On G_2 Holonomy Riemannian Metrics on Deformations of Cones over $S^3 \times S^3$

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Let (M^n, g) be Riemannian manifold. Metric g generates Levi-Chivita connection ∇ and parallel transportation operation

$$P_\gamma : T_\gamma(0)M \to T_\gamma(1)M$$

along arbitrary curve $\gamma : [0,1] \to M$.
Fix $p \in M$. Then any loop $\gamma : [0,1] \to M$, $\gamma(0) = \gamma(1) = p$ gives isometric transformation

$$P_\gamma \in O(n) = Iso(T_pM).$$

The set of all such transformation generates Holonomy Group $Hol_p = Hol$ of Riemannain manifold M which is a subgroup of $O(n)$:

$$Hol_p = \{P_\gamma | \gamma : [0,1] \to M, \gamma(0) = \gamma(1) = p\} \subset O(n).$$

It is easy to see that conjugation class of Hol_p in $O(n)$ does not depend of p.
Question: what are the subgroups of $O(n)$ which can be holonomy groups of some Riemannian manifold M^n?

If M is simply connected then $Hol \subset SO(n)$ and we have:

Borel-Lichnerowicz, 1952

Holonomy group of simply connected n-dimensional Riemannian manifold is compact Lie subgroup of $SO(n)$

If M is not simply connected then the analogous result is not true (Wilking, 1999).
The action of P_γ on T_pM is called holonomy representation. The following theorem shows that holonomy group characterizes global structure of complete Riemannian manifold.

de Rham, 1952

Let holonomy representation of holonomy group G of complete Riemannian manifold M can be reduced to product of two representation of groups G_1 and G_2 (in particular G is isomorphic to product $G_1 \times G_2$). Then M is isometric to Cartesian product of two compete Riemannian manifolds M_1 and M_2 with holonomy groups G_1 and G_2 (and with corresponding holonomy representations).

The first important examples of special holonomy groups were shown by Cartan:

E. Cartan, 1923-1926

Let M be symmetric space and G be the group generated by all reflections of M which turn geodesics and $M = G/H$. Then holonomy group of M is isomorphic to H.

Cartan obtained list of all symmetric spaces with corresponding holonomy groups.

Berger, 1955

Let M be simply connected irreducible (in the since of de Rham theorem) Riemannian n-dimensional manifold which is not a symmetric. Then one of the following case holds:

1) $Hol(M) = SO(n)$ - general case,
2) $n = 2m$, where $m \geq 2$, $Hol(M) = U(m) \subset SO(2m)$ - Kähler manifolds,
3) $n = 2m$, where $m \geq 2$, $Hol(M) = SU(m) \subset SO(2m)$ - special Kähler manifolds,
4) $n = 4m$, where $m \geq 2$, $Hol(M) = Sp(m) \subset SO(4m)$ - hyper-Kähler manifolds,
5) $n = 4m$, where $m \geq 2$, $Hol(M) = Sp(m)Sp(1) \subset SO(4m)$ - quaternionic Kähler manifolds,
6) $n = 7$, $Hol(M) = G_2 \subset SO(7)$,
7) $n = 8$, $Hol(M) = Spin(7) \subset SO(8)$.

Riemannian manifold M is Einstein manifold with cosmological constant Λ if it satisfies Einstein equation:

$$R_{ij} = \Lambda g_{ij}.$$

The following facts give method of finding solutions of Einstein equation.

- Riemannian metrics with $Hol(M) = SU(m), Sp(m), G_2, Spin(7)$ satisfy Einstein equation for $\Lambda = 0$ (Ricci-flat case: $R_{ij} = 0$). For G_2, $Spin(7)$ it was proved by Bonan, 1966.
- Riemannian metrics with $Hol(M) = Sp(m)Sp(1)$ satisfy Einstein equation for some $\Lambda \neq 0$.

Ricci-flat manifolds are the most difficult for constructing.

Fact

All known compact simply connected Ricci-flat Einstein manifolds have special holonomy group $SU(m), Sp(m), G_2$ or $Spin(7)$.

Existing of Riemannian manifolds with special holonomy growps:

- holonomy $U(m)$, $Sp(m)Sp(1)$ — classical spaces;
- holonomy $Sp(m)$, $SU(m)$ — Calabi, 1978;
- holonomy G_2, $Spin(7)$ — Bryant, Salamon, 1987 (first examples), 1989 (first complete examples); Joyce, 1996 (first compact examples).

Further development in G_2 holonomy: Kovalev, 2003 (compact examle), noncompact examples by mathematical physicists.

- Construction of Joyce: to consider flat orbifold T^7/Γ, discrete group Γ act by isometries with disjoint copies of T^3 as fixed points set; then make blowup of each copy.
- Construction of Kovalev: to glue two copies of product of S^1 and special $SU(3)$ holonomy noncompact space.

In each construction we need to make some «surgery» of noncompact Riemannian manifolds with holonomy group $H \subset G_2$ and then to disturb resulting Riemannian metric to produce special holonomy.

Lie group G_2 can be defined as group of all automorphisms of octonions $\mathbb{C}a$ (the Cayley numbers). Another representation of G_2 can be considered if we produce skew symmetric three-form apply standard scalar product in $\mathbb{C}a$ to multiplication operation of octonions.

Let $\{e^i\}, i = 0, 1, 2, \dots, 7$ be orthonormal coframe in $\mathbb{R}^7$. Putting $e^{ijk} = e^i \wedge e^j \wedge e^k$ consider three-form Φ_0 and Hodge star dual four-form $\star\Phi_0$ on $\mathbb{R}^7$:

$$\Phi_0 = e^{123} + e^{147} + e^{165} + e^{246} + e^{257} + e^{354} + e^{367},$$

$$\star\Phi_0 = e^{4567} + e^{2356} + e^{2374} + e^{1357} + e^{1346} + e^{1276} + e^{1245},$$

Then $G_2 = \{A \in GL(7) | A^*\Phi_0 = \Phi_0, A^*(\star\Phi_0) = (\star\Phi_0)\}$.

G_2 structure

Differential three-form Φ on oriented 7-dimensional manifold M defines G_2 structure if in a neighborhood of every point $p \in M$ there exits orientation preserving isometry $\phi_p : T_pM \to \mathbb{R}^7$ such that $\phi_p^* \Phi_0 = \Phi_p$.

Moreover it possible to show that form Φ defines unique Riemanian metric $g = g_\Phi$ such that $g_\Phi(v, w) = \langle \phi_p v, \phi_p w \rangle$ for $v, w \in T_pM$.
If differential form Φ is parallel with respect to Levi-Chivita connection (that is $\nabla \Phi = 0$) then holonomy group of Riemannian manifold (M, g) is a subgroup of the group G_2. Parallelism condition is very restrictive and bring to over-defined differential equations.
Gray proved that form Φ is parallel if and only if it is closed and coclosed:

$$d\Phi = 0, d \star \Phi = 0.$$

The last two equations are correct in some geometrical cases.

G_2 structure on the cone over $S^3 \times S^3$

$S^3 = SU(2) = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, |a|^2 + |b|^2 = 1\}$ with standard biinvariant metric.

$$\langle X, Y \rangle = -\mathrm{tr}\,(XY),$$

where $X, Y \in \mathbf{su}(2)$. On the S^3 consider three Killing vector fields:

$$\xi^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ \xi^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \xi^3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$[\xi^i, \xi^{i+1}] = 2\xi^{i+2},$$

where indexes $i = 1, 2, 3$ are reduced $mod 3$. Let η_1, η_2, η_3 be dual co-frame, $\eta_i(\xi^j) = \delta_i^j$. Then

$$d\eta_i = -2\eta_{i+1} \wedge \eta_{i+2}.$$

G_2 structure on the cone over $S^3 \times S^3$

Let $M = S^3 \times S^3$ then there exist six Killing vector fields ξ^i, $\tilde{\xi}^i$, $i = 1, 2, 3$ on M tangent to first and second factor, respectively, and there exist six dual one-forms η_i, $\tilde{\eta}_i$. Consider cone $\overline{M} = \mathbb{R}_+ \times M$ with a metric

$$d\bar{s}^2 = dt^2 + \sum_{i=1}^{3} A_i(t)^2 \left(\eta_i + \tilde{\eta}_i\right)^2 + \sum_{i=1}^{3} B_i(t)^2 \left(\eta_i - \tilde{\eta}_i\right)^2,$$

where $A_i(t)$ and $B_i(t)$ are some positive functions which define deformation of standard cone metric.
If we consider orthonormal co-frame

$$\begin{array}{ll} e^1 = A_1\left(\eta_1 + \tilde{\eta}_1\right), & e^4 = B_1\left(\eta_1 - \tilde{\eta}_1\right), \\ e^2 = A_2\left(\eta_2 + \tilde{\eta}_2\right), & e^5 = B_2\left(\eta_2 - \tilde{\eta}_2\right), \\ e^3 = A_3\left(\eta_3 + \tilde{\eta}_3\right), & e^6 = B_3\left(\eta_3 - \tilde{\eta}_3\right), \\ e^7 = dt, & \end{array}$$

then we can define the following three-form:

G_2 structure on the cone over $S^3 \times S^3$

$$\Phi = e^{564} + e^{527} + e^{513} + e^{621} + e^{637} + e^{432} + e^{417},$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$. Form Φ defines G_2 structure on $\overline{M}$ and we have first oder equations which guarantee parallelism of this form:

$$d\Phi = 0, d \star \Phi = 0. \quad (1)$$

We consider particular case $A_2 = A_3$, $B_2 = B_3$.

Equations (1) can be write down explicitly:

$$
\begin{aligned}
\frac{dA_1}{dt} &= \frac{1}{2}\left(\frac{A_1^2}{A_2^2} - \frac{A_1^2}{B_2^2}\right)\\
\frac{dA_2}{dt} &= \frac{1}{2}\left(\frac{B_2^2-A_2^2+B_1^2}{B_1B_2} - \frac{A_1}{A_2}\right)\\
\frac{dB_1}{dt} &= \frac{A_2^2+B_2^2-B_1^2}{A_2B_2}\\
\frac{dB_2}{dt} &= \frac{1}{2}\left(\frac{A_2^2-B_2^2+B_1^2}{A_2B_1} + \frac{A_1}{B_2}\right)
\end{aligned}
\tag{2}
$$

Now if we want to obtain smooth Riemannian manifold with regular metric we have to resolve cone singularity. We have two ways to produce this.

- Type 1:

$$\begin{aligned} A_1(0) = A_2(0) = 0, \\ B_1(0) = B_2(0) \neq 0, \\ |A_1'(0)| = |A_2'(0)| = \tfrac{1}{2}, \\ B_1'(0) = B_2'(0) = 0. \end{aligned}$$

  In this case $\overline{M}$ is diffeomorphic to $S^3 \times \mathbb{R}^4$.

- Type 2:

$$\begin{aligned} &B_1(0) = 0, |B_1'(0)| = 2, \\ &A_2(0) = B_2(0) \neq 0, A_2'(0) = -B_2'(0), \\ &A_1(0) \neq 0, A_1'(0) = 0. \end{aligned}$$

  In this case we obtain manifold $\overline{M}$ which is diffeomorphic to $S^3 \times H^4$, where H^4 is the forth tensor power of canonical complex line bundle H over two-sphere S^2.

I will discuss in my talk only Type 2 of resolving singularity.

ALC metrics

Riemannian metric $d\bar{s}^2$ is called asymptotically locally conical off there exist affine over t functions $\tilde{A}_i(t)$, $\tilde{B}_i(t)$ such that

$$\left|1 - \frac{A_i}{\tilde{A}_i}\right| \to 0, \ \left|1 - \frac{B_i}{\tilde{B}_i}\right| \to 0, \text{ при } \ t \to \infty$$

The metric which is defined by such functions $\tilde{A}_i(t)$, $\tilde{B}_i(t)$ is called locally conical.

Let $R(t) = (A_1(t), A_2(t), B_1(t), B_2(t)) \in \mathbb{R}^4$ and $V : \mathbb{R}^4 \to \mathbb{R}^4$ is the right side of system (2). Then our system of differential equations has form

$$\frac{dR}{dt} = V(R).$$

We see that V is invariant with respect to homotheties of $\mathbb{R}^4$ so we can put $R(t) = f(t)S(t)$, where

$$|S(t)| = 1, f(t) = |R(t)|,$$
$$S(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t)).$$

In such way our system (2) splits into «radial» and «tangential» parts:

$$\frac{dS}{du} = V(S) - \langle V(S), S\rangle S = W(S), \qquad (3)$$

$$\tfrac{1}{f}\tfrac{df}{du} = \langle V(S), S\rangle,$$
$$dt = fdu.$$

Consequently we need firstly to solve autonomous system (3) on three-sphere $S^3 = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)| \sum_{i=1}^4 \alpha_i^2 = 1\}$ and then solutions of (2) can be found by integrating last two equations.

Лемма

Stationary solutions of (3) on S^3 correspond to zeros of vector field W in $S^3 \subset \mathbb{R}^4$ with respect to symmetries of our system:

$$\left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right), \quad \left(0, \frac{\sqrt{3}}{\sqrt{10}}, \frac{\sqrt{2}}{\sqrt{5}}, \frac{\sqrt{3}}{\sqrt{10}}\right).$$

Lemma

Stationary solutions of (3) correspond to locally conical metrics on $\overline{M}$ and trajectories of (3) which converges to stationary solutions correspond to asymptotically locally conical metrics on $\overline{M}$.

Solutions of system (2) with initial data of Type 2 correspond to solutions of autonomous system (3) with initial point $S_0 = (\mu, \lambda, 0, \lambda)$, where $2\lambda^2 + \mu^2 = 1$ (with respect to symmetries of our system).
Remark that right hand of system (2) has singularity at point S_0, so classical Cauchy problem does not work here.

Lemma

For any above considered point $S_0 = (\mu, \lambda, 0, \lambda)$ there exists unique smooth trajectory of system (3) escaping point S_0.

Lemma

The trajectory of system (3) defined by initial point $S_0 = (\mu, \lambda, 0, \lambda)$, $\lambda, \mu > 0$, $2\lambda^2 + \mu^2 = 1$, converges to stationary point $S_\infty = \left(0, \frac{\sqrt{3}}{\sqrt{10}}, \frac{\sqrt{2}}{\sqrt{5}}, \frac{\sqrt{3}}{\sqrt{10}}\right)$ when $u \to \infty$.

So we obtain the following:

Theorem (B.-Bogoyavlenskaya)

There exist one-parameter family of pairwise nonhomothetic complete Riemannian metrics of form $d\bar{s}^2$ with holonomy group G_2 over $H^4 \times S^3$. More over these metrics can be parametrize by initial data set $(A_1(0), A_2(0), B_1(0), B_2(0)) = (\mu, \lambda, 0, \lambda)$, where $\lambda, \mu > 0$ and $\mu^2 + \lambda^2 = 1$. When $t \to \infty$ the metrics of this family are asymptotically isometric to Cartesian product $S^1 \times C(S^2 \times S^3)$, where $C(S^2 \times S^3)$ is cone over spheres product.

THANK YOU FOR YOUR ATTENTION!