# Universal formulae in Lie groups and Chern-Simons theory

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# Plan

- **Lie geography: Vogel's map**

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References


**R.L. Mkrtchyan, A.N. Sergeev, A.P.V.** JMP, 53, 102106 (2012).

**R.L. Mkrtchyan, A.P.V.** JHEP 08 (2012) 153.

**R.L. Mkrtchyan, A.P.V.** arXiv:1304.3031 (2013).

# Vogel's plane

**Vogel's plane** is a quotient $\mathbb{P}^2/S_3$ of the projective plane with homogeneous coordinates $\alpha, \beta$ and $\gamma$. It is a moduli space of a tensor category, which is meant to be a model of the **universal simple Lie algebra [Vogel, 1999].**

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Table: Vogel's parameters for simple Lie algebras

| Type | Lie algebra | $\alpha$ | $\beta$ | $\gamma$ | $t = h^\vee$ |
|---|---|---|---|---|---|
| $A_n$ | $\mathfrak{sl}_{n+1}$ | $-2$ | $2$ | $(n+1)$ | $n+1$ |
| $B_n$ | $\mathfrak{so}_{2n+1}$ | $-2$ | $4$ | $2n-3$ | $2n-1$ |
| $C_n$ | $\mathfrak{sp}_{2n}$ | $-2$ | $1$ | $n+2$ | $n+1$ |
| $D_n$ | $\mathfrak{so}_{2n}$ | $-2$ | $4$ | $2n-4$ | $2n-2$ |
| $G_2$ | $\mathfrak{g}_2$ | $-2$ | $10/3$ | $8/3$ | $4$ |
| $F_4$ | $\mathfrak{f}_4$ | $-2$ | $5$ | $6$ | $9$ |
| $E_6$ | $\mathfrak{e}_6$ | $-2$ | $6$ | $8$ | $12$ |
| $E_7$ | $\mathfrak{e}_7$ | $-2$ | $8$ | $12$ | $18$ |
| $E_8$ | $\mathfrak{e}_8$ | $-2$ | $12$ | $20$ | $30$ |

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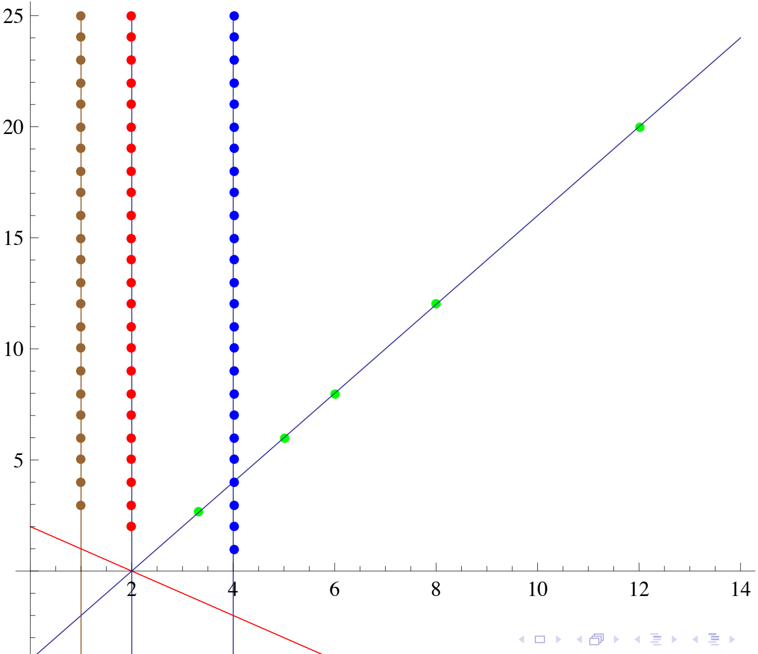
Motivations: Knot theory (**Vassiliev** invariants), **Deligne**'s study of exceptional Lie algebras

# Vogel's map



# Universal formulae for dimension

**Vogel, 1999**:

$$\dim \mathfrak{g} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma}$$

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In the decomposition

$$S^2\mathfrak{g} = \mathbb{C} \oplus Y(\alpha) \oplus Y(\beta) \oplus Y(\gamma)$$

the Casimir values $C_2$ are respectively $4t - 2\alpha, 4t - 2\beta, 4t - 2\gamma$ (which can be used as a definition of Vogel's parameters) and

$$dim\, Y_2(\alpha) = -\frac{(3\,\alpha - 2\,t)\;(\beta - 2\,t)\;(\gamma - 2\,t)\;t\;(\beta + t)\;(\gamma + t)}{\alpha^2\;(\alpha - \beta)\;\beta\;(\alpha - \gamma)\;\gamma}.$$

# Casimir operators

For any simple complex Lie algebra $\mathfrak{g}$ define the Casimir operators $C_p$ as the following elements of the centre of the corresponding universal enveloping algebra $U\mathfrak{g}$ as

$$C_p = g_{\mu_1 \dots \mu_p} X^{\mu_1} \dots X^{\mu_p}, \ p = 0, 1, 2, \dots$$

where $X^\mu$ are the generators of $\mathfrak{g}$,

$$g^{\mu_1 \cdots \mu_n} = Tr(\hat{X}^{\mu_1} \dots \hat{X}^{\mu_n}),$$

where the trace is taken in the adjoint representation of $\mathfrak{g}$ and the indices are lowered using the Cartan-Killing metric

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Consider the values of $C_p$ on the adjoint representation. We claim that they can be expressed rationally in the terms of the universal Vogel's parameters $\alpha, \beta, \gamma$.

**Theorem (Mkrtchyan, Sergeev, V, 2011)**

The generating function $C(z) = \sum_{p=1}^{\infty} C_p z^p$ has the form

$$C(z) = z^2 \frac{96t^3 + 168t^3 z + 6(14t^3 + tt_2 - t_3)z^2 + (13t^3 + 3tt_2 - 4t_3)z^3}{6(2t + \alpha z)(2t + \beta z)(2t + \gamma z)(2 + z)(1 + z)}$$

where

$$t_2 = \alpha^2 + \beta^2 + \gamma^2, \; t_3 = \alpha^3 + \beta^3 + \gamma^3.$$

# Universal formula for Casimirs

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In particular, the first few Casimirs are

$$C_1 = 0, \;\; C_2 = 1, \;\; C_3 = -\frac{1}{4}, \;\; C_4 = \frac{3tt_2 - t_3}{16t^3}.$$

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Proof uses the results of **Okubo (1977)** and **Landsberg-Manivel (2004)**.

# Calculations: the quartic Casimir

**Cvitanovic**: for the orthogonal group $SO(n)$

$$C_4 = \frac{(n-2)(n^3 - 9n^2 + 54n - 104)}{8}.$$

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The universal parameters of $SO(n)$ are

$$\alpha = -2,\ \beta = 4,\ \gamma = n - 4;\ t = n - 2$$

Assume that the numerator is a symmetric polynomial of $\alpha, \beta, \gamma$ :

$$n^3 - 9n^2 + 54n - 104 = At^3 + Btt_2 + Ct_3,$$

$$t_2 = \alpha^2 + \beta^2 + \gamma^2 = n^2 - 8n + 36,\ t_3 = \alpha^3 + \beta^3 + \gamma^3 = n^3 - 12n^2 + 48n - 8.$$

This gives 4 relations on three constants $A, B$ and $C$, which in general should not be consistent.

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In our case however we do have a solution: $A = 0,\ B = 3/2,\ C = -1/2$ which leads to our previous formula.

# Chern–Simons theory

**Chern, Simons (1974), A.S. Schwarz (1978), S.P. Novikov (1981), Deser, Jackiw, Templeton (1983), Witten (1989)**

Let $M$ be 3-dimensional manifold, $G$ is a simply connected simple compact Lie group with Lie algebra $\mathfrak{g}$.

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**Chern-Simons** action is

$$S(A) = \frac{\kappa}{4\pi} \int_M Tr \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

where $A$ is $\mathfrak{g}$-valued 1-form on $M$ and $Tr$ denotes some invariant bilinear form on a simple Lie algebra $\mathfrak{g}$.

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The universal Chern-Simons theory depends on 4 parameters $\alpha, \beta, \gamma, \kappa$ defined up to a common multiple, where $\alpha, \beta, \gamma$ are Vogel's parameters. In fact it is more convenient to replace $\kappa$ by

$$\delta = \kappa + t = \kappa + \alpha + \beta + \gamma.$$

## Chern-Simons partition function of $S^3$

The Chern-Simons partition function

$$Z(M) = \int DA exp \left( \frac{ik}{4\pi} \int_M Tr \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right)$$

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It can be written as the product $Z(S^3) = Z_1 Z_2$, where

$$Z_1 = Vol(Q^\vee)^{-1}\delta^{-r/2}\prod_{\mu\in R_+}\frac{2\pi(\mu,\rho)}{\delta} = \frac{(2\pi\delta^{-1/2})^{dim\,\mathfrak{g}}}{Vol(G)},$$

where $Vol(G)$ is the volume of the corresponding compact simply connected group $G$ (**Ooguri, Vafa; MV**) and

$$Z_2 = \prod_{\mu\in R_+}\sin\frac{\pi(\mu,\rho)}{\delta}\Big/\frac{\pi(\mu,\rho)}{\delta}.$$

# Perturbative part

Consider the corresponding free energy $F_2 = -\ln Z_2$. Using

$$\sin \pi x = \pi x \prod_{n=1}^{\infty}(1 - (\frac{x}{n})^2)$$

we have

$$\ln \frac{\sin \pi x}{\pi x} = \sum_{n=1}^{\infty} \ln(1 - (\frac{x}{n})^2) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m}\frac{x^{2m}}{n^{2m}} = \sum_{m=1}^{\infty} \frac{\zeta(2m)}{m} x^{2m},$$

where $\zeta(z)$ is the Riemann zeta-function.

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where $\zeta(z)$ is the Riemann zeta-function.

Thus the perturbative part of free energy is

$$F_2 = \sum_{m=1}^{\infty} \frac{\zeta(2m)}{m} \sum_{\mu \in R_+} \left( \frac{(\mu, \rho)}{\delta} \right)^{2m}.$$

To show its universality we should express the sums $\sum_{\mu \in R} (\mu, \rho)^{2m}$ in terms of Vogel's parameters.

# Weyl formula and universality

Consider the exponential generating function of $p_k = \sum_{\mu \in R} (\mu, \rho)^k$ :

$$F(x) = \sum_{k=1}^{\infty} \frac{p_k}{k!} x^k = \sum_{\mu \in R} (e^{x(\mu,\rho)} - 1).$$

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**Theorem (Mkrtchyan, V, 2012).**

$$F(x) = \frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(\frac{x\alpha}{4})} \frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})} \frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})} - \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}$$

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Idea of the proof: use Weyl's character formula for the adjoint representation

$$\chi_\theta(x\rho) = \prod_{\mu \in R_+} \frac{\sinh(x(\mu, \theta + \rho)/2)}{\sinh(x(\mu, \rho)/2)}$$

and Key lemma.

# Corollary: Freudenthal-de Vries strange formulae

Expanding the previous formula in $x$ we have in the leading order

$$\sum_{\mu \in R_+} (\mu, \rho)^2 = \frac{t^2}{12} dim\, \mathfrak{g},$$

which is a homogeneous form of the **Freudenthal-de Vries strange formula**

$$< \rho, \rho >= \frac{1}{24} dim\, \mathfrak{g}.$$

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In the next orders we have

$$\sum_{\mu\in R_+} (\mu,\rho)^4 = \frac{t(18t^3 - 3tt_2 + t_3)}{480} dim\,\mathfrak{g},$$

where $t_2 = \alpha^2 + \beta^2 + \gamma^2$, $t_3 = \alpha^3 + \beta^3 + \gamma^3$, and

$$\sum_{\mu\in R_+} (\mu,\rho)^6 = \frac{t(396t^5 - 157t^3 t_2 + 15tt_2^2 + 39t^2 t_3 - 5t_2 t_3)}{16128} dim\,\mathfrak{g}.$$

**Expectation value of the unknotted Wilson loop** $C$ in $S^3$

$$< W(C) >= \frac{1}{Z} \int dA e^{iS(A)} W(C), \quad W(C) = TrP(\exp \int A_\mu dx^\mu)$$

with $A_\mu$ taken in adjoint representation of $\mathfrak{g}$ can be given as

$$< W(C) >= \frac{sin(\frac{\pi(\alpha-2t)}{2\delta})}{sin(\frac{\pi\alpha}{2\delta})} \frac{sin(\frac{\pi(\beta-2t)}{2\delta})}{sin(\frac{\pi\beta}{2\delta})} \frac{sin(\frac{\pi(\gamma-2t)}{2\delta})}{sin(\frac{\pi\gamma}{2\delta})}.$$

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**Central charge** $c$ can be expressed universally as

$$c = \frac{\kappa(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma(\kappa+\alpha+\beta+\gamma)} = \frac{(\delta-t)(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma\delta}.$$

# More universal formulae in Chern-Simons theory

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Proof is based on explicit formulae given by **Witten**.

# Universal formula for the volume of $G$

Let $G$ be a connected and simply-connected compact Lie group with simple Lie algebra $\mathfrak{g}$. Supply $G$ with the Riemannian metric defined by the canonical Cartan-Killing form. What is the volume of $G$ ?

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$$Vol(G) = Vol(\mathfrak{g}/\mathfrak{g}_{\mathbb{Z}}) \prod_{i=1}^{r} \frac{2\pi^{m_i+1}}{m_i!}.$$

where $r$ is rank and $m_i$ are the exponents of $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{Z}}$ is the Chevalley lattice.

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**Mkrtchyan, V. (2013)**:

$$Vol(G) = (2\sqrt{2}\pi)^{\dim G} e^{-\Phi(\alpha,\beta,\gamma)}, \quad \Phi(\alpha,\beta,\gamma) = \int_0^{\infty} \frac{F(x;\alpha,\beta,\gamma)dx}{x(e^x-1)},$$

$$F(x;\alpha,\beta,\gamma) = \frac{\sinh(x\frac{\alpha-2t}{4t})}{\sinh(\frac{x\alpha}{4t})} \frac{\sinh(x\frac{\beta-2t}{4t})}{\sinh(x\frac{\beta}{4t})} \frac{\sinh(x\frac{\gamma-2t}{4t})}{\sinh(x\frac{\gamma}{4t})} - \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma},$$

$$\dim G = \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}.$$

For the unitary groups we have

$$Vol(SU_n) = \frac{2^{\frac{n^2-1}{2}} n^{\frac{n^2}{2}} (2\pi)^{\frac{n^2+n-2}{2}}}{1!2!3!\ldots(n-1)!}.$$

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$$\Phi(-2,2,z) = \ln\, G(z+1) - \frac{1}{2}z^2 \ln\, z + \frac{1}{2}(z^2-z)\ln\, 2\pi$$

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Thus, the analytic continuation given by the universal formula agrees with Barnes.

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What are the universal formulae for the symmetrised choice (**Gelfand**) of the Casimir values ?

Which characteristics of simple Lie (super)algebras can be expressed in terms of universal Vogel's parameters ?

Which sectors of Chern-Simons theory are universal ?

What does this all mean for other values of parameters?

# Maryland-1994

**Dunkl (1989)**: the operators

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where $R$ is a root system of a Coxeter group $G$ and $\hat{s}_\alpha f(x) = f(s_\alpha x)$, commute:

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$$L = -Res\, \nabla^2 = -\Delta + \sum_{\alpha \in R_+} \frac{k_\alpha (k_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2}$$

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# Dunkl operators and Calogero-Moser problem

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Generalisations to trigonometric case: **Heckman (1991), Cherednik (1991)**

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$$f_\alpha(z) = k_\alpha \sigma_{(\alpha^\vee, \lambda)}(z), \quad \sigma_\lambda = \frac{\theta_1(z-\lambda)\theta_1'(0)}{\theta_1(z)\theta_1(-\lambda)}$$

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**BFV (1994); Etingof, Felder, Ma, V (2011)**: quantum integrals of elliptic CM system

$$\hat{H}_i = Res \lim_{\lambda \to 0} H_i(\nabla, \lambda),$$

where $H_i(p, q)$ is a **classical integral** of CM system !

**Dear Sergey Petrovich and Victor Matveevich,**

**HAPPY BIRTHDAY !**