac{d^2}{dx^2} + x^3 - \frac{g_2}{4} x + \frac{g_3}{4} \right)^3 + 6x \frac{d^2}{dx^2} + 6 \frac{d}{dx} - 6v(x)x.$$

The functional parameter corresponding to the Dixmier example of rank 2, genus 1 among all the Krichever–Novikov commuting operators of rank 2, genus 1 was found by Grinevich. The class of self-adjoint Krichever–Novikov commuting operators of rank 2, genus 1 was singled out by Grinevich and Novikov. Various other approaches and representations for commuting operators of rank 2, genus 1 were also proposed by Dehornoy, Grünbaum, Latham, Previato and Wilson. We found explicit formulae for self-adjoint commuting operators of rank 2, genus 2 depending on two functional parameters connected by a third-order nonlinear ordinary differential relation, but this relation was not resolved.

The general form of commuting ordinary scalar differential operators of rank 3 for an arbitrary elliptic spectral curve (the general commuting operators of rank 3, genus 1 are parametrized by two arbitrary functions) was found by Mokhov, where also the functional parameters corresponding to the Dixmier example of rank 3, genus 1 among all the commuting operators of rank 3, genus 1 were found and all commuting operators of rank 3, genus 1 with rational coefficients were singled out.

Moreover, examples of commuting ordinary scalar differential operators of genus 1 with polynomial coefficients were constructed for any rank r (some commutative subalgebras of the Weyl algebra W_1).

The description of commuting ordinary scalar differential operators with polynomial coefficients (commutative subalgebras of the Weyl algebra W_1) is a separate nontrivial problem, and this problem has not been solved completely yet even for the Krichever–Novikov commuting operators of rank 2, genus 1.

Rank $r = 4$, $g = 1$.
Gelfand question on existing of commuting scalar ordinary differential operators with polynomial or rational coefficients for any given genus g and any given rank r (the Gelfand seminar, 1981).
Recently Mironov proved that for commuting operators L, M of rank 2 with hyperelliptic spectral curve Γ of arbitrary genus g,

$$\Gamma: \ \mu^2 = \lambda^{2g+1} + a_{2g}\lambda^{2g} + \ldots + a_1\lambda + a_0,$$

where a_i are some constants, the operator L of order 4 is self-adjoint if and only if

$$\chi_1(x, P) = \chi_1(x, \sigma(P)),$$

where σ is the involution

$$\sigma(\lambda, \mu) = (\lambda, -\mu)$$

on the hyperelliptic curve Γ.

Any self-adjoint operator L of order 4 in the standard canonical form can be represented as

$$L = \left(\frac{d^2}{dx^2} + V(x) \right)^2 + W(x). \qquad (\star)$$

Mironov proved that if the operator L of order 4 from a pair of commuting operators L, M of rank 2 with hyperelliptic spectral curve Γ of arbitrary genus g is self-adjoint, then the operator L, obviously, has the form $(\star)$ and the corresponding functions $\chi_0(x, P), \chi_1(x, P)$ have the form

$$\chi_0(x, P) = -\frac{1}{2} \frac{Q_{xx}}{Q(x, \lambda)} + \frac{\mu}{Q(x, \lambda)} - V(x), \quad \chi_1(x, P) = \frac{Q_x}{Q(x, \lambda)},$$

where $P = (\lambda, \mu)$ and $Q(x, \lambda)$ is a polynomial in λ of degree g with coefficients depending on x,

$$Q(x, \lambda) = (\lambda - \lambda_1(x)) \cdots (\lambda - \lambda_g(x)),$$

such that the following relation holds:

$$(\lambda - W(x))(Q(x,\lambda))^2 - V(x)(Q_x)^2 + \frac{1}{4}(Q_{xx})^2 - \frac{1}{2}Q_x Q_{xxx} +$$

$$+Q\left(V_x Q_x + 2V(x)Q_{xx} + \frac{1}{2}Q_{xxxx}\right) = \lambda^{2g+1} + a_{2g}\lambda^{2g} + \ldots + a_1\lambda + a_0.$$

Using this approach Mironov constructed for any genus $g > 1$ remarkable examples of commuting ordinary scalar differential operators of ranks 2 and 3 with polynomial coefficients that generalize naturally the Dixmier examples of ranks 2 and 3, genus 1:

$$L = \left(\left(\frac{d}{dx}\right)^2 + x^3 + \alpha\right)^2 + g(g+1)x,$$

$$M^2 = L^{2g+1} + a_{2g}L^{2g} + \ldots + a_1 L + a_0,$$

where a_i are some constants, α is an arbitrary nonzero constant, L and M are the Mironov commuting operators of rank 2, genus g (the orders of the operators L and M are 4 and $4g+2$, respectively), the coefficients of the operator M are expressed polynomially in terms of the coefficients of the operator L and their derivatives, $[L, M] = 0$;

$$L = \left(\left(\frac{d}{dx}\right)^3 + x^2 + \alpha\right)^2 + g(g+1)\frac{d}{dx},$$

$$M^2 = L^{2g+1} + a_{2g}L^{2g} + \ldots + a_1 L + a_0,$$

where a_i are some constants, α is an arbitrary nonzero constant, L and M are the Mironov commuting operators of rank 3, genus g (the orders of the operators L and M are 6 and $6g+3$, respectively), the coefficients of the operator M are expressed polynomially in terms of the coefficients of the operator L and their derivatives, $[L, M] = 0$.

Using Mironov's results, we constructed examples of commuting ordinary scalar differential operators with polynomial coefficients that are related to a spectral curve of an arbitrary genus g for an arbitrary even rank $r = 2k$, $k > 1$, and for an arbitrary rank of the form $r = 3k$, $k \geq 1$.

The operators L and M of orders $4k$ and $4kg + 2k$, respectively,

$$L = \left(\left(\frac{d}{dx} \right)^{2k} - 2x \left(\frac{d}{dx} \right)^{k} - k \left(\frac{d}{dx} \right)^{k-1} + \left(\frac{d}{dx} \right)^{3} + x^2 + \alpha \right)^2 +$$

$$+ g(g+1) \frac{d}{dx},$$

$$M^2 = L^{2g+1} + a_{2g} L^{2g} + \ldots + a_1 L + a_0,$$

where a_i are some constants, α is an arbitrary nonzero constant, are commuting operators of rank $r = 2k$, $k > 1$, genus g, $[L, M] = 0$, the coefficients of the operator M are expressed polynomially in terms of the coefficients of the operator L and their derivatives. For $k > 2$ the commuting operators L and M have the standard canonical form.

The operators L and M of orders $6k$ and $6kg + 3k$, respectively,

$$L = \left(\left(\frac{d}{dx}\right)^{3k} - 3x\left(\frac{d}{dx}\right)^{2k} - 3k\left(\frac{d}{dx}\right)^{2k-1} + 3x^2\left(\frac{d}{dx}\right)^{k} + \right.$$

$$\left. +3kx\left(\frac{d}{dx}\right)^{k-1} + k(k-1)\left(\frac{d}{dx}\right)^{k-2} + \left(\frac{d}{dx}\right)^{2} - x^3 + \alpha \right)^2 -$$

$$-g(g+1)x,$$

$$M^2 = L^{2g+1} + a_{2g}L^{2g} + \ldots + a_1 L + a_0,$$

where a_i are some constants, α is an arbitrary nonzero constant, are commuting operators of rank $3k$, $k \geq 1$, genus g, $[L, M] = 0$, the coefficients of the operator M are expressed polynomially in terms of the coefficients of the operator L and their derivatives. For $k > 1$ the commuting operators L and M have the standard canonical form.

Moreover, Mironov proved the following theorem: the self-adjoint operator

$$L = \left(\frac{d^2}{dx^2} + \alpha_1 S(x) + \alpha_0 \right)^2 + \alpha_1 c_2 g(g+1) S(x),$$

where the function $S(x)$ satisfies the equation

$$(S_x)^2 = c_2 (S(x))^2 + c_1 S(x) + c_0,$$

α_1 and c_2 are arbitrary nonzero constants, α_0, c_1 and c_0 are arbitrary constants, form a pair of commuting operators L, M of rank 2 with a certain hyperelliptic spectral curve Γ of genus g. In particular, if $S(x) = \cosh x$, $c_2 = 1$, $c_1 = 0$, $c_0 = -1$, then all the conditions are satisfied and the operator

$$L = \left(\frac{d^2}{dx^2} + \alpha_1 \cosh(x) + \alpha_0 \right)^2 + \alpha_1 g(g+1) \cosh x, \qquad (\star\star)$$

is the first operator of order 4 from the commuting pair L, M of genus g and rank 2 with a hyperelliptic spectral curve.

Let us consider the change of variable

$$x = \ln P(z),$$

where $P(z)$ is an arbitrary nonconstant function. Then the operator $(\star\star)$ is transformed to the form

$$L = \left(\frac{(P(z))^2}{(P_z)^2} \frac{d^2}{dz^2} + \frac{P(z)((P_z)^2 - P(z)P_{zz})}{(P_z)^3} \frac{d}{dz} + \right.$$

$$\left. + \alpha_1 \frac{(P(z))^2 + 1}{2P(z)} + \alpha_0 \right)^2 + \alpha_1 g(g+1) \frac{(P(z))^2 + 1}{2P(z)}.$$

In particular, if $P(z)$ is an arbitrary nonconstant rational function, then we get an operator of order 4 from the commuting pair of genus g and rank 2 with rational coefficients, but this operator is not in the standard canonical form.

Let us consider the special case

$$P(z) = (z + \sqrt{z^2 - 1})^r, \quad r = \pm 1, \pm 2, \ldots$$

Then the operator L can be represented as

$$L = \left((1 - z^2)\frac{d^2}{dz^2} - z\frac{d}{dz} + aT_r(z) + b\right)^2 - ar^2 g(g+1) T_r(z),$$

where r is an arbitrary nonvanishing integer, $T_r(z)$ is the Chebyshev polynomial of the first kind, of degree r or $-r$, a is an arbitrary nonzero constant, b is an arbitrary constant. Recall that

$$T_0(z) = 1, \quad T_1(z) = z, \quad T_r(z) = 2zT_{r-1}(z) - T_{r-2}(z), \quad T_{-r}(z) = T_r(z)$$

It is very interesting that the Chebyshev polynomials of the first kind $T_r(z)$ are commuting polynomials

$$T_n(T_m(z)) = T_{nm}(z) = T_m(T_n(z)).$$

Theorem. *The operators L_{2r} and $M_{(2g+1)r}$ of orders $2r$ and $(2g+1)r$, respectively,*

$$L_{2r} = \left(aT_r\left(\frac{d}{dx}\right) - x^2\frac{d^2}{dx^2} - 3x\frac{d}{dx} + x^2 + b\right)^2 - ar^2g(g+1)T_r\left(\frac{d}{dx}\right)$$

$$M^2_{(2g+1)r} = L^{2g+1}_{2r} + a_{2g}L^{2g}_{2r} + \ldots + a_1L_{2r} + a_0,$$

where a_i are some constants, $T_r(x)$ is the Chebyshev polynomial of the first kind, of degree r, $r > 1$, the notation $T_r\left(\frac{d}{dx}\right)$ means the ordinary differential operator, which is the Chebyshev polynomial T_r of $\frac{d}{dx}$, a is an arbitrary nonzero constant, b is an arbitrary constant, are commuting operators of rank r, genus g, $[L_{2r}, M_{(2g+1)r}] = 0$, the coefficients of the operator $M_{(2g+1)r}$ are expressed polynomially in terms of the coefficients of the operator L_{2r} and their derivatives. For $r > 3$ the commuting operators L_{2r} and $M_{(2g+1)r}$ have the standard canonical form (for $a = 1/2^{r-1}$). For $r = 1$ this pair of operators is commuting one of rank 2 and genus g.

Examples.

1) Rank 4, genus g:

$$L_8 = \left(\left(\frac{d}{dx} \right)^4 - (x^2+1) \left(\frac{d}{dx} \right)^2 - 3x \frac{d}{dx} + x^2 + \alpha \right)^2 - $$
$$-16g(g+1) \left(\left(\frac{d}{dx} \right)^4 - \left(\frac{d}{dx} \right)^2 \right).$$

2) Rank 5, genus g:

$$L_{10} = \left(\left(\frac{d}{dx} \right)^5 - \frac{5}{4} \left(\frac{d}{dx} \right)^3 - \right.$$
$$\left. -x^2 \left(\frac{d}{dx} \right)^2 - \left(3x - \frac{5}{16} \right) \frac{d}{dx} + x^2 + \alpha \right)^2 -$$
$$-25g(g+1) \left(\left(\frac{d}{dx} \right)^5 - \frac{5}{4} \left(\frac{d}{dx} \right)^3 + \frac{5}{16} \frac{d}{dx} \right).$$

3) Rank 6, genus g:

$$L_{12} = \left(\left(\frac{d}{dx}\right)^6 - \frac{3}{2}\left(\frac{d}{dx}\right)^4 - \right.$$

$$\left. - \left(x^2 - \frac{9}{16}\right)\left(\frac{d}{dx}\right)^2 - 3x\frac{d}{dx} + x^2 + \alpha \right)^2 -$$

$$-36g(g+1)\left(\left(\frac{d}{dx}\right)^6 - \frac{3}{2}\left(\frac{d}{dx}\right)^4 + \frac{9}{16}\left(\frac{d}{dx}\right)^2 \right).$$

4) Rank 7, genus g:

$$L_{14} = \left(\left(\frac{d}{dx}\right)^7 - \frac{7}{4}\left(\frac{d}{dx}\right)^5 + \frac{7}{8}\left(\frac{d}{dx}\right)^3 - \right.$$

$$\left. -x^2\left(\frac{d}{dx}\right)^2 - \left(3x + \frac{7}{64}\right)\frac{d}{dx} + x^2 + \alpha \right)^2 -$$

$$-49g(g+1)\left(\left(\frac{d}{dx}\right)^7 - \frac{7}{4}\left(\frac{d}{dx}\right)^5 + \frac{7}{8}\left(\frac{d}{dx}\right)^3 - \frac{7}{64}\frac{d}{dx} \right).$$