Amoebas, Ronkin function and Monge-Ampère measures of algebraic curves with marked points.

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June 21, 2013

The moduli space

$$\mathcal{S} := \{\Gamma, d\zeta_1, d\zeta_2\} \quad (1)$$

of smooth algebraic curves with a pair of *real (imaginary)* normalized differentials are central for many aspects of the algebraic-geometrical integration theory. They provide a unifying framework for

- the Hamiltonian theory of soliton equations,
- the Whitham equations,
- WDVV equations,
- Siberg-Witten solution of $N = 2$ SUSY gauge models.
- Laplacian growth problem

Real normalized differentials

By definition a real normalized meromorphic differential is a differential whose periods over any cycle on the curve are real.

The universality of this notion is that:

Lemma

For any fixed singular parts of poles with pure imaginary residues, there exists a unique meromorphic differential $d\zeta$, having prescribed singular part at p_α and such that all its periods on Γ are real, i.e.

$$\mathrm{Im}\left(\oint_c d\zeta\right) = 0, \ \forall\, c \in H^1(\Gamma, \mathbb{Z}).$$

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Motivation

The imaginary (real) normalized differentials of the third kind per se are not new. They were probably known to Maxwell (the real part of such differential is a single valued harmonic function on Γ equal to the potential of electromagnetic filed on Γ created by charged particles at the marked points); they were used in the, so-called, light-cone string theory, and played a crucial role in joint works of S. Novikov and the author on Laurent-Fourier theory on Riemann surfaces and on operator quantization of bosonic strings.

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The Whitham equations

The Whitham method is a generalization to the case of partial differential equations of the classical Bogolyubov-Krylov method of averaging. It is applicable to nonlinear equations which have a family of exact solutions of the form $u_0(Ux + Vy + Wt + Z|I)$. Here $u_0(z_1, \ldots, z_g|I)$ is a periodic function of the variables z_i; $U,\ V,\ W$ are vectors which like u_0 itself, depend on the parameters $I = (I_1, \ldots, I_N)$

These exact solutions can be used as a leading term for the construction of asymptotic solutions

$$u(x, y, t) = u_0(\varepsilon^{-1}S(X, Y, T) + Z(X, Y, T)|I(X, Y, T))+$$

$$+\varepsilon u_1(x, y, t) + \varepsilon^2 u_2(x, y, t) + \cdots,$$

where I depend on *slow* variables $X = \varepsilon x$, $Y = \varepsilon y$, $T = \varepsilon t$ and ε is a small parameter.

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If the vector-valued function $S(X, Y, T)$ is defined by the equations

$$\partial_X S = U(X, Y, T),\ \partial_Y S = V(X, Y, T),\ \partial_T S = W(X, Y, T),$$

then the leading term of the series satisfies the original equation up to first order one in ε.

All other terms of the asymptotic series are obtained from non-homogeneous linear equations

$$\mathcal{L}[u_0]u_i = F_i(u_0, u_1, \ldots, u_{i-1})$$

- The linear operator $\mathcal{L}[u_0]$ is *integrable (!)*

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In general, the asymptotic series becomes unreliable on scales of the original variables x and t of order ε^{-1}. In order to have a reliable approximation, one needs to require a special dependence of the parameters $I(X, Y, T)$. The equations which describe this drift are in general called *Whitham equations*, although there is no systematic scheme to obtain them.

The Whitham equation for the finite gap solutions of the KdV equation were *postulated* by Flashka, Forrest, McLaughlin (1979):

$$\partial_T \bar{P}_n = \partial_X \bar{J}_n,$$

where $P_n = P_n(u, u_x, \ldots)$ are densities of the KdV integrals and $Q_n = Q_n(u, u_x, \ldots)$ are the corresponding currents, i.e. $\partial_t P_n = \partial_x J_n$.

- Integrability of these equation via generalized hodograth transform was proved by Tsarev.
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The universal Whitham hierarchy

FFM approach is not applicable for (2+1) integrable systems (KP,2D Toda) (integrals are not local). Equations of the universal Whitham hierarchy are *necessary* condition for the first term u_1 of the asymptotic series to be *bounded* for all T_i.

Recall that u_1 is a solution of a non-homogeneous linear equation $\mathcal{L}[u_0]u_1 = F_1[u_0]$, with (quasi) periodic coefficients. It is bounded if and only if the non-homogeneous term is orthogonal to a kernel of the formal adjoint linear operator $\mathcal{L}^*$, i.e. $<< \psi^+ F_1[u_0]\psi >>= 0$.

It turned out (highly not trivial derivation) that the later equation is equivalent to the equations:

$$\partial_i \Omega_j - \partial_j \Omega_j + \{\Omega_i, \Omega_j\} = 0$$

where $\Omega_j = \Omega_j(p, T)$ are certain abelian integrals on $\Gamma = \Gamma(T)$, and $\{f, g\} := \partial_p f \partial_x g - \partial_p g \partial_x = g$.

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The Whitham tau-function

The Whitham equations can be transformed to FFM form.

Let E(p,T) be a solution

$$\partial_i E(p,T) = \{\Omega_i, E\}$$

Then we can invert the equation $E(p,T) = E$ and define $p = p(E,T)$. After that $\Omega_i = \Omega_i(E,T)$ and the Whitham equations take the form

$$\partial_i \Omega_j(E,T) = \partial_j \Omega_i(E,T) \Rightarrow \partial_i S(E,T) = \Omega_i$$

The generating differential dS can be written as

$$dS = QdE, \quad \{Q,E\} = 1$$

The Whitham tau-function:

$$\mathcal{F} = \log \tau = \int_\Gamma \bar{d}S \wedge dS$$

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Recent results

Recently constructions with real normalized differentials have found new applications to the study of geometry of moduli spaces of curves with punctures. Among results are:

- a new proof the Diaz' theorem on dimension of complete subvarieties of $\mathcal{M}_g$, and the proof of the vanishing of a certain tautological classes in $\mathcal{M}_{g,n}$ (Grushevsky-Kr, 2011).
- The proof of Arbarello's conjecture: the statement that any complete complex subvariety of $\mathcal{M}_g$ of dimension $g - n$ intersects the locus W_n of curves that have a Weierstrass point of weight at most n (i.e. such admitting a degree n to $\mathbb{P}^1$ fully ramified at this point) (Kr 2011).

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The main goal of the talk is to present new constructions and notions associated with algebraic curves with a pair of real normalized differentials. They generalize concepts of amoebas, Ronkin functions associated with plane curves. The later have played crucial role in the recent progress in the theory of real algebraic curves (Mikhalkin) and in the theory of limiting shapes of random surfaces (Kenyon, Okounkov, Sheffield).

Amoebas and the Ronkin function of plane curves

- The amoeba $\mathcal{A}_f$ of a holomorphic function $f : (\mathbb{C}^*)^n \to \mathbb{C}$ (where $\mathbb{C}^* = \mathbb{C} \setminus 0$) is, by definition, the image in $\mathbb{R}^n$ of the zero locus of f under the mapping $\mathrm{Log} : (\mathrm{z}_1, \ldots, \mathrm{z}_\mathrm{n}) \to (\log |\mathrm{z}_1|, \ldots, \log |\mathrm{z}_\mathrm{n}|)$. The terminology was introduced by Gelfand, Kapranov and Zelevinsky and reflects the geometric shape of typical amoebas, that is a semianalytic closed subset of $\mathbb{R}^n$ with tentacle-like asymptotes going off to infinity.
- All connected components of the amoeba complement $\mathcal{A}_f^c = \mathbb{R}^n \setminus \mathcal{A}_f$, are convex. When f is a Laurent polynomial, then there is a natural injective map from the set of connected components of $\mathcal{A}_f^c$ to the set of integer points of Newton polytop Δ_f of f (Forsberg, Passare, Tsikh).

This injective map is defined by the gradient $\nabla\mathcal{R}_f$ of, the so-called Ronkin function:

$$\mathcal{R}_f(x) = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(\mathrm{x})} \frac{\log|f(z_1,\dots,z_n)|\,dz_1\cdots dz_n}{z_1\cdots z_n} \qquad (2)$$

The Ronkin function $\mathcal{R}_f(x)$ is convex.

Recall, that each convex function u defines the associated Monge-Ampère measure Mu. If u is a smooth convex function on $\mathbb{R}^n$, then $Mu = \det(\mathrm{Hess}(u))v$, where $\mathrm{Hess}(u)$ is the Hessian matrix and v denotes the Lebesque measure on $\mathbb{R}^n$. If u is convex but not necessary smooth ∇u can still be defined as a multifunction, and the Monge-Ampère measure of u is defined as in the smooth case.

Since $\mathcal{R}_f(x)$ is affine linear in a connected component of $\mathcal{A}_f^c$, the support of the associated Monge-Ampère measure $\mu := M\mathcal{R}_f$ is in $\mathcal{A}_f$.

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Extremal properties of Harnack curves

For $n = 2$ the area of amoeba is always bounded (Passare, H. Rullgard):

$$\mathrm{Area}(\mathcal{A}_f) \leq \pi^2 \mathrm{Area}(\Delta_f) \quad (3)$$

Theorem (Mikhalkin, Rullgard)

Suppose that $\mathrm{Area}(\Delta) > 0$. Then the following conditions are equivalent.
1.

$$\mathrm{Area}(\mathcal{A}) = \pi^2 \mathrm{Area}(\Delta)$$

2. The map Log is at most 2:1 and the curve $f(z_1, z_2) = 0$ is real up to multiplication by a constant.
3. The curve $f(z_1, z_2) = 0$ is real up to multiplication by a constant and its real part is a (possibly singular) simple Harnack curve for the Newton polygon Δ.

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Amoebas of algebraic curves with punctures

Let us fix a pair of n-tuples of real numbers $a_j = \{a_{\alpha,j}\}$, $j = 1, 2$. Then for each smooth algebraic curve Γ with n marked points p_α we have two associated imaginary normalized differential $d\zeta_j$ with $\mathrm{Res}_{p_\alpha} d\zeta_j = a_{\alpha,j}$.

Definition

The amoeba $\mathcal{A}_\mathcal{S} \subset \mathbb{R}^2$ associated with the data $\mathcal{S} = \{\Gamma, p_\alpha, a_{\alpha,j}\}$ is the image of the map $\chi : \Gamma_0 \to \mathbb{R}^2$, $\chi(p) = (x_1(p), x_2(p))$, where $x_j(p)$ are harmonic functions on Γ_0 defined by the imaginary normalized meromorphic differentials $d\zeta_j$, i.e.

$$x_a(p) = \mathrm{Re}\left(\int^p d\zeta_a + c\right) \tag{4}$$

The following result shows that the geometric shape of generalized amoebas is the same as that of amoebas of plane curves:

Theorem

All connected components of the complement $\mathcal{A}_S^c$ are convex. There are n unbounded components separated by tentacle-like asymptotes of the amoeba.

Open problem: upper bound on the number of *connected components of $\mathcal{A}_S^c$*

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Critical points of the amoeba map

Lemma

The locus $\gamma \subset \Gamma$ of critical points of the amoeba map is a union of the locus γ_0, where the function $R(p) = \frac{d\zeta_2}{d\zeta_1}$ is real and the finite set (possibly empty) of the common zeros of the differentials $d\zeta_j$.

Let p be a regular point of the map χ. Then in the neighborhood of p we can write $\zeta_1 = x_1 + iy_1(x_1, x_2), \;\; \zeta_2 = x_2 + iy_2(x_1, x_2)$. Using the fact that the ratio R is a meromorphic function on Γ it is easy to prove the equations:

$$\partial_1 y_2 = -\frac{1}{\mathrm{Im}\, R}\,, \quad \partial_2 y_2 = -\partial_1 y_1 = \frac{\mathrm{Re}\, R}{\mathrm{Im}\, R}, \quad \partial_2 y_1 = \frac{|R|^2}{\mathrm{Im}\, R} \tag{5}$$

The later imply

$$4dx_1 \wedge dx_2 = -2i(\mathrm{Im} R)\, d\zeta_1 \wedge d\bar{\zeta}_1 = -2i \mathrm{Im} R^{-1}\, d\zeta_2 \wedge d\bar{\zeta}_2\,. \tag{6}$$

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Generalized Ronkin function

For each $x = (x_1, x_2) \in \mathbb{R}^2$ let us define the closed subset $\Gamma_x \subset \Gamma$:

$$\Gamma_x := \{p \in \Gamma_x | \, x_1(p) \leq x_1, \; x_2(p) \leq x_2\} . \qquad (7)$$

Definition

The generalized Ronkin function $\rho_{\mathcal{S}}$, associated with a smooth algebraic curve with two imaginary normalized differentials, i.e. associated with the data $\mathcal{S}$ is given by:

$$\rho_{\mathcal{S}}(x) = \frac{1}{8\pi i} \int \int_{\Gamma_x} \mathrm{sgn}(\mathrm{Im} R)(d\zeta_1 \wedge d\bar{\zeta}_2 - d\zeta_1 \wedge d\bar{\zeta}_2). \qquad (8)$$

Theorem

The generalized Ronkin function $\rho_{\mathcal{S}}(x)$ given by (8) is a convex function on $\mathbb{R}^2$. It is affine linear in each connected component of $\mathcal{A}_{\mathcal{S}}^c$. It is smooth at the regular points of the amoeba, i.e. outside of the set F of critical values of χ, and furthermore at $x \in \mathcal{A}_{\mathcal{S}} \setminus F$

$$\operatorname{Hess} \rho_{\mathcal{S}}(x) = \frac{1}{2\pi} \sum_{p \in \chi^{-1}(x)} \frac{1}{|\operatorname{Im} R(p)|} \begin{pmatrix} 1 & \operatorname{Re} R(p) \\ \operatorname{Re} R(p) & |R(p)|^2 \end{pmatrix}. \tag{9}$$

Corollary

The area of the amoeba $\mathcal{A}_{\mathcal{S}}$ is not greater than π^2 times the area of the polygon $\Delta_{\mathcal{S}}$,

$$\operatorname{Area}(\mathcal{A}_{\mathcal{S}}) \leq \pi^2 \operatorname{Area}(\Delta_{\mathcal{S}}) \tag{10}$$

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Newton polygon

The image of the gradient map $\nabla\rho_{\mathcal{S}}$ is the polygon $\Delta_{\mathcal{S}}$ that is a convex hull of the set of points $(v_{\alpha,1}, v_{\alpha,2}) \in \mathbb{R}^2$, which are the image under $\nabla\mathcal{R}_{\mathcal{S}}$ of the unbounded components of $\mathcal{A}_{\mathcal{S}}$.
If the vectors a_α and a_β are not collinear i.e., $\varepsilon_{\alpha,\beta} := a_{\beta,1}a_{\alpha,2} - a_{\beta,2}a_{\alpha,1} \neq 0$, then the vertices of $\Delta_{\mathcal{S}}$ equal

$$\pm a_{\alpha,2} > 0 \Rightarrow v_{\alpha,1} = \pm \sum_{\beta \in I_\alpha^\pm} a_{\beta,2},\, I_\alpha^\pm := \{\beta \in I_\alpha^\pm \,|\, \pm a_{\beta,2} > 0, \pm\varepsilon_{\alpha,\beta} > 0\} \tag{11}$$

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The formulae (11) and (12) continuously extended to the general case when some of vectors a_α might be collinear.

Amoebas of M-curves

A smooth genus g algebraic curve Γ with antiholomorphic involution $\tau:\Gamma\to\Gamma$ is M-curve if τ has $g+1$ (maximal possible number) of fixed ovals.

Definition

The set of data $(\Gamma, p_\alpha, a_{\alpha,j})$ is called Harnack: (i) Γ is a M-curve; (ii) the marked points p_α are on one of the fixed ovals $A_0,\ldots A_g$ of the antiinvolution τ, say $p_\alpha\in A_0$; (iii) The cyclic order p_α along the cycle A_0 coincides with counterclockwise order of the vertices of the polygon $\Delta_{\mathcal{S}}$.

The antiinvolution τ on an M-curve is always of the separating type, i.e. $\Gamma=\Gamma^+\cup\Gamma^-$, $\tau:\Gamma^+\to\Gamma^-$, $\Gamma^+\cap\Gamma^-=\cup_{j=0}^{g}A_j$,

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Lemma

The amoeba map defined by imaginary normalized differentials $d\zeta_j$ corresponding to Harnack data restricted to $\Gamma^+ \subset \Gamma$ is a diffeomorphism of $\Gamma^+ \setminus \{p_\alpha\}$ with $\mathcal{A}_S$.

Theorem

The following conditions are equivalent.

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$$\mathrm{Area}(\mathcal{A}_S) = \pi^2 \mathrm{Area}(\Delta_S)$$

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Spectral theory of $2D$ difference operators

In the framework of the spectral theory of two-dimensional difference operators

$$(L\psi)_{n,m} = \psi_{n+1,\,m} + \psi_{n,\,m+1} + u_{n,\,m}\psi_{n,\,m} \tag{13}$$

the plane curves of degree d arise as the spectral curves of d-periodic operators, $u_{n,\,m} = u_{n+d,\,m} = u_{n,\,m+d}$.

The points of the spectral curve parameterize Bloch solution of the equation $L\psi = 0$, i.e.

$$\psi_{n+d,\,m} = z_1\psi_{n,\,m}, \quad \psi_{n,\,m+d} = z_2\psi_{n,\,m} \tag{14}$$

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Quasi-periodic integrable operators

Theorem (Kr 1985)

Let p be a point of Γ. Then for any g-dimensional vector Z the function

$$\psi_{m,n} = \frac{\theta(A(p) + mU + nV + Z)}{\theta(mU + nV + Z)}\, e^{m\Omega_1(p)+n\Omega_2(p)}, \tag{16}$$

where $U = A_(p_1) - A(p_3),\ V = A(p_2) - A(p_3)$, satisfies the difference equation

$$\psi_{m,n+1} = \psi_{m+1,n} + u_{m,n}\psi_{m,n} \tag{17}$$

with

$$u_{m,n} = \frac{\tau_{m+1,n+1}\tau_{m,n}}{\tau_{m+1,n}\tau_{m,n+1}} \tag{18}$$

where

$$\tau_{m,n} = c_1^m c_2^n\, \theta(mU + nV + Z) \tag{19}$$

- In general the coefficient of the difference equation is a complex quasiperiodic function (possibly singular) of the variables (m, n).
- If the vectors U and V are d-periodic points of the Jacobain, then $u_{m,n}$ are d-periodic. (The later condition is equivalent to the condition that on Γ there exist functions z_1 and z_2 having pole of order d at p_3 and zeros of order d at p_1 and p_2, respectively).
- If on Γ there is an antiholomorphic involution and if the marked points p_j are fixed by the involution, then u is real for real Z (but still might be singular).
- If Γ is M-curve, and the marked points are on one of the fixed ovals of the involution, then for all real Z the coefficients of the difference equation are non-singular for all (n, m).

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Under the gauge transformation $\Psi_{m,n} = (-1)^n \tau_{m,n} \psi_{m,n}$, equation (17) takes the following form

$$\tau_{m+1,n}\Psi_{m,n+1} + \tau_{n,n+1}\Psi_{m+1,n} + \tau_{m+1,n+1}\Psi_{m,n} = 0 \qquad (20)$$

Corollary

The coefficients of equation (20) are real positive numbers if and only if the corresponding set of algebraic-geometrical data is a Harnack set.

In the most general form the amoeba map $\chi : \Gamma_0 \to \mathbb{R}^2$ of a smooth algebraic curve Γ with marked points p_α can be defined for any pair of harmonic function $x_j(p)$ on $\Gamma_0 = \Gamma \setminus \{p_\alpha\}$.

Let $x(p)$ be a harmonic function, then locally there exists a unique up to an additive constant conjugate harmonic function $y(p)$. Hence, $x(p)$ uniquely defines the differential $d\zeta = dx + idy$, which by construction is *imaginary normalized* holomorphic differential on Γ_0. One can specify asymptotic behavior of $x(p)$ near the marked point by the requirement that $d\zeta$ is meromorphic on Γ and has a fixed singular part at the marked points.

Amoebas versus mushrooms and other creatures.

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Example 1. Let $d\zeta_1, d\zeta_2$ be imaginary normalized differentials on Γ having pole at a marked point p_0 of the form $(z^{-2} + O(1))dz$ and $i(z^{-2} + O(1))dz$. Notice, that a different choice of the local coordinate z corresponds to a linear transformation of the pair of differentials.

Example 2. KP2 hierarchy

The following example of a pair of imaginary normalized differentials having poles of the form $d\zeta_1 = (-z^{-2} + O(1))$ and $d\zeta_2 = (-2z^{-3} + O(1))dz)$ is connected with the spectral theory of nonstationary Shrödinger equation.

- The corresponding map χ is of degree zero.
- There is one infinite connected component of a complement of the image of χ. It is bounded by a curve which asymptotically is the parabola $x_2 = x_1^2$.
- For the case of *M*-curves and one puncture fixed under anti-involution τ the map χ is $2:1$ outside of images of fixed ovals, which are boundaries of compact connected components of $\mathcal{A}^c$. The gradient map $\nabla\rho$ restricted to Γ^+ is one-to-one with the upper half plane of $\mathbb{R}^2$ with g points removed.

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