Self-adjoint commuting ordinary differential operators of rank two

Andrey Mironov

Algebraic Topology and Abelian Functions, Moscow 2013

- Introduction: commuting ordinary differential operators of rank one

- Commuting higher rank ordinary differential operators

- Evolution equations of Krichever–Novikov type

- Difference Krichever–Novikov operators

- Rank two Schrödinger operators in magnetic fields

- Open problems

$$L_n = \partial_x^n + \sum_{i=0}^{n-2} u_i(x)\partial_x^i, \quad L_m = \partial_x^m + \sum_{i=0}^{m-2} v_i(x)\partial_x^i.$$

There is a classification of commuting operators (I.M. Krichever)

Examples

1. u_i, v_j are constant

2. $L_n = F_1(L)$, $L_m = F_2(L)$, F_1, F_2 are polynomials

Wallenberg, 1903:

1. $n = 1$, $L_2 = F(L_1)$.

2. $n = 2, m = 3$.

$$L_1 = \partial_x^2 + u(x), \qquad L_2 = \partial_x^3 + \frac{3}{2}u(x)\partial_x + \frac{3}{4}u'(x),$$

where $u(x)$ satisfies the equation

$$(u')^2 + 2u^3 + s_1 u + s_0 = 0.$$

Lemma (Schur, 1905)

If $L_1L_2 = L_2L_1$ and $L_1L_3 = L_3L_1$ ($L_1 \neq const$), then

$$L_2L_3 = L_3L_2.$$

There is a formal operator $S = 1 + v_1(x)\partial^{-1} + v_2(x)\partial^{-2} \ldots$

$\tilde{L}_1 = SL_1S^{-1} = \partial^n$, $\tilde{L}_2 = SL_2S^{-1}$

$\tilde{L}_1\tilde{L}_2 = \tilde{L}_2\tilde{L}_1 \Rightarrow \tilde{L}_2$ has constant coefficients

$\tilde{L}_1\tilde{L}_3 = \tilde{L}_3\tilde{L}_1 \Rightarrow \tilde{L}_3$ has constant coefficients

$\tilde{L}_2\tilde{L}_3 = \tilde{L}_3\tilde{L}_2 \Rightarrow L_2L_3 = L_3L_2$

Lemma (Burchnall, Chaundy, 1923)

If $L_1L_2 = L_2L_1$, then there exist a non-trivial polynomial $Q(\lambda, \mu)$ of two commuting variables such that $Q(L_1, L_2) = 0$.

Example

$$L_1 = \frac{d^2}{dx^2} - \frac{2}{x^2}, \quad L_2 = \frac{d^3}{dx^3} - \frac{3}{x^2}\frac{d}{dx} + \frac{3}{x^3}$$

$$L_1^3 = L_2^2, \quad Q(\lambda, \mu) = \lambda^3 - \mu^2.$$

Spectral curve

$$\Gamma = \{(\lambda, \mu) \in \mathbb{C}^2 : Q(\lambda, \mu) = 0\}.$$

If $L_1\psi = \lambda\psi$ and $L_2\psi = \mu\psi$, then $(\lambda, \mu) \in \Gamma$.

rank of L_1 and L_2 is

$$l = \dim\{\psi : L_1\psi = \lambda\psi, \ L_2\psi = \mu\psi\}.$$

Baker–Akhiezer function $\psi(x, P)$

Spectral data

$$\{\Gamma, q, k^{-1}, \gamma_1, \ldots, \gamma_g\}$$

Γ is algebraic curve, $q \in \Gamma$, k^{-1} is a local parameter near q, $k^{-1}(q) = 0$, $\gamma_1, \ldots \gamma_g \in \Gamma$.

The Baker–Akhiezer function has the property:

1. $\psi = e^{kx}\left(1 + \frac{f(x)}{k} + \ldots\right)$

2. on $\Gamma \backslash q$ the BA-function ψ is meromorphic with the poles in $\gamma_1, \ldots, \gamma_g$

For $\gamma_1, \ldots, \gamma_g$ in general position the BA-function ψ there exists and unique.

Let f(P) be a meromorphic function on Γ with a unique pole in q of order n

$$f = k^n + c_{n-1}k^{n-1} + \cdots + c_0 + \frac{c_{-1}}{k} + \ldots$$

$$\partial_x^n \psi + u_{n-1}(x)\partial_x^{n-1}\psi + \cdots + u_0(x)\psi = f\psi + e^{kx}\left(O\left(\frac{1}{k}\right)\right).$$

From the uniqueness of BA-function it follows that

$$L_1\psi(x,P) = f(p)\psi(x,P).$$

Let g(P) be a meromorphic function with unique pole in q of order m

$$L_2\psi(x,P) = g(P)\psi(x,P).$$

We have

$$(L_1L_2 - L_2L_1)\psi(x,P) = 0 \Rightarrow L_1L_2 = L_2L_1.$$

Example $\Gamma = \mathbb{C}P^1,\ q = \infty,\ k = z$

Baker–Akhiezer function $\psi = e^{xz}$

$$f = z^n + c_{n-1} z^{n-1} + \cdots + c_0,$$

$$\partial_x^n \psi + c_{n-1} \partial_x^{n-1} \psi + \cdots + c_0 \psi = f \psi.$$

Example

$$\Gamma = \mathbb{C}/\{2\omega\mathbb{Z} + 2\omega'\mathbb{Z}\}, \;\; q = 0,$$

$$\psi = e^{-x\zeta(z)}\frac{\sigma(z+x)}{\sigma(x)\sigma(z)},$$

$$(\partial_x^2 - 2\wp(x))\psi(x,z) = \wp(z)\psi(x,z),$$

$$\left(\partial_x^3 - 3\wp(x)\partial_x - \frac{3}{2}\wp'(x)\right)\psi(x,z) = \frac{1}{2}\wp'(z)\psi(x,z).$$

Lame operators

$$L_2 = -\partial_x^2 + g(g+1)\wp(x)$$

Rank $l > 1$

Spectral data (Krichever)

$$\{\Gamma, q, k^{-1}, \gamma_1, \ldots, \gamma_{lg}, \alpha_1, \ldots, \alpha_{lg}\}$$

$\alpha_i = (\alpha_{1i}, \ldots, \alpha_{il-1})$ — vector

(γ, α) — Turin parameters define stable (in the sense of Mumford) vector bundle of rank l degree lg on Γ with holomorphic sections $\eta_1, \ldots, \eta_l$

$$\eta_l(\gamma_i) = \sum_{j=1}^{l-1} \alpha_{ij}\eta_j(\gamma_i).$$

Vector Baker–Akhiezer function $\psi(x,P)=(\psi_0(x,P),\dots,\psi_{l-1}(x,P))$:

1. $\psi(x,P)=\left(\sum_{s=0}^{\infty}\xi_s(x)k^{-s}\right)\Psi_0(x,P)$, $\xi_0=(1,0,\dots,0)$, $\frac{d}{dx}\Psi_0=A\Psi_0$,

$$A=\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ k+u_0(x) & u_1(x) & u_2(x) & \dots & u_{l-1}(x) & 0 \end{pmatrix}$$

2. on $\Gamma-\{q\}$ ψ is meromorphic with the simple poles in $\gamma_1,\dots,\gamma_{lg}$

3. $\mathrm{Res}_{\gamma_i}\psi_j=\alpha_{ij}\mathrm{Res}_{\gamma_i}\psi_{l-1}$.

If f(P) is meromorphic function with the pole in q of order n, then there exist $L(f)$ such that

$$L(f)\psi(x,P)=f(P)\psi(x,P),\ \mathrm{ord}L(f)=ln.$$

Method of Turin parameters deformation (Krichever–Novikov method)

$$\frac{d^l}{dx^l}\psi_j = \chi_{l-1}\frac{d^{l-1}}{dx^{l-1}}\psi_j + \cdots + \chi_0\psi_j$$

χ_s — meromorphic on Γ, χ_s has lg simple poles $P_1(x), \ldots, P_{lg}(x)$. In the neighbourhood of q the functions χ_s have the form

$$\chi_0(x, P) = k + g_0(x) + O(k^{-1}),$$

$$\chi_j(x, P) = g_j(x) + O(k^{-1}), \quad j < l - 1,$$

$$\chi_{l-1}(x, P) = O(k^{-1}).$$

At the point $P_i(x)$

$$\chi_j = \frac{c_{ij}(x)}{k - \gamma_i(x)} + d_{ij}(x) + O(k - \gamma_i(x)).$$

Theorem *Parameters* $\gamma_i(x), \alpha_{ij}(x) = \frac{c_{ij}(x)}{c_{i,l-1}(x)}$, *and* $d_{ij}(x), 0 \leq j \leq l - 2, 1 \leq i \leq lg$ *satisfy the equation*

$$c_{i,l-1}(x) = -\gamma_i'(x),$$

$$d_{i0}(x) = \alpha_{i0}(x)\alpha_{i,l-2}(x) + \alpha_{i0}(x)d_{i,l-1}(x) - \alpha_{i0}'(x),$$

$$d_{ij}(x) = \alpha_{ij}(x)\alpha_{i,l-2}(x) - \alpha_{i,j-1}(x) + \alpha_{ij}(x)d_{i,l-1}(x) - \alpha_{ij}'(x), j \geq 1.$$

Krichever, Novikov: $g=1,\ l=2$ $\Gamma: \mu^2 = P_3(\lambda) = 4\lambda^3 + g_2\lambda + g_3$

$$L_1 = \left(\partial_x^2 + u\right)^2 + 2c_x(\wp(\gamma_2) - \wp(\gamma_1))\partial_x + (c_x(\wp(\gamma_2) - \wp(\gamma_1)))_x - \wp(\gamma_2) - \wp(\gamma_1),$$

$$\gamma_1(x) = \gamma_0 + c(x),\ \gamma_2(x) = \gamma_0 - c(x),$$

$$u(x) = -\frac{1}{4c_x^2} + \frac{1}{2}\frac{c_{xx}^2}{c_x^2} + 2\Phi(\gamma_1,\gamma_2)c_x - \frac{c_{xxx}}{2c_x} + c_x^2(\Phi_c(\gamma_0+c,\gamma_0-c) - \Phi^2(\gamma_1,\gamma_2)),$$

$$\Phi(\gamma_1,\gamma_2) = \zeta(\gamma_2 - \gamma_1) + \zeta(\gamma_1) - \zeta(\gamma_2).$$

Operator L_2 can be find from the equation $L_2^2 = P_3(L_1)$.

Dixmier: $g=1,\ l=2$

$$L_1 = \left(\frac{d^2}{dx^2} - x^3 - \alpha\right)^2 - 2x,$$

$$L_2 = \left(\frac{d^2}{dx^2} - x^3 - \alpha\right)^3 - \frac{3}{2}\left(x\left(\frac{d^2}{dx^2} - x^3 - \alpha\right) + \left(\frac{d^2}{dx^2} - x^3 - \alpha\right)x\right).$$

Theorem (Grinevich)

Commuting operators L_1 and L_2 corresponding to an elliptic curve have rational coefficients if and only if

$$c(x) = \int_{q(x)}^{\infty} \frac{dt}{\sqrt{P_3(t)}},$$

where $q(t)$ is a rational function.

If $\gamma_0 = 0$, and $q(x) = x$, we have the Dixmier operators.

Theorem (Grinevich, Novikov) *Operator L_1 is formally self-adjoint if and only if $\wp(\gamma_1) = \wp(\gamma_2)$.*

Mokhov: $g = 1, \ l = 3$

Rank $l = 2,\ g > 1$: self-adjoint case

Let L be an operator of the forth order of rank 2, then

$$\Gamma : w^2 = F(z) = z^{2g+1} + c_{2g} z^{2g} + \dots + c_0,$$

q is a branch point,

$$\sigma : \Gamma \to \Gamma,\ \sigma(z, w) = (z, -w).$$

We have

$$\psi'' = \chi_0 \psi + \chi_1 \psi',$$

where $\psi = (\psi_1, \psi_2)$ is a Baker–Akhiezer function.

Theorem (M.) *The operator L_4 is self-adjoint if and only if*

$$\chi_1(x, P) = \chi_1(x, \sigma(P)).$$

At $g = 1$ the Theorem was proved by Grinevich and Novikov.

Theorem (M.) *If L_4 is self-adjoint*

$$L_4 = (\partial_x^2 + V(x))^2 + W(x),$$

then

$$\chi_0 = -\frac{1}{2}\frac{Q''}{Q} + \frac{w}{Q} - V, \qquad \chi_1 = \frac{Q'}{Q},$$

where

$$Q = (z - \gamma_1(x)) \dots (z - \gamma_g(x)).$$

Function Q satisfies the equation

$$4F_g(z) = 4(z - W)Q^2 - 4V(Q')^2 + (Q'')^2 - 2Q'Q^{(3)}$$

$$+2Q(2V'Q' + 4VQ'' + Q^{(4)}),$$

where $Q', Q'', Q^{(k)}$ mean $\partial_x Q, \partial_x^2 Q, \partial_x^k Q$.

Corollary (M.) The function Q satisfies the linear equation

$$\mathcal{L}_5 Q = \left(\partial_x^5 + 2\langle V, \partial_x^3\rangle + 2\langle z - W - V'', \partial_x\rangle\right) Q = 0,$$

where $\langle A, B\rangle = AB + BA$. Potentials V, W have the form

$$V = \left(\frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2}\right) |_{z=\gamma_j},$$

$$W = -2(\gamma_1 + \cdots + \gamma_g) - c_{2g}.$$

The functions $\gamma_1(x), \ldots, \gamma_g(x)$ satisfy the equations

$$\frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2} |_{z=\gamma_j} = \frac{(Q'')^2 - 2Q'Q^{(3)} - 4F_g(z)}{4(Q')^2} |_{z=\gamma_k} .$$

Finite-gap Schrödinger operator

$$(-\partial_x^2 + u(x))\psi = z\psi, \qquad L_{2g+1}\psi = w\psi,$$

$$-\partial_x^2 + u(x) - z = (-\partial_x - i\chi_0)(\partial_x - i\chi_0), \qquad \chi_0 = \frac{Q_x}{2iQ} + \frac{w}{Q},$$

$$4F_g(z) = 4(z-u)Q^2 - (Q')^2 + 2QQ'',$$

$$Q_{xxx} - 4Q_x(u(x) - z) - 2u_x(x)Q = 0.$$

The potential u has the form

$$u = -2(\gamma_1 + \cdots + \gamma_g) - c_{2g}.$$

Dubrovin equations

$$\gamma_j' = \pm \frac{2i\sqrt{F_g(\gamma_j)}}{\prod_{k\neq j}(\gamma_k - \gamma_j)}.$$

Examples.

1. $L_4 = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g+1)\alpha_3 x,$

2. $L_4 = (\partial_x^2 + \alpha_1 \cos(x) + \alpha_0)^2 - \alpha_1 g(g+1)\cos(x),$

3. $L_4 = (\partial_x^2 + \alpha_1 e^x + \alpha_0)^2 + \alpha_1 g(g+1) e^x$ (V.Davletshina),

4. $L_4 = (\partial_x^2 + \alpha_1 \wp(x) + \alpha_0)^2 + s_1 \wp(x) + s_2 \wp^2(x),$

$\alpha_1 = \frac{1}{4} - 2g^2 - 2g, s_2 = -4g(g+2)(g^2-1), s_1 = \frac{1}{4} g(g+1)(16\alpha_0 + 5g)2),$

$(\wp'(x))^2 = 4\wp^3(x) + g_2 \wp^2(x) + g_1 \wp(x) + g_0.$

At $g = 1$ we have

$$V = \frac{-16F_1(\gamma) + W''^2 - 2W'W'''}{4W'^2}, \ \gamma = \frac{-c_2 - W}{2},$$

$$L_4 = (\partial_x^2 + V(x))^2 + W(x).$$

Evolution equations of Krichever–Novikov type

$$[L_4, \partial_{t_n} - A_n] = 0,$$

$$L_4 = (\partial_x^2 + V(x,t_n))^2 + W(x,t_n),$$

$$-A_n^* = A_n = \partial_x^{2n+1} + ...$$

$$A_3 = \partial_x^3 + \frac{3}{2}V(x,t)\partial_x + \frac{3}{2}V^{'}(x,t),$$

$$V_{t_1} = \frac{1}{4}(6VV^{'} + 6W^{'} + V^{'''}), \quad W_{t_1} = \frac{1}{2}(-3VW^{'} - W^{'''}),$$

Drinfeld and Sokolov found solutions of rank 1

$$[L_4, L_{2g+1}] = 0.$$

Solutions of rank two

$$[L_4, \partial_{t_n} - A_n] = 0, \quad [L_4, L_{4g+2}] = 0$$

Theorem (Davletshina, M.)

$$Q_{t_1} = \frac{1}{2}(-3VQ' - Q'''),$$

$$Q_{t_2} = \frac{1}{8}(-4QW' + 2V'Q'' + Q'(8z - 5V^2 + 2W - V'') - 2VQ''').$$

These equations give symmetries of

$$4F_g(z) = 4Q^2(z - W) - 4VQ'^2 + Q''^2 - 2Q'Q''' + 2Q(2Q'V' + 4VQ'' + Q^4).$$

At $g = 1, n = 1$ we have the Krichever–Novikov equation

$$W_{t_1} = \frac{48F_1(\gamma) - W''^2 + 2W'W'''}{8W'}, \ \gamma = \frac{-c_2 - W}{2}$$

Krichever–Novikov operators up to the conjugations are self-adjoint operators.

Theorem (Latham, Previato)

$$L_4 = (\partial_x^2 + V(x))^2 + W(x), \;\; g = 1$$

$$L_4 - z_0 = A_2 T, \qquad L_6 - w_0 = A_4 T, \qquad T = \partial_x^2 - \chi_1 \partial_x - \chi_0.$$

We have

$$L_{KN} = TA_2 = T(L_4 - z_0)T^{-1}, \qquad \tilde{L}_{KN} = TA_4 = T(L_6 - w_0)T^{-1}$$

To prove an analog of the Theorem for $g > 1$.

Kadomtsev–Petviashvili equation

$$\frac{3}{4}U_{yy} = \partial_x(U_t + \frac{3}{2}UU_x - \frac{1}{4}U_{xxx})$$

is equivalent to

$$[\partial_y - M, \partial_t - A] = 0,$$

where

$$M = \partial_x^2 - U(x,y,t), \quad A = \partial_x^3 - \frac{3}{2}U\partial_x + S(x,y,t),$$

$$S_x = -\frac{3}{4}U_y - \frac{3}{4}U_{xx}, \quad S_y = -U_t - \frac{3}{4}U_{xy} + \frac{U_{xxx}}{4} - \frac{3}{2}UU_x$$

Krichever found rank one solutions of KP

$$U = 2\partial_x^2 \log\theta(V_1x + V_2y + V_3t + V_4, \Omega).$$

Shiota proved the Novikov conjecture.

Rank two, $g=1$ solutions of KP were found by Krichever and Novikov

$$[L_{KN}, \partial_y - M] = 0,$$

$$L_{KN} = (\partial_x^2 - U)^2 + f_1\partial_x + \partial_x f_1 + f_0,$$

$$U = -V - \frac{-2W'^2 + 2(c_2 + W + 2z(y))}{(c_2 + W + 2z(y))^2},$$

$$V = \frac{-16F_1(\gamma) + W''^2 - 2W'W'''}{4W'^2}, \ \gamma = \frac{-c_2 - W}{2},$$

$W = W(x,t)$ satisfies the Krichever–Novikov equation

$$W_t = \frac{48F_1(\gamma) - W''^2 + 2W'W'''}{8W'},$$

$z(y)$ satisfies the following equation

$$(z')^2 = 4F_1(z).$$

To find rank two solutions of KP ($g>1$).

Difference commuting operators

$$L_n = \sum_{N_-}^{N_+} u_i(n)T^i, \quad L_m = \sum_{M_-}^{M_+} v_i(n)T^i,$$

where $n \in \mathbb{Z}$, T — shift operator

$$Tf(n) = f(n+1),$$

$$L_n\psi(n,P) = z\psi(n,P), \quad L_m\psi(n,P) = w\psi(n,P), \quad P = (z,w) \in \Gamma.$$

Theorem (I.M.Krichever, S.P.Novikov)

$$L_{KN} = L_2^2 - \wp(\gamma_n) - \wp(\gamma_{n-1}),$$

where

$$L_2 = T + v_n + c_n T^{-1},$$

$$4c_{n+1} = (s_n^2 - 1)F(\gamma_{n+1}, \gamma_n)F(\gamma_{n-1}, \gamma_n),$$

$$2v_{n+1} = s_n F(\gamma_{n+1}, \gamma_n) - s_{n+1} F(\gamma_n, \gamma_n + 1),$$

$$F(u, v) = \zeta(u + v) - \zeta(u - v) - 2\zeta(v),$$

s_n, γ_n — *functional parameters.*

Rank two one-point operators. Spectral curve

$$w^2 = F_g(z) = z^{2g+1} + c_{2g} z^{2g} + c_{2g-1} z^{2g-1} + ... + c_0,$$

$$L_4 = \sum_{i=-2}^{2} u_i(n) T^i, \quad L_{4g+2} = \sum_{i=-(2g+1)}^{2g+1} v_i(n) T^i,$$

$$L_4 \psi = z\psi, \quad L_{4g+2}\psi = w\psi.$$

Common eigenfunctions satisfy

$$\psi(n+1, P) = \chi_1(n, P)\psi(n-1, P) + \chi_2(n, P)\psi(n, P).$$

Functions $\chi_1(n, P)$, $\chi_2(n, P)$ are rational on Γ.

Theorem (G.Mauleshova, M.) *If*

$$\chi_1(n,P)=\chi_1(n,\sigma(P)),\quad \chi_2(n,P)=-\chi_2(n,\sigma(P)),$$

then L_4 *has the form*

$$L_4=(T+V(n)T^{-1})^2+W(n),$$

$$\chi_1(n)=-\frac{V(n)Q(n+1)}{Q(n)},\quad \chi_2(n)=\frac{w}{Q(n)},$$

where $Q(n)=(z-\gamma_1(n))\dots(z-\gamma_g(n))$, $V(n),W(n),Q(n)$ *satisfy*

$$F_g(z)=V(n)(Q(n-1)-Q(n))Q(n+1)+$$

$$+Q(n)(V(n+1)Q(n+2)+Q(n+1)(z-V(n+1)-W(n))),$$

$$Q(n-1)V(n)+Q(n)(z-V(n)-V(n+1)-W(n))+Q(n+2)(-z+V(n+1)+$$

$$+V(n+2)+W(n+1))-Q(n+3)V(n+2)=0.$$

Theorem (G.Mauleshova, M.) *The operator*

$$L_4 = (T + (r_3 n^3 + r_2 n^2 + r_1 n + r_0)T^{-1})^2 + g(g+1)r_3 n, \quad r_3 \neq 0.$$

commutes with an operator L_{4g+2}.

Theorem (G.Mauleshova, M.) *The operator*

$$L_4 = (T + (r_1 a^n + r_0)T^{-1})^2 + g(g+1)r_1 a^n, \quad r_1 \neq 0.$$

commutes with an operator L_{4g+2}.

Def. (I.M. Krichever, S.P. Novikov)

2D-Schrödinger operator H is called *rank two operator*, if there are operators L_1, L_2 such that

$$[H, L_i] = B_i H, \qquad [L_1, L_2] = B_3 H,$$

common eigenvalues

$$H\psi = E\psi, \qquad L_1\psi = z\psi, \qquad L_2\psi = w\psi$$

satisfy $R(z, w) = 0$, and the dimension of solutions is two.

Let $\wp(z)$ be an elliptic function satisfying

$$(\wp'(z))^2 = b_3\wp(z)^3 + b_2\wp(z)^2 + b_1\wp(z) + b_0,$$

$$b_3 = -\frac{1}{2a^2}, \qquad b_2 \neq 0, \qquad b_1 = -\frac{7b_0}{10b_2a^2} - \frac{20b_2^2a^2}{49},$$

Theorem (M., B.Saparbaeva) *Operator*

$$H=\frac{\partial^2}{\partial z\partial \bar z}+\frac{7a\wp'(az+b\bar z)}{20ab_2-14\wp(az+b\bar z)}\frac{\partial}{\partial \bar z}+\frac{b\wp(az+b\bar z)}{2a}$$

is rank two operator of rank two with an elliptic spectral curve, and

$$\begin{cases}[H,L_4]=B_1H,\\ [H,L_6]=B_2H,\\ [L_4,L_6]=0,\end{cases}\qquad \begin{cases}[H,\tilde L_4]=\tilde B_1H,\\ [H,\tilde L_6]=\tilde B_2H\\ [\tilde L_4,\tilde L_6]=0,\end{cases}\qquad [L_4,\tilde L_4]=BH,$$

where

$$L_4=(\frac{\partial^2}{\partial z^2}+\wp(az+b\bar z))^2+\frac{15}{14}b_2a^2\wp(az+b\bar z)+\frac{3}{8}\wp(az+b\bar z)^2,$$

$$\tilde L_4=\frac{\partial^4}{\partial \bar z^4}-\frac{14b\wp'(az+b\bar z)}{10b_2a^2-7\wp(az+b\bar z)}\frac{\partial^3}{\partial \bar z^3}+$$

$$+\frac{b^2(686b_0-100b_2a^4+1540b_2^2a^2\wp(az+b\bar z)-735b_2\wp(az+b\bar z)^2)}{70a^2b_2(10b_2a^2-7\wp(az+b\bar z))}\frac{\partial^2}{\partial \bar z^2}+$$

$$+\frac{b^3(343b_0-100b^3a^4+980b_2^2a^2\wp(az+b\bar z)-735b_2\wp(az+b\bar z)^2)\wp'(az+b\bar z))}{10b_2(10b_2a^2-7\wp(az+b\bar z))^2}\frac{\partial}{\partial \bar z}.$$

The group of automorphisms of the first Weyl algebra $Aut(A_1)$ acts on the moduli spaces of operators with polynomial coefficients. For example, with the help of the automorphism

$$\varphi_1(x) = \alpha x + \beta\partial_x, \qquad \varphi_1(\partial_x) = \gamma x + \delta\partial_x, \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathsf{SL}_2$$

one can get from $L_4^\sharp, L_{4g+2}^\sharp$ the operators of rank 3

$$L_4^\sharp = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g+1)\alpha_3 x, \qquad \alpha_3 \neq 0.$$

Another example of automorphisms are

$$\varphi_2(x) = x + P_1(\partial_x), \qquad \varphi_2(\partial_x) = \partial_x,$$

$$\varphi_3(x) = x, \qquad \varphi_3(\partial_x) = \partial_x + P_2(x),$$

where P_1, P_2 are polynomials. Dixmier proved that $Aut(A_1)$ is generated by φ_i. It would be very interesting to understand how $Aut(A_1)$ acts on the spectral data.

The equation

$$Y^2 = X^{2g+1} + c_{2g}X^{2g} + \cdots + c_0$$

has nonconstant solutions $X = L_4^\sharp, Y = L_{4g+2}^\sharp \in A_1$ for some c_i.

It is easy to see that the group $Aut(A_1)$ preserves the space of all such solutions, i.e. if (X, Y) is a solution to the polynomial equation above, with $X, Y \in A_1$, then $(\varphi(X), \varphi(Y))$ is also a solution for any $\varphi \in Aut(A_1)$. Then, a natural question is to describe the orbits of $Aut(A_1)$ in the space of solutions under the action of $Aut(A_1)$.

Yu. Berest has proposed the following conjecture: If $g > 1$, then there are only finitely many such orbits, i.e. the equation $f(X, Y) = \sum_{i,j=0}^{k} \alpha_{ij} X^i Y^j = 0$ with generic $\alpha_{ij} \in \mathbb{C}$ has at most finitely many solutions in A_1 up to the action of $Aut(A_1)$.