Toric geometry of the action of compact torus on complex Grassmannians

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Content of the talk

- Formulation of the problem
- Classes of manifolds
- Orbit space $G_{4,2}/T^3$
- Orbit space CP^5/T^3
- On differentiable structure of the orbit space $G_{4,2}/T^3$

- M^{2n} - projective toric variety, $T^n \hookrightarrow M^{2n}$;
- $\mu : M^{2n} \to R^n$ — moment map, T^n - invariant;
- $Im\mu = P$ — simple convex polytope.

μ splits as:

$$M^{2n} \xrightarrow{\pi} M^{2n}/T^n \xrightarrow{\cong} P$$

- M^{2n} — smooth, closed, oriented manifold; $T^n \hookrightarrow M^{2n}$;
- The action is locally isomorphic to the standard action $T^n \hookrightarrow C^n$;
- There exists $\mu : M \to P$, P – simple convex polytope;
- $\mu^{-1}(x)$ is an orbit of T^n-action for any $x \in P$.

Again μ splits as:

$$M^{2n} \xrightarrow{\pi} M^{2n}/T^n \xrightarrow{\cong} P$$

We consider the following setting:

- M^{2n} — smooth, closed manifold;
- $T^k \hookrightarrow M^{2n}, \quad k < n$ — effective action;
- There exists $\mu : M^{2n} \to R^k, \quad T^k$ - invariant;
- $Im\mu = P$ — convex polytope (not necessarily simple!).

It is defined a map

$$f : M^{2n}/T^k \longrightarrow P \text{ by } \mu = \pi \circ f$$

<u>Note:</u> f — is not a homeomorphism

Problem: Provide the class of manifolds M^{2n} with a moment map μ for which M^{2n}/T^k and the map f can be effectively described.

Results

We demonstrate our approach in the case of $G_{4,2}$ and CP^5:

For the canonical action $T^4 \hookrightarrow G_{4,2}$ we obtain effective action of T^3

We prove:

- $P = \Delta_{4,2}$ — octahedron
- $$G_{4,2}/T^4 \cong \partial\Delta_{4,2} * CP^1,$$
- $$f : \partial\Delta_{4,2} * CP^1 \to \Delta_{4,2} \quad - - - \quad \text{projection}$$
- Moreover from topological results:

$$G_{4,2}/T^4 \cong S^5.$$

Results

For the action $T^4 \hookrightarrow CP^5$

defined by the representation $T^4 \to T^6$:

$$(t_1, t_2, t_3, t_4) \to (t_1t_2, t_1t_3, t_1t_4, t_2t_3, t_2t_4, t_3t_4)$$

we obtain effective action $T^3 \hookrightarrow CP^5$

We prove:

- $P = \Delta_{4,2}$
-
$$CP^5/T^4 \cong \partial\Delta_{4,2} * CP^2,$$
$$f : \partial\Delta_{4,2} * CP^2 \to \Delta_{4,2} \quad --- \quad \text{projection}$$

- $M^{2n} = CP^n$;
- $T^{n+1} \hookrightarrow CP^n$ — canonical;
- Assume that representation $\rho : T^k \to T^{n+1}$, $k < n$ is a regular and given by $t \to (\alpha_0(t), \ldots, \alpha_n(t))$;
- $\alpha_j : T^k \to S^1$ are characters of T^k;
- The weight vectors we denote by the same letters α_j, $0 \leq j \leq n$
- It is defined the action $T^k \hookrightarrow CP^n$;
- The moment map $\mu_{T^k} : CP^n \to R^k$,

$$\mu_{T^k}(z) = \frac{1}{\|z\|^2} \sum_{0 \leq j \leq n} |z_j|^2 \alpha_j,$$

$z = [(z_0, \ldots, z_n)]$,

- $Im\mu = P = convh(\alpha_0, \ldots, \alpha_n)$

- $G_{n,k}$ - Grassmann manifold (k-dimensional subspaces in C^n);
- $T^n \hookrightarrow G_{n,k}$ acts by the canonical action on C^n;
- $T^{n-1} \subset T^n$ acts effectively

$G_{n,k}$ are represented by $n \times k$ matrices A, $rangA = k$, for a fixed base in C^n

$$P(A) = (P^J(A)) = (\det A_J) \ - \ \text{Plücker coordinates},$$

$J \subset \{1, \ldots, n\}, \quad |J| = k$,
J are ordered by $J = \{j_1 < j_2\} < \bar{J} = \{\bar{j_1} < \bar{j_2}\} \Leftrightarrow j_1 < \bar{j_1}$,
A^J is $k \times k$-matrix given by the columns of A indexed by J.

- Plücker coordinates, up to constant, are uniquely defined;
- Plücker coordinates give the embedding of $G_{n,k}$ into $CP^{\binom{n}{k}-1}$.

Gel'fand-Serganova moment map $\mu : G_{n,k} \to R^k$ is defined by

$$\mu(X) = \frac{\sum_J |P^J(X)|^2 \delta_J}{\sum_J |P^J(X)|^2},$$

($P^J(X)$ are Plčker coordinates for X and $\delta_J \in R^n$ is given by

$$(\delta_J)_i = 1, \quad i \in J, \quad (\delta_J)_i = 0, \quad i \notin J.$$

— μ is T^n-invariant;
— $Im\mu = convh(\delta_J)$ – denoted by $\Delta_{n,2}$ – called hypersimplex.

$T^n \hookrightarrow G_{n,k}$ extends to $(C^*)^n \hookrightarrow G_{n,k}$. The well known results for the orbits of $(C^*)^n$-action:

- $\overline{(C^*)^n \cdot X}$ is a compact algebraic manifold;
- $\overline{(C^*)^n \cdot X} - (C^*)^n \cdot X$ consists of finitely many $(C^*)^n$ - orbits of lower dimension;
- $(C^*)^n \cdot X$ is unique everywhere dense open orbit in $\overline{(C^*)^n \cdot X}$;
- $\overline{(C^*)^n \cdot X}$ is a *toric manifold*.

The classical result (Atiyah, Guillemin-Sternberg, Gelfand-MacPherson):

Theorem

1. $\mu(\overline{(C^*)^n \cdot X}) = convh(V_I), \quad \{V_I\} \subseteq \{\delta_J\};$
2. *μ gives a bijection between p-dimensional $(C^*)^n$-orbits in $\overline{(C^*)^n \cdot X}$ and p-dimensional open faces of the polytope $\mu(\overline{(C^*)^n \cdot X})$.*

- The subpolytope $P \subseteq \Delta_{n,2}$ is called admissible if $\mu(\overline{(C^*)^n \cdot X}) = P$ for some $X \in G_{n,k}$;
- $X_1, X_2 \in G_{n,k}$ are said to belong to the same stratum Γ of $G_{n,k}$ if they have the same admissible polytopes.
- All strata $\{\Gamma\}$ gives partition of $G_{n,k}$ — stratification;
- Admissible polytopes $\equiv$ the polytopes of strata;
- $P = \Delta_{n,k}$ — Γ is called main stratum (generic orbits).

Atlas for $G_{n,k}$: (M_J, u_J), $J \subset \{1, \ldots, n\}$, $|J| = k$ – given by

$$M_J = \{X \in G_{n,k} \mid P^J(X) \neq 0\}, \;\; u_J : M_J \to C^{k(n-k)}.$$

$X \in M_J \Rightarrow$ it can be represented by matrix A such that $A_J = I_d$ and

$$u_J(X) = (a_{ij}(X)) \in C^{k(n-k)}, \;\; i \notin J$$

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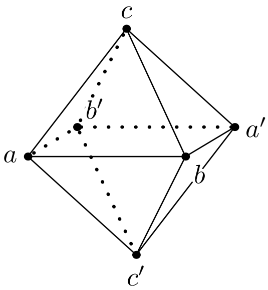
Note:

- the charts M_J are invariant under the action of $(C^*)^n$ and $(C^*)^{n-1}$ acts effectively on M_J,
- $(C^*)^n$-action, by the homeomorphism u_J, induces the action of $(C^*)^n$ on $C^{k(n-k)}$.

- $\mu(G_{4,2}) = Conv(\delta_{ij}) = \Delta_{4,2}$ — octahedron;
- $$\delta_{12} = (1,1,0,0), \delta_{13} = (1,0,1,0), \delta_{14} = (1,0,,0,1),$$
$$\delta_{23} = (0,1,1,0), \delta_{24} = (0,1,0,1), \delta_{34} = (0,0,1,1).$$
- δ_{ij}, $1 \le i < j \le 4$ belong to the hyperplane in R^4 given by the equation $x_1 + x_2 + x_3 + x_4 = 0$;
- P admissible — its vertices are some of δ_{ij}.



Lemma

The admissible polytopes are:

1. $\Delta_{4,2}$;
2. *any four-sided pyrhamid;*
3. *three diagonal squares;*
4. *any face on the boundary for* $\Delta_{4,2}$.

To summarize — the number of admissible polytopes in each dimension:

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 7 & 11 & 12 & 6 \end{bmatrix}.$$

In local chart M_{12} the orbit of (a_1, a_2, a_3, a_4) is given by

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ t_3 a_1 & t_3 a_3 \\ t_4 a_2 & t_4 a_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{t_3}{t_1} a_1 & \frac{t_3}{t_2} a_3 \\ \frac{t_4}{t_1} a_2 & \frac{t_4}{t_2} a_4 \end{pmatrix}.$$

The strata with admissible polytopes:

-
    1. $(0,0,0,0)$ – if $a_i = 0$ for all i — point δ_{12};
    2. C^* – if $a_i = 0$ for three i — edge on $\partial\Delta_{4,2}$ having vertex δ_{12} ;
    3. $(C^*)^2$ – if $a_i = 0$ for two i — triangle on $\partial\Delta_{4,2}$ or diagonal square having vertex δ_{12};
    4. $(C^*)^3$ – if $a_i = 0$ for one i — pyramid having vertex δ_{12} at the base;
    5. $\frac{z_1 z_4}{z_2 z_3} = 1$ – if $a_1 a_4 = a_2 a_3$ — pyramid having top-vertex δ_{12}.
- The hypersurface:

$$\frac{z_1 z_4}{z_2 z_3} = c \quad \text{for} \quad c = \frac{a_1 a_4}{a_2 a_3}, \quad a_i \neq 0, i = 1, \ldots, 4, \quad c \neq 1$$

Admissible polytope is $\Delta_{4,2}$ — main stratum;

We obtain the description of stratification for $G_{4,2}$.

Corollary

The strata on Grassmannian $G_{4,2}$, dimension and the number:

$$\begin{bmatrix} 8 & 6 & 4 & 2 & 0 \\ 1 & 6 & 11 & 12 & 6 \end{bmatrix}.$$

1. *the strata of dimension 8 is an open everywhere dense in $G_{4,2}$;*
2. *each strata of dimension ≤ 6 consists of one orbit.*

Orbit space $G_{4,2}/T^3$

1. $((C^*)^4 \cdot X)/T^3 \cong Int\Delta_{4,2}$ for a generic point X ;
2. these orbits (of the main stratum Γ) are parametrized by $c \in C - \{0, 1\}$;
3. $\Gamma \cong Int\Delta_{4,2} \times (C - \{0, 1\})$.

Using this we continuously parametrize by $c \in CP^1$ all orbits for $(C^*)^4$-action on $G_{4,2}$.

Orbit space $G_{4,2}/T^3$

Proposition

The points of 6-dimensional non-generic $(C^*)^3$-orbit with admissible polytope K_{ij} can be continuously parametrized by $c = 0, 1, \infty$:

- *for* $K_{14} = \Delta_{4,2} - \delta_{14}$ or $K_{23} = \Delta_{4,2} - \delta_{23} \longrightarrow c = 0$;
- *for* $K_{13} = \Delta_{4,2} - \delta_{13}$ or $K_{24} = \Delta_{4,2} - \delta_{24} \longrightarrow c = \infty$;
- *for* $K_{12} = \Delta_{4,2} - \delta_{12}$ or $K_{34} = \Delta_{4,2} - \delta_{34} \longrightarrow c = 1$.

Proposition

The points of $2l$-dimensional orbit, where $l \leq 2$, with admissible polytope K can be continuously parametrized using the parametrization of 6-dimensional orbits:

- *if* $l = 0, 1 \longrightarrow$ *any* $c \in CP^1$;
- *if* $l = 2$ *and* K *is a triangle* $\longrightarrow$ *any* $c \in CP^1$;
- *if* $l = 2$ *and* K *is a square:*
  1. $K_{14,23} = \Delta_{4,2} - \{\delta_{14}, \delta_{23}\} \longrightarrow c = 0$,
  2. $K_{13,24} = \Delta_{4,2} - \{\delta_{13}, \delta_{24}\} \longrightarrow c = \infty$,
  3. $K_{12,34} = \Delta_{4,2} - \{\delta_{12}, \delta_{34}\} \longrightarrow c = 1$.

Orbit space $G_{4,2}/T^3$

Corollary

- $K_{14} \cup K_{23} \cup K_{14,23} = \Delta_{4,2} \longrightarrow c = 0,$
- $K_{13} \cup K_{24} \cup K_{13,24} = \Delta_{4,2} \longrightarrow c = \infty,$
- $K_{12} \cup K_{34} \cup K_{12,34} = \Delta_{4,2} \longrightarrow c = 1.$

This leads to:

Theorem

$X = G_{4,2}/T^3$ is homeomorphic to the quotient space

$$(\Delta_{4,2} \times CP^1)/\approx \ \textit{where}\ (x, c) \approx (y, c^{'}) \Leftrightarrow x = y \in \partial\Delta_{4,2}.$$

Orbit space $G_{4,2}/T^3$

Corollary

- $K_{14} \cup K_{23} \cup K_{14,23} = \Delta_{4,2} \longrightarrow c = 0,$
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Corollary

$G_{4,2}/T^3$ is homeomorphic to the join $S^2 * S^2$.

Orbit space $G_{4,2}/T^3$

Generalized Poincare conjecture:

Theorem

$G_{4,2}/T^3$ is a topological manifold without boundary, and, thus, $G_{4,2}/T^3$ is homeomorphic to the sphere S^5.

We have:

$$f : G_{4,2}/T^3 \cong \partial\Delta_{4,2} * S^2 \longrightarrow \Delta_{4,2} --- \text{projection}$$

$$\mu = \pi \circ f.$$

Orbit space CP^5/T^3

The action $T^4 \hookrightarrow CP^5$ is defined by the representation $\rho : T^4 \to T^6$:

$$(t_1, t_2, t_3, t_4) \to (t_1 t_2, t_1 t_3, t_1 t_4, t_2 t_3, t_2 t_4, t_3 t_4)$$

and the canonical action $T^6 \hookrightarrow CP^5$.
By this action we obtain effective action T^3 on CP^5.
The moment map:

$$\mu(z) = \frac{1}{||z||^2}(|z_1|^2\delta_{12} + |z_2|^2\delta_{13} + |z_3|^2\delta_{14} + |z_4|^2\delta_{23} + |z_5|^2\delta_{24} + |z_6|^2\delta_{34})$$

- This action extends to the action of $(C^*)^3$;
- Easy to see: any polytope spanned by some subset of vertices for $\Delta_{4,2}$ is admissible polytope;

$M_0 = \{z_0 \neq 0\}$ – chart on CP^5 – coordinates $(z_1, \ldots, z_5)$;

$$(C^*)^4 \cdot (a_1, \ldots, a_5) = (\frac{t_3}{t_2}a_1, \frac{t_4}{t_2}a_2, \frac{t_3}{t_1}a_3, \frac{t_4}{t_1}a_4, \frac{t_3 t_4}{t_1 t_2}a_5) =$$

$$(\bar{t}_1 a_1, \bar{t}_2 a_2, \bar{t}_3 a_3, \frac{\bar{t}_2 \bar{t}_3}{\bar{t}_1} a_4, \bar{t}_2 \bar{t}_3 a_5).$$

The strata with admissible polytopes:

1. $(0,0,0,0)$ — point δ_{12};
2. C^* — edges having vertex δ_{12} ;
3. $(C^*)^2$ — triangles having vertex δ_{12};
4. surfaces:

   $$z_1 = z_4 = 0, \quad \frac{z_2 z_3}{z_5} = c \ \text{ or } \ z_2 = z_3 = 0, \quad \frac{z_1 z_4}{z_5} = c, \quad c \neq 0$$

   — squares having vertex δ_{12};
5. $(C^*)^3$ – if $a_i = a_j = 0$ for $\{i, j\} \neq \{1, 4\}, \{2, 3\}$ — tetrahedra having vertex δ_{12};

1. surfaces:

$$z_1 = 0 \ \wedge \ \frac{z_2 z_3}{z_5} = c, \quad \text{or} \ \ z_4 = 0 \ \wedge \ \frac{z_2 z_3}{z_5} = c,$$

$$z_2 = 0 \ \wedge \ \frac{z_1 z_4}{z_5} = c, \quad \text{or} \ z_3 = 0 \ \wedge \ \frac{z_1 z_4}{z_5} = c,$$

$$z_5 = 0 \ \wedge \ \frac{z_2 z_3}{z_1 z_4} = c, \quad c \neq 0$$

— four-sided pyramids having vertex δ_{12};

2. surfaces (main stratum) :

$$\frac{z_2 z_3}{z_5} = c_1 \ \wedge \ \frac{z_1 z_4}{z_5} = c_2, \quad c_1, c_2 \neq 0$$

— $\Delta_{4,2}$

The generic orbits of the main stratum are parametrized by $(c_1 : c_2 : 1)$, $c_1, c_2 \neq 0$,

Theorem

Using the parametrization of the main stratum, each non-generic orbit which are not on $\partial\Delta_{4,2}$

- *can be parametrized by $(0 : c_2 : 1)$, $c_1 \neq 0$ or $(c_1 : 0 : 1)$, $c_2 \neq 0$ or $(0 : 0 : 1)$ or $(c_1 : c_2 : 0)$, $c_1, c_2 \in C$, $(c_1, c_2) \neq (0, 0)$.*
- *They can be divided into four groups such that:*
  1. *All orbits from the same group are equally parametrized;*
  2. *The admissible polytopes for the orbits from the same groups glue together to give $\Delta_{4,2}$.*

The orbits on $\partial\Delta_{4,2}$ can be parametrized by CP^2.

Orbit space CP^5/T^3

This leads:

Theorem

$$CP^5/T^3 \cong (\Delta_{4,2} \times CP^2)/\approx \ \ where\ (x,c) \approx (y,c^{'}) \Leftrightarrow x = y \in \partial\Delta_{4,2}$$

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Corollary

$$CP^5/T^3 \cong S^2 * CP^2$$

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Corollary

$$CP^5/T^3 \cong S^2 * CP^2$$

Remark: Embedding $G_{4,2} \subset CP^5$ by the Plücker coordinates is equivariant for T^3-action $\Longrightarrow G_{4,2}/T^3 \subset CP^5/T^3$.
In homogeneous coordinates:

$$G_{4,2} \subset CP^5 : \quad z_1z_6 + z_3z_4 = z_2z_5$$

$$G_{4,2}/T^3 \subset CP^5/T^3 : \quad S^2 * CP^1 \subset S^2 \subset CP^2, \text{ where } CP^1 \subset CP^2$$

$$(c,1) \to (c:1:(1-c)), \ \ (1,0) \to (0,0,1).$$

Quotients of algebraic torus orbits

$X \in G_{4,2}, \quad (\overline{(C^*)^3 \cdot X})/T^3$ with a admissible polytope P :

1. has 6 or 1 singular points, otherwise it is a manifold with corners, if $P = \Delta_{4,2}$ or P is four-sided pyramid;
2. is a manifold with corners if P is a triangle, square or interval.

Differentiable structure on $G_{4,2}/T^3 \cong S^5$

- S^5 has unique differentiable structure, the standard one;
- suggests: no differentiable structure on $X = G_{4,2}/T^3$ such that $\pi : G_{4,2} \to X$ is a smooth map;
  otherwise X would be diffeomoirphic to the standard sphere S^5, $S^1 \hookrightarrow S^5$ smoothly, while it is not clear where such an action on X would come from.

— We prove the quotient structure is not differentiable;

— Describe the corresponding smooth and singular points;

We use the slice or equivariant tubular neighborhood theorem:
M — a smooth manifold, $G \hookrightarrow M$ — smooth action, G - compact group

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The slice theorem states:

Theorem

For a fixed point p there exist G-equivariant diffeomorphism from a neighbourhood of the origin in T_pM onto neioghbourhood of p in M.

We use the slice or equivariant tubular neighborhood theorem:
M — a smooth manifold, $G \hookrightarrow M$ — smooth action, G - compact group
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Theorem

For a fixed point p there exist G-equivariant diffeomorphism from a neighbourhood of the origin in T_pM onto neioghbourhood of p in M.

– If p is not a fixed point, let H be its stabilizer, a proper subgroup of G;
– The slice representation V for p:
the normal bundle in the tangent bundle for M along the points of orbit $G \cdot p$, to the tangent bundle $T(G \cdot p)$ of the orbit. It is taken related to some G-invariant metric on M. We obtain representation of H in V.
The general slice theorem states:

Theorem

There exists G-equivariant diffeomorphism from the vector bundle $G \times_H V$ onto neighbourhood of the orbit $G \cdot p$ in M.

$\pi : G_{4,2} \to G_{4,2}/T^4$ — a natural projection.

Theorem

The point $q \in G_{4,2}/T^4$ *is:*

- *a smooth point if* $\dim \pi^{-1}(q) = 3$;
- *a cone-like singularity point if* $\dim \pi^{-1}(q) \leq 2$ *which has the neighborhood of the form*
  1. $D^2 \times cone(S^2)$ *for* $\dim \pi^{-1}(q) = 2$;
  2. $D^1 \times cone(S^5/T^2)$ *for* $\dim \pi^{-1}(q) = 1$;
  3. $cone(S^7/T^3)$ *for* $\dim \pi^{1}(q) = 0$,

  with the induced actions of T^2 *on* S^5 *and* T^3 *on* S^7.

Proposition

The orbit space S^5/T^2 has three cone-like singular points, while all other points are smooth. Moreover, the singular points have a neghbourhood of the form cone(S^2).

Proposition

The points of the orbit space S^7/T^3 which correspond to the:

- *three-dimensional orbits are smooth points;*
- *two-dimensional orbits are cone - like singularities with a neioghbourhood of the form $D^1 \times$ cone(S^2);*
- *one-dimensional orbits are cone-like singularirities with a neighborhood of the form cone(S^5/T^2).*

$H \hookrightarrow V \Rightarrow V = V^H \oplus L$ related to some G-invariant metric, V^H is the subspace of the vectors fixed by H.

$$G \times_H V = G \times_H (V^H \oplus L) = V^H \times (G \times_H L).$$

If further implies that

$$(G \times_H V)/G = V^H \times L/H.$$

We have fixed some G-invariant Riemannian metric $\Rightarrow$ the action of H on L preserves the scalar product meaning that $H(S(L)) \subseteq S(L)$ where $S(L)$ is an unitary sphere whose center is at origin p of L. Therefore

$$L/H = ([0, \infty) \times S(L))/H = cone(S(L)/H),$$

what gives

$$(G \times_H V)/G = V^H \times cone(S(L)/H).$$