Cohomological localization towers

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Summary of the Talk

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- **Cohomological localizations** are not known to exist.
- In 2012 we proved that the existence of supercompact cardinals (a **large-cardinal axiom**) implies that cohomological localizations exist.
- In work in progress we show that, assuming that cohomological localizations of spectra exist, they can be **constructed** by means of homotopy inverse limits.

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Then $\boldsymbol{E_*}$ is a **homology theory** and $\boldsymbol{E^*}$ is a **cohomology theory.**

They are homotopy invariant, additive functors sending fibre sequences of spectra to long exact sequences of abelian groups.

Homotopical Localization

Let E be any spectrum.

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Hovey conjectured in 1995 that for each spectrum E there is another spectrum F such that the class of E^*-acyclics is equal to the class of F_*-acyclics. This is still an open problem.

Motivation

Localization with respect to ordinary homology $(H\mathbb{Z}_{(p)})_*$ with p-local coefficients gives rise to ***H*-space structures** on some spheres (Adams, Sullivan) and **homotopy splittings** (Brown–Peterson, Mimura–Nishida–Toda).

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Under suitable assumptions on X, the Adams–Novikov spectral sequence for a homology theory E_* converges to the homotopy groups of the E_*-localization of X:

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This has been extensively studied for $E = BP$ and for $E = E(n)$, where $E(n) = K(0) \vee \cdots \vee K(n)$ is a wedge of Morava K-theories. The resulting Adams–Novikov spectral sequence is called **chromatic spectral sequence.**

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What remains to be done: Show that the existence of arbitrary cohomological localizations is indeed a set-theoretical problem, that is, it **cannot** be proved using only the ZFC axioms.

Sets versus Classes

For each spectrum E, the class of E^*-acyclic spectra is a **proper class.** If it were a **set,** then we could let A be the coproduct of all its members, and then Farjoun's **nullification** functor P_A would be an E^*-localization.

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Strategy: Find a **set** of E^*-acyclic spectra such that every E^*-acyclic spectrum is a **filtered colimit** of spectra in this set.

Why this works for homology: Every spectrum is the union of its **finite** subspectra. Since $E \wedge -$ is a left adjoint functor, it preserves colimits. Hence, if X is E_*-acyclic (thus $E \wedge X = 0$), then for each finite subspectrum $Y \subseteq X$ there is a bigger finite subspectrum Y' with $Y \subseteq Y' \subseteq X$ such that $E \wedge Y' = 0$. In this way we construct a **cofinal** family of E_*-acyclic subspectra of X, so X is indeed a filtered colimit of E_*-acyclic finite subspectra.

Disclaimer

The content of the next slides, until further notice, is ***not*** *to be followed in detail. Its only purpose is to illustrate our method to prove the existence of cohomological localizations by means of set-theoretical techniques.*

Structures

A (finitary) S-sorted **signature** Σ consists of a set S of **sorts,** a set Σ_{op} of **operation symbols,** another set Σ_{rel} of **relation symbols,** and an **arity** function that assigns to each operation symbol a finite sequence $\langle s_i : i \leq n \rangle$ of **input sorts** and an **output sort** $s \in S$, and to each relation symbol a finite sequence of sorts $\langle s_j : j \leq m \rangle$.

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Given an S-sorted signature Σ, a **Σ-structure** X consists of a nonempty set X_s for each sort $s \in S$, a function $\sigma_X \colon \prod_{i \leq n} X_{s_i} \to X_s$ for each operation symbol σ of arity $\langle s_i : i \leq n \rangle \to s$, and a set $\rho_X \subseteq \prod_{j \leq m} X_{s_j}$ for each relation symbol ρ of arity $\langle s_j : j \leq m \rangle$.

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A class M is **transitive** if every element of an element of M is an element of M. All **ordinals** are transitive. We tacitly assume that models of ZFC are transitive.

Absoluteness

Given two ZFC models $M \subseteq N$, we say that a formula $\varphi(x_1, \ldots, x_k)$ of the language of set theory is **absolute between *M* and *N*** if, for all $a_1, \ldots, a_k$ in M,

$$N \models \varphi(a_1, \ldots, a_k) \text{ if and only if } M \models \varphi(a_1, \ldots, a_k).$$

A formula is **absolute** if it is absolute between any two models.

A formula $\varphi(x_1, \ldots, x_k)$ is **upward absolute** if, given any two ZFC models $M \subseteq N$ and given $a_1, \ldots, a_k \in M$ for which $\varphi(a_1, \ldots, a_k)$ is true in M, $\varphi(a_1, \ldots, a_k)$ is also true in N. And we say that φ is **downward absolute** if, in the same situation, if $\varphi(a_1, \ldots, a_k)$ holds in N then it also holds in M.

A **class** $\mathcal{C}$ is **absolute** if it is definable by an absolute formula, possibly with parameters.

The Lévy Hierarchy

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Following Lévy (1965), Σ_n formulas and Π_n formulas are defined inductively as follows:

- $\boldsymbol{\Pi_0}$ formulas are the same as Σ_0 formulas;
- $\boldsymbol{\Sigma_{n+1}}$ formulas are of the form $\boldsymbol{(\exists x_1 \ldots x_k)\, \varphi}$, where φ is Π_n;
- $\boldsymbol{\Pi_{n+1}}$ formulas are of the form $\boldsymbol{(\forall x_1 \ldots x_k)\, \varphi}$, where φ is Σ_n.

Complexity of Classes

We say that a class $\mathcal{C}$ is **$\boldsymbol{\Sigma_n}$-definable** (or, shortly, that $\mathcal{C}$ is Σ_n) if it can be defined with a Σ_n formula, and similarly with Π_n.

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If a class $\mathcal{C}$ is $\boldsymbol{\Sigma_1}$ with a set of parameters p, then it is **upward absolute** for transitive classes containing p. In fact, given a Σ_1 formula $\exists x\, \varphi(x, y)$ where φ is Σ_0, and a set p of parameters, suppose that $M \subseteq N$ are transitive classes with $p \in M$. Then, if $M \models \exists x\, \varphi(x, p)$, it follows that $N \models \exists x\, \varphi(x, p)$ as well.

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Conversely, if a class $\mathcal{C}$ is upward absolute for transitive models of (some finite fragment of) ZFC, then it is Σ_1.

Similarly, if a class $\mathcal{C}$ is defined by a $\boldsymbol{\Pi_1}$ formula with parameters, then it is **downward absolute** for transitive classes containing the parameters, and, if $\mathcal{C}$ is downward absolute for transitive models of some finite fragment of ZFC, then it is Π_1.

Elementary Embeddings

An **elementary embedding** of a structure X into another structure Y (where X and Y can be proper classes) is a function $j\colon X \to Y$ that preserves and reflects truth. That is, for every formula $\varphi(x_1, \ldots, x_n)$ of the language of Σ and all $\{a_i : i \leq n\}$ in X, the sentence $\varphi(a_1, \ldots, a_n)$ is satisfied in X if and only if $\varphi(j(a_1), \ldots, j(a_n))$ is satisfied in Y.

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In what follows, we consider elementary embeddings between models of ZFC (viewed as structures of the language of set theory, i.e., with a single relation symbol $\in$).

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In what follows, we consider elementary embeddings between models of ZFC (viewed as structures of the language of set theory, i.e., with a single relation symbol $\in$).

Let V denote the **universe** of all sets. Namely, we define, recursively on the class of ordinals, $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ for all α, where $\mathcal{P}$ denotes the power-set operation, and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ if λ is a limit ordinal. Then every set is an element of some V_α. The **universe** V of all sets is the union of V_α for all ordinals α.

Supercompact Cardinals

If $j\colon V \to M$ is a nontrivial elementary embedding of the universe V into a transitive class M, then its **critical point** (i.e., the least ordinal moved by j) is a **measurable** cardinal. In fact, the existence of a nontrivial elementary embedding of the universe into a transitive class is equivalent to the existence of a measurable cardinal.

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A cardinal κ is **supercompact** if for every cardinal λ there is an elementary embedding $j\colon V \to M$ with M transitive and with critical point κ, such that $j(\kappa) > \lambda$ and M is closed under λ-sequences; that is, every sequence $\langle X_\alpha : \alpha < \lambda \rangle$ of elements of M is an element of M.

Using Supercompact Cardinals

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Theorem 1 *Let $\mathcal{C}$ be a class of finitary structures definable with a Σ_2 formula with a set p of parameters. Suppose that there exists a supercompact cardinal κ such that p and the signature are in $H(\kappa)$. Then for every $Y \in \mathcal{C}$ there exists $X \in \mathcal{C} \cap H(\kappa)$ and an elementary embedding of X into Y.*

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Every category of structures embeds into the category of sets. Elementary embeddings of structures are **injective** functions, since $j(x) = j(y)$ implies that $x = y$. Moreover, injective functions are **monomorphisms** in every subcategory of sets.

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Every category of structures embeds into the category of sets. Elementary embeddings of structures are **injective** functions, since $j(x) = j(y)$ implies that $x = y$. Moreover, injective functions are **monomorphisms** in every subcategory of sets. Hence, the theorem concludes that **every object *Y* of $\mathcal{C}$ has a subobject *X* in *H*(κ),** if enough supercompact cardinals exist.

Proof

Proof. Suppose that κ is a supercompact cardinal such that $\{p, \Sigma\} \in H(\kappa)$. Let Y be an object of $\mathcal{C}$. Let μ be a cardinal bigger than κ such that $Y \in H(\mu)$ and such that $H(\mu) \preceq_2 V$ (that is, Σ_2 sentences are absolute between $H(\mu)$ and V).

Let $j : V \to M$ be an elementary embedding with M transitive and critical point κ, such that $j(\kappa) > \mu$ and M is closed under μ-sequences. Then $j(Y)$ is a structure and the restriction $j \restriction Y : Y \to j(Y)$ is an elementary embedding. Moreover, Y and $j \restriction Y$ are in M because $H(\mu) \in M$, and Y is a structure in M since being a (finitary) structure is absolute.

Since being a cardinal is downward absolute, μ is a cardinal in M, and this implies that $H(\mu)$ in the sense of M is the same as $H(\mu)$ in V. It follows that **$H(\mu) \preceq_1 M$,** since every Σ_1 sentence ψ which holds in M also holds in V (as Σ_1 sentences are upward absolute) and therefore ψ holds in $H(\mu)$ because $H(\mu) \preceq_2 V$. **Hence, Σ_2 formulas are upward absolute between $H(\mu)$ and M. Since $H(\mu) \preceq_2 V$ and the class $\mathcal{C}$ is defined by a Σ_2 formula $\varphi(x, y)$, we have that $H(\mu) \models \varphi(Y, p)$, and consequently $M \models \varphi(Y, p)$.**

Proof (continued)

Now $Y \in H(j(\kappa))$ in V since $j(\kappa) > \mu$, and also in M. Thus, **in M there exists a structure X such that $X \in H(j(\kappa))$ and $\varphi(X, p)$ holds, and there is an elementary embedding $X \to j(Y)$;** namely, Y witnesses this claim.
Note that $j(p) = p$ since $p \in H(\kappa)$.

By elementarity of j, the corresponding statement is true in V; that is, **there exists a structure X such that $X \in H(\kappa)$ and $\varphi(X, p)$ holds, and there is an elementary embedding $X \to Y$,** as we needed to prove. $\square$

Main Theorem

Theorem 2 *Let $\mathcal{C}$ be a λ-accessible subcategory closed under λ-filtered colimits in a category of structures for some regular cardinal λ and a finitary signature, and let $\mathcal{S}$ be a* $\mathbf{\Sigma_2}$ *full subcategory of $\mathcal{C}$. Suppose that there are arbitrarily large* ***supercompact*** *cardinals. Then there is a* ***dense small*** *full subcategory $\mathcal{D}$ of $\mathcal{S}$.*

Proof

Proof. Choose a Σ_2 formula defining $\mathcal{S}$ with a set p of parameters. Let $\mathcal{C}_\lambda$ be a set of representatives of all isomorphism classes of λ-presentable objects in $\mathcal{C}$.

Pick a supercompact cardinal $\kappa > \lambda$ such that each object in $\mathcal{C}_\lambda$ is in $H(\kappa)$ and $\{p, \Sigma\} \in H(\kappa)$ as well, and such that $\mathcal{C}$ is κ-accessible.

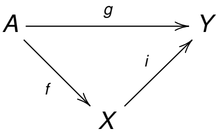
Let $\mathcal{D}$ be a full subcategory of $\mathcal{S}$ containing one representative of each isomorphism class of objects in the set $\mathcal{S} \cap H(\kappa)$. Since each object of $\mathcal{D}$ is in $H(\kappa)$, all objects of $\mathcal{D}$ are κ-presentable in $\mathcal{C}$.

Let $\mathcal{C}_\kappa$ be a set of representatives of all isomorphism classes of κ-presentable objects of $\mathcal{C}$, chosen so that $\mathcal{D} \subseteq \mathcal{C}_\kappa$. Let Y be any object of $\mathcal{S}$. Since $\mathcal{C}$ is κ-accessible, we know that Y is a colimit of the canonical diagram $(\mathcal{C}_\kappa \downarrow Y) \to \mathcal{C}$, which is κ-filtered. Therefore, if we prove that $(\mathcal{D} \downarrow Y)$ is **cofinal** in $(\mathcal{C}_\kappa \downarrow Y)$, it will then follow that Y is a colimit of the canonical diagram $(\mathcal{D} \downarrow Y) \to \mathcal{C}$, and that $(\mathcal{D} \downarrow Y)$ is κ-filtered. Moreover, since Y is in $\mathcal{S}$, we will be able to conclude that Y is also a colimit of the canonical diagram $(\mathcal{D} \downarrow Y) \to \mathcal{S}$, as we wanted to show.

Proof (continued)

Thus, towards proving that $(\mathcal{D} \downarrow Y)$ is cofinal in $(\mathcal{C}_\kappa \downarrow Y)$, let A be any object of $\mathcal{C}_\kappa$ and let a morphism $g\colon A \to Y$ be given. Since each object of $\mathcal{C}_\lambda$ is in $H(\kappa)$, we infer that $A \in H(\kappa)$.

Now we view $(A \downarrow \mathcal{S})$ as a full subcategory of $(A \downarrow \mathcal{C})$. **According to Theorem 1, there is an object $\langle X, f \rangle$ in $(A \downarrow \mathcal{S})$ which is in $H(\kappa)$, together with an elementary embedding $i\colon X \to Y$ with $i \circ f = g$.** We replace, if necessary, X by an isomorphic object of $\mathcal{S} \cap H(\kappa)$, so we may assume that $X \in \mathcal{D}$. We therefore have a commutative triangle



displaying the fact that $\langle X, f \rangle$ is a subobject of $\langle Y, g \rangle$, but where f can *also* be viewed as a morphism from $\langle A, g \rangle$ to $\langle X, i \rangle$ in $(\mathcal{C}_\kappa \downarrow Y)$. This tells us that $(\mathcal{D} \downarrow Y)$ is cofinal in $(\mathcal{C}_\kappa \downarrow Y)$, as we wanted to show. $\square$

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Theorem 3 *The class of* ***E_*-acyclic*** *simplicial sets for a spectrum E is* $\mathbf{\Sigma_1}$ *with E as a parameter.*

Back to Algebraic Topology

Note that the category of simplicial sets and the category of Bousfield–Friedlander spectra are accessible and embed into categories of finitary structures in such a way that the embedding preserves colimits.

Theorem 3 *The class of* ***E_*-acyclic*** *simplicial sets for a spectrum E is* $\boldsymbol{\Sigma_1}$ *with E as a parameter.*

Proof. If (X, p) is a pointed simplicial set and E is a spectrum with structure maps $\langle \sigma_n : n \geq 0 \rangle$, then $X \wedge E$ is a spectrum with $(X \wedge E)_n = X \wedge E_n$ and structure maps $(\mathrm{id} \wedge \sigma_n) \circ (\tau \wedge \mathrm{id})$ for all n, where $\tau \colon \mathbb{S}^1 \wedge X \to X \wedge \mathbb{S}^1$ is the twist map. Then the spectrum $X \wedge E$ is weakly contractible if and only if the following expression holds, where we need to define $F = X \wedge E$:

$$X \in \mathbf{sSet}_* \;\wedge\; \exists F\, [F \text{ is a spectrum} \;\wedge\; (\forall n < \omega)((F_n = X \wedge E_n) \\ \wedge\, \sigma_n^F = (\mathrm{id} \wedge \sigma_n^E) \circ (\tau \wedge \mathrm{id})) \;\wedge\; F \text{ is weakly contractible}].$$

This is a Σ_1 formula, so the theorem is proved. □

Cohomology Theories

Theorem 4 *The class of* ***E^*-acyclic*** *simplicial sets for an Ω-spectrum E is* $\mathbf{\Sigma_2}$ *with E as a parameter.*

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Proof. Let E be an Ω-spectrum, which will be used as a parameter. The definition of an Ω-spectrum is absolute; hence, every transitive model of ZFC containing E will agree with the fact that E is an Ω-spectrum.

A pointed simplicial set (X, p) is E^*-acyclic if and only if for all $n \geq 0$ the simplicial set $\mathrm{Map}_*(X, E_n)$ is weakly contractible (assuming that E is an Ω-spectrum). Thus, X is E^*-acyclic if and only if the following formula holds, where we need to define $M = \mathrm{Map}_*(X, E_n)$:

$$X \in \mathbf{sSet}_* \;\wedge\; (\forall n < \omega)\, \exists M\, [M \in \mathbf{sSet}_*$$
$$\wedge\, (\forall k < \omega)\, [(\forall f \in M_k)\, f \in \mathbf{sSet}_*(X \wedge \Delta[k]_+, E_n)$$
$$\wedge\, \forall g\, (g \in \mathbf{sSet}_*(X \wedge \Delta[k]_+, E_n) \to g \in M_k)] \;\wedge\; M \text{ is weakly contractible}].$$

This is indeed a Σ_2 formula. □

About Hovey's Conjecture

It follows from the above discussion that an important difference between homology acyclics and cohomology acyclics is that **classes of homology acyclics are Σ_1 while classes of cohomology acyclics are Σ_2.**

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This implies that the existence of **homological localizations** is provable in ZFC, while **cohomological localizations** can be shown to exist by assuming a suitable large-cardinal axiom.

This also leads us close to disproving **Hovey's conjecture,** according to which for every cohomology theory there is a homology theory with the same acyclics.

End of Set Theory

Constructing Cohomological Localizations

From now on, we assume that cohomological localizations exist.

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Assume that ***E*** **is a ring spectrum.** We are going to describe a **long tower** for each spectrum X,

$$\cdots \to T_{\alpha+1}X \to T_\alpha X \to \cdots \to T_2X \to T_1X,$$

indexed by all the ordinals α, together with compatible maps $\eta_\alpha \colon X \to T_\alpha$, and we will prove that for each X there is an ordinal $\alpha = \alpha(X)$ such that $\boldsymbol{T_\alpha X \simeq L^E X}$**,** where $L^E X$ denotes the E^*-localization of X.

E-completion

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Then one defines $\boldsymbol{\widehat{X}_E = \mathrm{Tot}(X^\circ)}$**.**

Next we define by transfinite induction, for each ordinal α,

$$T_{\alpha+1}X = T_1(T_\alpha X),$$

and $T_\lambda X = \mathrm{holim}_{i<\lambda} T_i X$ for each limit ordinal λ.

A Long Tower

For each α there is a map

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If $f \colon X \to Y$ is a map of spectra, then, for every α,

$$T_\alpha f \colon T_\alpha X \simeq T_\alpha Y \quad \Longleftrightarrow \quad f^* \colon E^*(Y) \cong E^*(X).$$

Local Complexity

The E^*-local spectra form a **colocalizing subcategory** of the homotopy category of spectra. It is the **smallest** colocalizing subcategory containing E.

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Hence, every E^*-local spectrum can be obtained from E by a (possibly transfinite) sequence of **homotopy inverse limits.**

This yields a **complexity** $\boldsymbol{c(X)}$ for each E^*-local spectrum X. Namely, $c(X) = 0$ if X is a retract of a product of copies of E, and $c(X) < \kappa + 1$ if $X \simeq \operatorname{holim}_{d \in D} X_d$ where $c(X_d) < \kappa$ for all d.

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Proof: Assume it true for $i < \kappa$, and suppose that $X = \operatorname{holim}_{d \in D} X_d$ with $c(X_d) < \kappa$ for all $d \in D$. Then, since homotopy inverse limits commute, $T_{\kappa+1}X \simeq \operatorname{holim}_{d \in D} T_{\kappa+1}X_d$. Thus our claim follows from the induction hypothesis. □

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Hence, for each spectrum X, if $c(L^E X) = \kappa$, then

$$T_{\kappa+1} L^E X \simeq L^E X.$$

Since $T_{\kappa+1} X \simeq T_{\kappa+1} L^E X$, our argument is complete.

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