Hopf algebras and homology of loop suspensions

Jelena Grbić

University of Southampton

joint work with Victor Buchstaber

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Motivation:

- Milnor - double suspension problem (1961):

  Let M^3 be a homology 3-sphere with $\pi_1(M^3) \neq 0$.

  Q: $\Sigma^2 M^3 \cong S^5$

  - Edwards (1975): partially proved
  - Cannon (1979): $\Sigma^2 S^n_H \cong S^{n+2}$
- homotopy analogue:

$$\Sigma^n X \simeq \Sigma^{n+1} Y$$

- Our question: For which X

$$\Sigma X \simeq \Sigma^2 Y \ ?$$

Hopf algebars

- Hopf algebras - homology of topological spaces with multiplication
- Borel (1953): expression Hopf algebra
  - honouring the fundamental work of Hopf on Γ-manifolds
- *H*-space:
  - X -pointed topological space
  - multiplication $\quad \mu\colon X \times X \longrightarrow X$

  Examples:

  topological groups, based loop spaces, Eilenberg-MacLane spaces, finite H-spaces (Lie groups, S^7)

Hopf and Borel:

- classified graded Hopf algebras which can be realised as the cohomology rings of H-spaces
- Examples:
  $\mathbb{C}P^n$, $BU(2)$
- existence of the comultiplication in a Hopf algebra restricts the multiplication structure considerably

Definition

A- Hopf algebra, $P(A)$ - primitives in A, $I(A)$ - augmentation ideal, $Q(A) = I(A)/I(A)^2$- indecomposables.
A is called *Lie-Hopf algebra* if it is primitively generated, that is, if the natural map

$$P(A) \longrightarrow I(A) \longrightarrow Q(A)$$

is onto.

Bott - Samelson

- necessary condition for graded Hopf algebras to be realised as the homology algebra of loop-suspension spaces

Theorem (Bott-Samelson)

R - PID

X - connected space s.t. $H^*(X \times X; R) \cong H^*(X; R) \otimes H^*(X; R)$

Then

$$H_*(\Omega\Sigma X) \cong T(\widetilde{H}_*(X))$$

as an algebra

The suspension map $E\colon X \longrightarrow \Omega\Sigma X$ induces the canonical inclusion

$$\widetilde{H}_*(X) \longrightarrow H_*(\Omega\Sigma X).$$

- This tensor algebra can be given many different “algebraic ” Hopf algebra structures
- The diagonal map on X

$$\Delta\colon X \longrightarrow X \times X$$

induces a coproduct in $H_*(X; R)$ and dually the cup product in $H^*(X; R)$.

Proposition

Let $T(\widetilde{H}_*(X; R))$ be the Hopf algebra where the coalgebra structure is generated by the comultiplication on $\widetilde{H}_*(X; R)$ induced by the diagonal $\Delta_X\colon X \longrightarrow X \times X$. This Hopf algebra structure coincides with the Hopf algebra structure of $H_*(\Omega\Sigma X; R)$ where the comultiplication is induced by the diagonal map $\Delta_{\Omega\Sigma X}\colon \Omega\Sigma X \longrightarrow \Omega\Sigma X \times \Omega\Sigma X$.

Corollary

Let C be a co-H-space. Then the Hopf algebra $H_*(\Omega\Sigma C; R)$ is a Lie-Hopf algebra, that is, it is primitively generated.

- the cup product in $H^*(X; R)$ presents an obstruction to $H_*(\Omega\Sigma X; R)$ be a Lie-Hopf algebra.

Definition

Define a homotopy invariant which to a topological space X associates the isomorphism class (over the integers) of the Hopf algebra $H_*(\Omega\Sigma X)$.

- we say that the invariant is trivial if the Hopf algebra $H_*(\Omega\Sigma X)$ is isomorphic to a Lie-Hopf algebra

Theorem

Let X be a topological space such that $\Sigma X \simeq \Sigma C$ where C is a co-H-space and assume that $H_*(X;\mathbb{Z})$ is torsion free. Then over $\mathbb{Z}$ the Hopf algebra $H_*(\Omega\Sigma X;\mathbb{Z})$ is isomorphic to a Lie-Hopf algebra, that is, to a primitively generated Hopf algebra.

- provides a necessary condition on the homology of $\Omega\Sigma X$ so that $\Sigma X \simeq \Sigma^2 Y$.

Proposition

The Hopf algebra $H_*(\Omega\Sigma X)$ for $X = \Omega\Sigma Z$ with Z a co-H-space is isomorphic to a Lie-Hopf algebra. The change of homology generators is given by Hopf invariants.

Let $\varphi\colon \Sigma X \to \Sigma^2 Y$ be a homotopy equivalence. Then by the Bott-Samelson theorem, we have

$$H_*(\Omega\Sigma X) \cong T(\widetilde{H}_*(X)), \qquad H_*(\Omega\Sigma^2 Y) \cong T(\widetilde{H}_*(\Sigma Y)).$$

Let $\{a_i\}$ be an additive basis for $\widetilde{H}_*(X)$ and let $\{b_i\}$ be an additive basis for $\widetilde{H}_*(\Sigma Y)$. Then the elements b_i are primitive, that is, $\Delta b_i = 1 \otimes b_i + b_i \otimes 1$.

Lemma

The comultiplication $\Delta_X\colon H_*(X) \to H_*(X) \otimes H_*(X)$ is determined by the Hopf algebra homomorphism

$$\varphi_*\colon H_*(\Omega\Sigma X) \longrightarrow H_*(\Omega\Sigma^2 Y)$$

by the formula

$$\Delta_{\Omega\Sigma^2 Y}\varphi_*(a) = \varphi_* \otimes \varphi_*(\Delta_{\Omega\Sigma X}\, a)$$

where $a \in H_*(X)$.

Classical results revisited

- using the invariant, we reformulate the Pontryagin-Whitehead and Freedman theorem

Proposition

Let M_1 and M_2 be closed simply-connected 4-manifolds. If $\Sigma M_1 \simeq \Sigma M_2$, then the manifolds M_1 and M_2 are homotopy equivalent (homeomorphic).

- the Hopf invariant is a classical obstruction to $\Sigma \mathbb{C}P^2$ being a double suspension.
- for $H_*(\Omega\Sigma\mathbb{C}P^2)$ to be isomorphic to a Lie-Hopf algebra,

$$1 + 2\lambda = 0$$

  needs to have a solution
- no Hopf invariant one complex X satisfies that $\Sigma X \simeq \Sigma^2 Y$ and thus as a consequence the suspension of the Hopf map is not null homotopic.

Structural properties of the cup product

- topological spaces X such that $\Sigma X \simeq \Sigma^2 Y$ are not rear: polyhedral products, complements of hyperplane arrangements, some simply connected 4-manifolds
- class of manifolds with this property seems to be quite narrow and hard to detect
- the cohomology ring of moment-angle complexes $\mathcal{Z}_K = (D^2, S^1)^K$ is the Stanley-Reisner ring $\mathbb{Z}[K]$
- necessary conditions for an algebra to be realised as the cohomology ring of a moment-angle complex

Proposition

For an algebra A to be the cohomology algebra of a moment-angle complex $\mathcal{Z}_K$, there must exist a change of basis in $T(A^*)$ such that $T(A^*)$ becomes a Lie-Hopf algebra, where A^* denotes the dual of A.

Quasi-symmetric polynomials

Definition

Let $t_1, t_2, \ldots$ be a finite or an infinite set of variables of degree 2.
For a composition $\omega = (j_1, \ldots, j_k)$, consider a quasi-symmetric monomial

$$M_\omega = \sum_{l_1 < \ldots < l_k} t_{l_1}^{j_1} \ldots t_{l_k}^{j_k}, \quad M_{()} = 1$$

whose degree is equal to $2|\omega| = 2(j_1 + \ldots + j_k)$.
A product of any two monomials $M_{\omega'}$ and $M_{\omega''}$ is defined in the ring of polynomials $\mathbb{Z}[t_1, t_2, \ldots]$
Finite integer combinations of quasi-symmetric monomials form the *ring of quasi-symmetric functions* denoted by $\mathrm{QSymm}[t_1, \ldots, t_n]$, where n is the number of variables.
In the case of an infinite number of variables it is denoted by $\mathrm{QSymm}[t_1, t_2, \ldots]$ or QSymm.

- The diagonal map $\Delta\colon \mathrm{QSymm} \longrightarrow \mathrm{QSymm} \otimes \mathrm{QSymm}$ given by

$$\Delta M_{(a_1,\ldots,a_k)} = \sum_{i=0}^{k} M_{(a_1,\ldots,a_i)} \otimes M_{(a_{i+1},\ldots,a_k)}$$

defines on QSymm the structure of a graded Hopf algebra.
- NSymm denotes the non-commutative analogue of the Hopf algebra of symmetric functions Symm
- $(\mathrm{NSymm})^* = \mathrm{QSymm}$
- Baker and Richter: $H_*(\Omega\Sigma\mathbb{C}P^\infty) \cong \mathrm{NSymm}$
- Ditter's conjecture: QSymm is a free commutative algebra

Lemma

(i) *Over the rationals,* $\mathrm{NSymm} \otimes \mathbb{Q} \cong H_*(\Omega\Sigma\mathbb{C}P^\infty; \mathbb{Q})$ *is isomorphic to a Lie-Hopf algebra.*

(ii) *Over the integers,* $\mathrm{NSymm} \cong H_*(\Omega\Sigma\mathbb{C}P^\infty)$ *is not isomorphic to a Lie-Hopf algebra.*

- using topological methods, we find a maximal subalgebra of $H_*(\Omega\Sigma\mathbb{C}P^\infty)$ which is over $\mathbb{Q}$ isomorphic to $H_*(\Omega\Sigma\mathbb{C}P^\infty;\mathbb{Q})$ but over $\mathbb{Z}$ is a Lie-Hopf algebra

Theorem

The Hopf algebra $H_*(\Omega\Sigma\Omega\Sigma S^2)$ is a maximal subHopf algebra of $H_*(\Omega\Sigma\mathbb{C}P^\infty)$ which is isomorphic to a Lie-Hopf algebra.

Obstructions to desuspending the homotopy equivalence $e\colon \Sigma\left(\Omega\Sigma S^2\right) \longrightarrow \Sigma\left(\bigvee_{i=1}^{\infty} S^{2i}\right)$

- The homotopy equivalence

$$a\colon \Omega\Sigma\left(\bigvee_{i=1}^{\infty} S^{2i}\right) \longrightarrow \Omega\Sigma(\Omega\Sigma S^2)$$

  induces an isomorphism of graded Hopf algebras

$$a_*\colon \mathbb{Z}\langle \xi_1, \xi_2, \ldots\rangle \longrightarrow \mathbb{Z}\langle w_1, w_2, \ldots\rangle$$

  and its algebraic form is determined by the conditions

$$\Delta a_*\xi_n = (a_* \otimes a_*)(\Delta\xi_n).$$

  For example, $a_*\xi_1 = w_1,\ a_*\xi_2 = w_2 - w_1|w_1,\ a_*\xi_3 = w_3 - 3w_2|w_1 + 2w_1|w_1|w_1$.
- interesting from the topological point of view, since the elements $(w_n - a_*\xi_n)$ for $n \geq 2$ are obstructions to the desuspension of e.