ON SOME RATIONAL AND INTEGRAL COMPLEX GENERA

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Abstract. We prove that on the rational complex bordism ring the Krichever-Höhn genus ϕ_{KH} is the composition $\psi \circ \kappa^{-1}$. Here the genus ψ corresponds to the formal group law whose invariant differential $\omega(t)$ equals $\sqrt{1+p_1t+p_2t^2+p_2t^3+p_4t^4}$ and κ classifies the formal group law strictly isomorphic to the universal formal group law under strict isomorphism $x\mathbf{CP}(x)$. We construct certain elements A_{ij} in the Lazard ring and give an alternative definition of the universal Krichever formal group law. This implies that the corresponding coefficient ring is the quotient of the Lazard ring by the ideal generated by all A_{ij}, $i, j \geq 3$.

1. Rational Krichever-Höhn genus

For the current state of complex cobordism and formal group laws we refer the reader to the excellent survey [5].

A formal group law over a commutative ring with unit R is a power series $F(x, y) \in R[[x, y]]$ satisfying

(i) $F(x, 0) = F(0, x) = x$,
(ii) $F(x, y) = F(y, x)$,
(iii) $F(x, F(y, z)) = F(F(x, y), z)$.

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Let F and G be formal group laws. A homomorphism from F to G is a power series $\nu(x) \in R[[x]]$ with constant term 0 such that

$$\nu(F(x, y)) = F(\nu(x), \nu(y)).$$

It is isomorphism if $\nu'(0)$ (the coefficient at x) is a unit in R, and a strict isomorphism if the coefficient at x is 1.

If F is a formal group law over a commutative $\mathbb{Q}$-algebra R, then it is strictly isomorphic to the additive formal group law $x + y$. In other words, there is a strict isomorphism $l(x)$ from F to the additive formal group law, called the logarithm of F, so that $F(x, y) = l^{-1}(l(x) + l(y))$. The inverse to logarithm is called the exponential of F.

The logarithm $l(x) \in R \otimes \mathbb{Q}[[x]]$ of a formal group law F is given by

$$l(x) = \int_0^x \frac{dt}{\omega(t)}, \quad \omega(x) = \frac{\partial F(x,y)}{\partial y}(x,0).$$

There is a ring L, called Lazard universal ring, and a formal group law $F(x,y) = \sum a_{ij}x^i y^j$ defined over L such that for any formal group law G over any commutative ring with unit R there is a unique ring homomorphism $r : L \to R$ such that $G(x,y) = \sum r(a_{ij})x^i y^j$.

The formal group law of geometric cobordism was introdiced in [12]. Following Quillen we will identify it with the universal Lazard formal group law as it is proved in [13] that the coefficient ring of complex cobordism $MU_* = \mathbb{Z}[x_1, x_2, ...]$, $|x_i| = 2i$ is naturally isomorphic as a graded ring to Lazard universal ring.

The Krichever-Höhn genus, or the general four variable complex elliptic genus [7],[8],

$$\phi_{KH} : MU_* \otimes \mathbb{Q} \to \mathbb{Q}[q_1, ..., q_4]$$

is a graded $\mathbb{Q}$-algebra homomorphism defined by the following property: if one denotes by $f_{Kr}(x)$ the exponential of the Krichever [6] universal formal group law $\mathcal{F}_{Kr}$, then the series

$$h(x) := \frac{f'_{Kr}(x)}{f_{Kr}(x)}$$

satisfies the differential equation

$$(h'(x))^2 = S(h(x)), \tag{1.1}$$

where $S(x) = x^4 + q_1x^3 + q_2x^2 + q_3x + q_4$, the generic monic polynomial of degree 4 with formal parameters q_i of weights $|q_i| = 2i$.

This genus is the universal genus on the rational bordism ring of SU-manifolds, which is multiplicative in fiber bundles of SU-manifolds with compact connected structure group.

To generalize naturally the Oshanin's elliptic genus from $\Omega_*^{SO} \otimes \mathbb{Q}$ to $\mathbb{Q}[\mu, \epsilon]$ (see [9]), new elliptic genus ψ is defined in [10],

$$\psi : MU_* \otimes \mathbb{Q} \to \mathbb{Q}[p_1, p_2, p_3, p_4],$$

to be the genus whose logarithm equals

$$\int_0^x \frac{dt}{\omega(t)}, \quad \omega(t) = \sqrt{1 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4}$$

and p_i are again formal parameters $|p_i| = 2i$.

It is proved in [10] that the ψ-genus is the universal genus on the rational complex bordism ring which is multiplicative in projectivizations $P(E)$ of complex vector bundles $E \to B$ over Calabi-Yau 3-folds B (i.e. B is a compact Kähler manifold with vanishing first Chern class).

Clearly to calculate the values of ψ on $\mathbf{CP}_i$, the generators of the rational complex bordism ring $MU_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbf{CP}_1, \mathbf{CP}_2, \dots]$ we need only the Taylor expansion of $(1+y)^{-1/2}$ as by above definition

$$(1+p_1x+p_2x^2+p_3x^3+p_4x^4)^{-1/2} = log'_\psi = \sum_{i\geq 1} \psi(\mathbf{CP}_i)x^i.$$

It is natural to ask if there exists similar elementary way, different from formulas in [3], for calculation of ϕ_{KH}.

Let κ be the classifying map of the formal group law $\tilde{F}$ over the rational Lazard ring defined as follows.

Let $\mathbf{CP}(x) = 1 + \sum_{i\geq 1} \mathbf{CP}_i x^i$, where $\mathbf{CP}_i$ is the bordism class of the complex projective space CP^i. Then let

$$\nu(x) := x\mathbf{CP}(x)$$

be the strict isomorphism $F \to \tilde{F}$, where F is the universal formal group law. So that

$$\nu(F(x,y)) = \tilde{F}(\nu(x), \nu(y)),$$

Now let

$$\kappa : MU_* \otimes \mathbb{Q} \to MU_* \otimes \mathbb{Q}$$

be the classifying map of $\tilde{F}$ and

$$t : \mathbb{Q}[p_1, ..., p_4] \to \mathbb{Q}[q_1, ..., q_4]$$

be the ring homomorphism defined by $t(p_i) = q_i$.

One has

Theorem 1.1. $\phi_{KH} = t \circ \psi \circ \kappa^{-1}$.

Let $log(x)$ and $mog(x)$ be the logarithm series of F and $\tilde{F}$ respectively then by definition

$$mog(x) = log(\nu^{-1}(x)), \quad \kappa(log(x)) = mog(x).$$

Therefore the value $\kappa(\mathbf{CP}_i)$ is determined by equating the coefficients at x^i in

$$\sum_{i\geq 1} \frac{\kappa(\mathbf{CP}_i)}{i+1} x^{i+1} = \sum_{i\geq 1} \frac{\mathbf{CP}_i}{i+1} (\nu^{-1}(x))^{i+1}. \tag{1.2}$$

For instance

$\kappa(\mathbf{CP}_1) = -\mathbf{CP}_1$;
$\kappa(\mathbf{CP}_2) = 3\mathbf{CP}_1^2 - 2\mathbf{CP}_2$;
$\kappa(\mathbf{CP}_3) = -10\mathbf{CP}_1^3 + 12\mathbf{CP}_1\mathbf{CP}_2 - 3\mathbf{CP}_3$;
$\kappa(\mathbf{CP}_4) = 35\mathbf{CP}_1^4 - 60\mathbf{CP}_1^2\mathbf{CP}_2 + 20\mathbf{CP}_1\mathbf{CP}_3 + 10\mathbf{CP}_2^2 - 4\mathbf{CP}_4$.

In the following two lemmas we prove that the pair $(t,\ \sum_{i\geq 0}\phi_{KH}(\mathbf{CP}_i)x^{i+1})$ is the strict isomorphism from $\phi_{KH}(F)$ to $\psi(F)$, where $F = F(x,y)$ is the universal formal group law. In other words the series $\sum_{i\geq 0}\phi_{KH}(\mathbf{CP}_i)x^{i+1}$ is the strict isomorphism from $\phi_{KH}(F)$ to $t\psi(F)$ in the usual sense.

Lemma 1.2. *Let F be the universal formal group law and $\tilde{F}$ be its isomorphic under the strict isomorphism $\nu(x) = x\mathbf{CP}(x)$ as above. Let exp be the exponent of F. The series $\frac{1}{h(x)} = exp(x)/exp'(x)$ is invertible, and let $j(x)$ be its inverse, then $j(x) = mog(x)$, the logarithmic series of $\tilde{F}$.*

Proof. Note that

$$j^{-1}(log(x)) = \frac{exp(log(x))}{exp'(log(x))} = x\mathbf{CP}(x).$$

Hence by definition of $\nu(x)$ one has

$$j^{-1}(mog(x)) = j^{-1}(log(\nu^{-1}(x))) = \nu^{-1}(x)\mathbf{CP}(\nu^{-1}(x)) = x,$$

as $x\mathbf{CP}(x) = \nu(x)$. □

Lemma 1.3. *Let* $\tilde{\omega} = \frac{1}{mog'(x)}$ *be the invariant differential form of the formal group law* $\tilde{F}$ *above. Then the condition of Krichever-Höhn* (1.1) *is satisfied if and only if* $\tilde{\omega}(x)^2$ *is a polynomial of degree 4.*

Proof. Since the series $j(x)$ is invertible, the condition (1.1) is equivalent to

$$(h'(j(x)))^2 = S(h(j(x))).$$

But $h(j(x)) = 1/x$, hence

$$(h \circ j)'(x) = h'(j(x))j'(x) = -1/x^2,$$

so that by Lemma 1.2

$$h'(j(x)) = -\tilde{\omega}(x)/x^2.$$

It follows that the condition (1.1) is equivalent to

$$\tilde{\omega}(x)^2 = x^4 S(1/x),$$

where on the right hand side one clearly has a degree four polynomial. □

Now let us prove Theorem 1.1. Lemma 1.3 implies that $\phi_{KH}(\tilde{F}))$ is of type $\psi(F)$, that is corresponding invariant form $\omega(x)^2$ is a polynomial of degree 4. By definition the formal group law $\psi(F)$ is universal with this property. Therefore there is classifying map of $\phi_{KH}(\tilde{F})$, that is unique ring homomorphism $t : \mathbb{Q}[p_1, ..., p_4] \to \mathbb{Q}[q_1, ..., q_4]$ such that $t(\psi(F)) = \phi_{KH}(\tilde{F}) = \phi_{KH}(\kappa(F)$. Therefore $t \circ \psi = \phi_{KH} \circ \kappa$. By definition κ is isomorphism and we get $t \circ \psi \circ \kappa^{-1} = \phi_{KH}$. Finally $t(p_i) = q_i$ as t is unique and it sends $1 + p_1 t + p_2 t^2 + p_2 t^3 + p_4 t^4$ to $1 + q_1 t + q_2 t^2 + q_3 t^3 + p_4 t^4$. $\square$

2. Integral Krichever genus

Now we turn to the universal Krichever formal group law F_{Kr} [8] and prove that it coincides with the universal formal group law by Buchstaber F_B (with a minor specialisation that does not affects on the formal group law).

In [4] V. M. Buchstaber has given the analytical solution of a functional equation for the exponent of the formal group law of the form

$$\mathcal{F}_B(x,y) = \sum \alpha_{ij} x^i y^j = \frac{A(y)x^2 - A(x)y^2}{B(y)x - B(x)y}. \tag{2.1}$$

As before let $\omega(x) = \frac{\partial F(x,y)}{\partial y}(x,0)$ be the invariant form of $F(x,y)$.

Lemma 2.1. *Let $\mathcal{F}$ be the formal group law of the form $\mathcal{F}_B$, with $B'(0) = A'(0)$. Then $B(x)$ coincides with restricted $\omega(x)$, that is $B(x)$ is the image of $\omega(x)$ under the ring homomorphism of the Lazard ring classifying the formal group law $\mathcal{F}$.*

For this note that if our series A and B have form

$$A(t) = A_0 + A_1 t + A_2 t^2 + O(t^3)$$

and

$$B(t) = B_0 + B_1 t + B_2 t^2 + O(t^3),$$

then

$$\mathcal{F}(x,y) = \frac{A_0}{B_0}(x+y) + O(xy).$$

So if $\mathcal{F}(x,y)$ is a formal group law we must have $A_0 = B_0$, and after dividing the numerator and denominator appropriately we may assume that $A_0 = B_0 = 1$. We then furthermore calculate

$$\mathcal{F}(x,y) = x + y + A_1 xy + \sum_{i=2}^{\infty} B_i(x^i y + xy^i) + O(x^2y^2).$$

We thus have

$$\alpha_{1i} = B_i,\ i \geq 2,\ \ \alpha_{11} = A_1 = A'(0) = B'(0)$$

and

$$1 + \sum_{i \geqslant 1} \alpha_{1i} t^i$$

is restricted $(1 + \sum_{i\geqslant 1} [\mathbf{CP}^i] t^i)^{-1} = \omega(t)$. □

Let us now present a minor modification of the analysis of $\mathcal{F}_B$ performed in [11] and as Lemma 2.1 suggests to introduce

$$A(x,y)=\sum A_{ij}x^iy^j=F(x,y)(x\omega(y)-y\omega(x)). \tag{2.2}$$

We define the universal Nadiradze formal group law $\mathcal{F}_N$ by the obvious classifying map of the Lazard ring to its quotient ring by the ideal generated by all A_{ij} with $i,j\geq 3$.

Proposition 2.2. *Let L be the Lazard ring. Then*

i) $\omega'(x) - \omega'(0) = 2x\hat{\omega}(x)$ *in* $L[[x]]$, *where* $\hat{\omega}(x) = \sum_{i \geqslant 1} \omega_i x^{i-1}$.

ii) One has in $L[[x,y]]/(xy)^3$

$$A(x,y) = (x\omega(y) + y\omega(x) - \omega'(0)xy)\,(x\omega(y) - y\omega(x)) + (\omega(x)\hat{\omega}(x) - \omega(y)\hat{\omega}(y))\,x^2y^2.$$

Proof. Let f and g be the exponent and logarithm of F respectively. Hence $F(x,y)=f(g(x)+g(y))$ and

$$f'(x)=1/g'(f(x))=\omega(f(x)),\ \ f'(g(x))=\omega(f(g(x))=\omega(x). \tag{2.3}$$

Let $\omega(x)=1+b_1x+b_2x^2+\cdots$. Then as $g''(0)=-f''(0)=-\omega'(0)=-b_1$ we have

$$\frac{\partial^2 F}{\partial y^2}(x,0)=f^{''}(g(x))+f'(g(x))g''(0)=\omega'(x)\omega(x)-\omega'(0)\omega(x). \tag{2.4}$$

This implies i) as the left hand side of (2.4) has factor 2, and $\omega(x)$ is invertible. ii) Because of antisymmetry we have modulo $(xy)^3$

$$A(x,y)=A(y)x^2-A(x)y^2=\sum(A_{i2}x^2y^i-A_{i2}x^iy^2).$$

We want to calculate $-\sum A_{i2}x^i$ in terms of $\omega(x)$.

Applying $\frac{\partial^2}{\partial y^2}(x,0)$ to (2.2) and taking into account (2.3) and (2.4) one obtains

$$-2\sum A_{i2}x^i=x\omega(x)\omega'(x)-x\omega'(0)\omega(x)+2x\omega'(0)\omega(x)-2\omega^2(x)+2b_2x^2.$$

As the coefficients are in the Lazard ring this implies

$$-\sum A_{i2}x^i=x\omega(x)\frac{\omega'(x)-\omega'(0)}{2}+x\omega'(0)\omega(x)-\omega^2(x)+b_2x^2.$$

Therefore

$\sum(A_{i2}x^2y^i-A_{i2}x^iy^2)=$
$(x\omega(y)+y\omega(x))(x\omega(y)-y\omega(x))-\omega'(0)xy(x\omega(y)-y\omega(x))+\omega(x)\hat{\omega}(x)x^2y^2-\omega(y)\hat{\omega}(y)x^2y^2.$ □

In order to compute the Krichever genus on the coefficients of the formal group law of geometric cobordism in [6] the universal Krichever formal group law $\mathcal{F}_{Kr}$ is defined as

$$\mathcal{F}_{Kr}(x,y) = xb(y) + yb(x) - b'(0)xy + \frac{b(x)\beta(x) - b(y)\beta(y)}{xb(y) - yb(x)}x^2y^2, \tag{2.5}$$

where $\beta(x) = \frac{b'(x)-b'(0)}{2x}$. Also it is proved in [6] that

$$b(x) = \frac{\partial \mathcal{F}_{Kr}}{\partial y}(x,0).$$

Lemma 2.1 and Proposition 2.2 ii) imply that $\mathcal{F}_{Kr}$ can be alternatively defined by the classifying map of $\mathcal{F}_N$ if relabel the restriction of the invariant form $\omega(x)$ as $b(x)$. Thus $\mathcal{F}_N = \mathcal{F}_{Kr}$.

Following [4] in [3] the authors consider the following formal group law corresponding to Krichever genus

$$\mathcal{F}_b(u_1,u_2) = u_1c(u_2) + u_2c(u_1) - au_1u_2 - \frac{d(u_1) - d(u_2)}{u_1c(u_2) - u_2c(u_1)}u_1^2u_2^2.$$

It follows from Lemma 2.1 and Proposition 2.2 ii) that (if specify $c'(0) = a$) $\mathcal{F}_b(x,y)$ coincides with (2.5), that is $c(x) = b(x)$ and $d(x) = -b(x)\beta(x)$.

Theorem 2.3. *$\mathcal{F}_N = \mathcal{F}_{Kr} = \mathcal{F}_b$ and the coefficient ring is the the quotient of the Lazard ring by the ideal generated by all A_{ij} with $i, j \geq 3$.*

In [1] we calculated on MAPLE the coefficient ring of Nadiradze formal group law $\mathcal{F}_N$ up to dimension 26. Namely there is a set of polynomial generators $z_1, z_2, ...$ of the Lazard ring for which the low degree defining relations are

$$5z_5 = z_2z_3 + 2z_1z_4,\ 2z_6 = 0,\ z_1z_6 = 0,\ z_3z_6 = 0,\ z_{10} = 0,\ z_5z_6 = 0,\ z_{12} = 0,$$

and $7z_7$, $2z_8$, $3z_9$, $11z_{11}$, and $13z_{13}$ are decomposable. For the space reason we omit here these long decompositions. Note that our calculations agree with the results in [6] on the structure of the coefficient ring of $\mathcal{F}_{Kr}$ obtained in terms of associativity equation. The new information here concerning the Krichever group, and hence the Krichever genus, is that as the relations $z_{10} = 0$, $z_{12} = 0$ imply in dimensions 20 and 24 there is no indecomposable elements. The question arises in which dimensions any element is multiplicatively decomposable.

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