Combinatorial group theory and the homotopy groups of finite complexes

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combinatorial description of $\pi_*(S^2)$

- Let
$$F_n = \langle x_0, x_1, \ldots, x_n \mid x_0 x_1 \cdots x_n \rangle$$
be the one-relator group generated by $x_0, \ldots, x_n$ with the defining relation $x_0 \cdots x_n = 1$. (Note that F_n is a free group of rank n with a basis given by $\{x_1, \ldots, x_n\}$.)
- Let $R_i = \langle x_i \rangle^{F_n}$ be the normal closure of x_i in F_n for $0 \leq i \leq n$. We can form a symmetric commutator subgroup
$$[R_0, R_1, \ldots, R_n]_S = \prod_{\sigma \in \Sigma_{n+1}} [\ldots [R_{\sigma(0)}, R_{\sigma(1)}], \ldots, R_{\sigma(n)}],$$
where the symmetric group Σ_{n+1} acts on $\{0, 1, \ldots, n\}$. The symmetric group Σ_{n+1} permutes the indices of the subgroups R_i.

combinatorial description of $\pi_*(S^2)$

- There is an action of the braid group B_{n+1} on $F_n = \langle x_0, x_1, \ldots, x_n \mid x_0 x_1 \cdots x_n \rangle$ by the Artin representation, which induces an action of B_{n+1} on the quotient group $F_n/[R_0, R_1, \ldots, R_n]_S$.
- **Theorem (Wu, 2002)** The center of $F_n/[R_0, R_1, \ldots, R_n]_S$ is exactly given by the fixed set of the pure braid group P_{n+1} action on $F_n/[R_0, R_1, \ldots, R_n]_S$ for $n \geq 3$.
- **Theorem (Wu, 1994, published version 2001).** For $n \geq 1$, there is an isomorphism

$$\pi_{n+1}(S^2) \cong \frac{R_0 \cap \cdots \cap R_n}{[R_0, \ldots, R_n]_S}$$

  This quotient group is isomorphic to the center of the group $F_n/[R_0, R_1, \ldots, R_n]_S$. □

Philosophy of centers of groups

- People have studied the question of how to realize a given abelian group as the center of a finitely-generated or finitely-presented group, for instance,
- **Baumslag**, *Finitely presented groups* in: Proceedings of the International Conference on the Theory of Groups, Canberra, August, 1965 (1967), pp. 37-50.
- **A. O. Houcine**, *Embeddings in finitely presented groups which preserve the center*, J.Algebra **307**, 1–23.
- **Brown-Loday**, let G be a group with trivial center. Then there is a natural isomorphism

$$\pi_3(\Sigma K(G,1)) \cong Z(G \otimes G),$$

where $G \otimes G$ is the non-abelian tensor square in the sense of Brown-Loday.

Question on $\pi_*(S^k)$

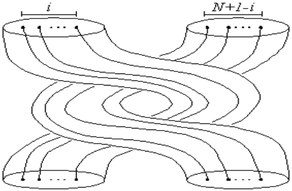
- It has been the concern of many people whether one can give a combinatorial description of the homotopy groups of higher dimensional spheres, ever since a description of $\pi_*(S^2)$ was announced in 1994.
- Technically the proof of this theorem was obtained by determining the Moore boundaries of Milnor's $F[K]$-construction on the simplicial 1-sphere S^1, which is a simplicial group model for ΩS^2. A canonical approach is to study Milnor's construction $F[S^k] \simeq \Omega S^{k+1}$ for $k > 1$. Although there have been some attempts to study this question using $F[S^k]$, technical difficulties arise in handling Moore boundaries of $F[S^k]$ in a good way, and combinatorial descriptions of the homotopy groups of higher dimensional spheres using the simplicial group model $F[S^k]$ would be very messy.

our construction

- In this article, we give a combinatorial description of $\pi_*(S^k)$ for any $k \geq 3$ by using the free product with amalgamation of pure braid groups.
- Our construction is as follows. Given $k \geq 3,\ n \geq 2$, let P_n be the n-strand Artin pure braid group with the standard generators $A_{i,j}$ for $1 \leq i < j \leq n$. We construct a (free) subgroup $Q_{n,k}$ of P_n from cabling as follows.
- Our cabling process starts from $P_2 = \mathbb{Z}$ generated by the 2-strand pure braid $A_{1,2}$.

The construction of $Q_{n,k}$: Step 1

Consider the 2-strand pure braid $A_{1,2}$. Let ξ_i be $(k-1)$-strand braid obtained by inserting i parallel strands into the tubular neighborhood of the first strand of $A_{1,2}$ and $k-i-1$ parallel strands into the tubular neighborhood of the second strand of $A_{1,2}$ for $1 \leq i \leq k-2$. [From Cohen-Wu 2004, 2011]



Where N+1=k-1

The construction of $Q_{n,k}$: Step 2

- Let $\alpha_k = [\ldots[[\xi_1^{-1}, \xi_1\xi_2^{-1}], \xi_2\xi_3^{-1}], \ldots, \xi_{k-3}\xi_{k-2}^{-1}, \xi_{k-2}]$ be a fixed choice of $(k-1)$-strand braid, which is a nontrivial $(k-1)$-strand Brunnian braid.

- For a group G and $g, h \in G$, we use the notation $[g, h] := g^{-1}h^{-1}gh$.

The construction of $Q_{n,k}$: Step 3

- By applying the cabling process as in Step 1 to the element α_k, we insert parallel strands into the tubular neighborhood of the strands of α_k in any possible way to obtain n-strand braids. As the order in which the strands are inserted is arbitrary, there are $\binom{n-1}{k-2}$ ways of doing this. Label the $\binom{n-1}{k-2}$ n-strand braids obtained in this way by y_j for $1 \leq j \leq \binom{n-1}{k-2}$.

- **It is too difficult to draw a picture for y_j now!**

- Let $Q_{n,k}$ be the subgroup of P_n generated by y_j for $1 \leq j \leq \binom{n-1}{k-2}$.

Free Product of Pure Braid Groups with Amalgamation

Now consider the free product with amalgamation

$$P_n *_{Q_{n,k}} P_n.$$

Namely this amalgamation is obtained by identifying the elements y_j in two copies of P_n. Let $A_{i,j}$ be the generators for the first copy of P_n and let $A'_{i,j}$ denote the generators $A_{i,j}$ for the second copy of P_n. Let $R_{i,j} = \langle A_{i,j}, A'_{i,j} \rangle^{P_n *_{Q_{n,k}} P_n}$ be the normal closure of $A_{i,j}, A'_{i,j}$ in $P_n *_{Q_{n,k}} P_n$. Let

$$[R_{i,j} \mid 1 \le i < j \le n]_S = \prod_{\{1,2,\ldots,n\}=\{i_1,j_1,\ldots,i_t,j_t\}} [[R_{i_1,j_1}, R_{i_2,j_2}], \ldots, R_{i_t,j_t}]$$

be the product of all commutator subgroups such that each integer $1 \le j \le n$ appears as one of indices at least once.

Our Main Theorem 1

Let $k \geq 3$. The homotopy group $\pi_n(S^k)$ is isomorphic to the center of the group

$$(P_n *_{Q_{n,k}} P_n)/[R_{i,j} \mid 1 \leq i < j \leq n]_S$$

for any n if $k > 3$ and any $n \neq 3$ if $k = 3$.

- **Note.** The only exceptional case is $k = 3$ and $n = 3$. In this case, $\pi_3(S^3) = \mathbb{Z}$ while the center of the group is $\mathbb{Z}^{\oplus 4}$.

Mahowald's Question

- Mark Mahowald asked in 1995 whether one can give a combinatorial description of the homotopy groups of the suspensions of real projective spaces.

- In this article, we also give a combinatorial description of the homotopy groups of Moore spaces as the first step for attacking Mahowald's question.

- Let $M(\mathbb{Z}/q, k)$ be the $(k+1)$-dimensional Moore space. Namely $M(\mathbb{Z}/q, k) = S^k \cup_q e^{k+1}$ is the homotopy cofibre of the degree q map $S^k \to S^k$.

description of $\pi_*(M(\mathbb{Z}/q, k)$ for $k \geq 3$

- If $k \geq 3$, we give a combinatorial description of $\pi_*(M(\mathbb{Z}/q, k))$ given as the centers of quotient groups of threefold self free product with amalgamation of pure braid groups, which is similar to the description for the homotopy groups of spheres.

- This description is less explicit then the one given in the Theorem for k-spheres, but it leads to combinatorial descriptions of homotopy groups of finite complexes from iterated self free products with amalgamations of pure braid groups.

description of $\pi_*(M(\mathbb{Z}/q, 2)$

- For the homotopy groups of 3-dimensional Moore spaces, there is an explicit combinatorial description that deserves to be described here as it arises in certain divisibility questions concerning braids.

- Let $\xi_1, \ldots, \xi_{n-1}$ be n-strand braid obtained by cabling $A_{1,2}$ as described in step 1 of the construction for the group $Q_{n,k}$.

- It was proved in by [Cohen-Wu, 2004] that the subgroup of P_n generated by $\xi_1, \ldots, \xi_{n-1}$ is a free group of rank $n-1$ with a basis given by $\xi_1, \ldots, \xi_{n-1}$.

The construction: free product with amalgamation

Let $F_{n-1}=\langle \xi_1,\ldots,\xi_{n-1}\rangle \leq P_n$ be the subgroup generated by $\xi_1,\ldots,\xi_{n-1}$. Given an integer q, since $F_{n-1}=\langle \xi_1,\ldots,\xi_{n-1}\rangle$ is free, there is a group homomorphism $\phi_q\colon F_{n-1}\to F_{n-1}$ such that $\phi_q(\xi_j)=\xi_j^q$ for $1\leq j\leq n-1$. Now we form a free product with amalgamation by the push-out diagram

$$\begin{array}{ccc} F_{n-1} & \hookrightarrow & P_n \\ \downarrow{\scriptstyle \phi_q} & & \downarrow \\ F_{n-1} & \longrightarrow & P_n *_{\phi_q} F_{n-1}, \end{array}$$

namely the group $P_n *_{\phi_q} F_{n-1}$, which is the free product given by identifying the subgroup F_{n-1} with the subgroup of F_{n-1} generated by $\xi_1^q,\ldots,\xi_{n-1}^q$ in a canonical way.

The construction: The subgroups

- Let y_j denote the generator ξ_j for F_{n-1} as the second factor in the free product $P_n *_{\phi_q} F_{n-1}$ for $1 \leq j \leq n-1$. Let

$$R_1 = \langle y_1 \rangle^{P_n *_{\phi_q} F_{n-1}}, R_j = \langle y_{j-1} y_j^{-1} \rangle^{P_n *_{\phi_q} F_{n-1}}, R_n = \langle y_{n-1} \rangle^{P_n *_{\phi_q} F_{n-1}}$$

  be the normal closure of $y_1, y_{j-1}y_j^{-1}, y_{n-1}$ in $P_n *_{\phi_q} F_{n-1}$, respectively, for $2 \leq j \leq n-1$.
- Let $R_{s,t} = \langle A_{s,t} \rangle^{P_n *_{\phi_q} F_{n-1}}$ be the normal closure of $A_{s,t}$ in $P_n *_{\phi_q} F_{n-1}$ for $1 \leq s < t \leq n$.

The construction: The symmetric commutator subgroup

- Define the index set $\mathrm{Index}(R_j) = \{j\}$ for $1 \le j \le n$ and
- $\mathrm{Index}(R_{s,t}) = \{s, t\}$ for $1 \le s < t \le n$.

Now define the symmetric commutator subgroup

$$[R_i, R_{s,t} \mid 1 \le i \le n, 1 \le s < t \le n]_S$$

$$= \prod_{\{1,2,\ldots,n\} = \bigcup_{j=1}^{t} \mathrm{Index}(C_j)} [[C_1, C_2], \ldots, C_t],$$

where each $C_j = R_i$ or $R_{s,t}$ for some i or (s, t).

Our Main Theorem 2

- The homotopy group $\pi_n(M(\mathbb{Z}/q, 2))$ is isomorphic to the center of the group

$$(P_n *_{\phi_q} F_{n-1})/[R_i, R_{s,t} \mid 1 \leq i \leq n, 1 \leq s < t \leq n]_S$$

for $n \neq 3$.

- **Note.** For the exceptional case $n = 3$, $\pi_3(M(\mathbb{Z}/q, 2))$ is contained in the center but the equality fails.

THANK YOU!