Quotients of affine spherical varieties by unipotent subgroups

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on the occasion of his 70-th birthday
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Motivation

Stratification of a topological space/algebraic variety X into "nice" cells $\rightsquigarrow$ nice homological/intersection-theoretic properties of X.

Source of stratifications in algebraic geometry:

Suppose that a connected solvable algebraic group B acts on X with finitely many orbits.

Each B-orbit is isomorphic to $\mathbb{C} \times \cdots \times \mathbb{C} \times (\mathbb{C} \setminus 0) \times \cdots \times (\mathbb{C} \setminus 0) \implies$ B-orbital decomposition yields a nice stratification of X.

Main goal: provide more natural examples of algebraic varieties stratified by orbital decomposition.

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Outline

Algebraic varieties

We consider algebraic varieties over $\mathbb{C}$.

An *affine algebraic variety* $X \subseteq \mathbb{C}^n$ is given by polynomial equations

$$p_1(x_1, \dots, x_n) = \cdots = p_k(x_1, \dots, x_n) = 0.$$

Zarisky topology: closed subsets = affine subvarieties.

Coordinate algebra:

$$\mathbb{C}[X] = \{\text{polynomial functions in } x_1, \dots, x_n \text{ restricted to } X\}.$$

Conversely:
given any finitely generated commutative $\mathbb{C}$-algebra A without nilpotents
$\implies A \simeq \mathbb{C}[x_1, \dots, x_n]/(p_1, \dots, p_k) \rightsquigarrow$ affine variety $X = \operatorname{Spec} A \subseteq \mathbb{C}^n$.

Arbitrary algebraic varieties are patched from affine open pieces.

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Orbital stratification

Suppose that a connected solvable linear algebraic group B acts on an algebraic variety X.

X affine $\implies$ may assume $X \subseteq \mathbb{C}^n$, $H \subseteq \left\{ \begin{pmatrix} * & & \mathbf{*} \\ & \ddots & \\ \mathbf{0} & & * \end{pmatrix} \right\}$

How to guarantee that B has finitely many orbits on X?

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Spherical varieties

Suppose $B \subseteq G$, a Borel (=maximal connected solvable) subgroup in a bigger reductive group acting on X (e.g.: $G = GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $O_n(\mathbb{C})$, $SO_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$, any semisimple group, any finite group, $(\mathbb{C} \setminus 0)^n$, etc).

Example

$$G = GL_n(\mathbb{C}), \qquad B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}.$$

Definition

A G-variety X is *spherical* if B has a dense open orbit in X.

Affine case: X spherical $\iff$ representation $G \curvearrowright \mathbb{C}[X]$ multiplicity free

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Examples:

- toric varieties;
- (partial, generalized) flag varieties $X = G/P$, $P \supset B$;
- determinantal varieties $X = \{x \in \mathrm{Mat}_{m\times n}(\mathbb{C}) \mid \mathrm{rk}(x) \leq r\}$;
- (algebraic) symmetric spaces $X = G/G^{\theta}$, $\theta \in \mathrm{Aut}\, G$, $\theta^2 = \mathrm{id}$;
- in particular, (complex) sphere $S^{n-1} = \{x \in \mathbb{C}^n \mid x_1^2 + \cdots + x_n^2 = 1\} \simeq SO_n(\mathbb{C})/SO_{n-1}(\mathbb{C})$.

Theorem (M. Brion, E. B. Vinberg, 1986)

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Problem

Extend to a wider class of varieties having finite B-orbital decomposition.
Specifically: push down the finiteness property to quotients.

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Categorical quotients

Let a linear algebraic group H act on an affine algebraic variety X.

May assume: $H \subseteq GL_n(\mathbb{C})$, $X \subseteq \mathbb{C}^n$.

Suppose that $\mathbb{C}[X]^H = \mathbb{C}[f_1, \ldots, f_m]$ is finitely generated.

Definition

Categorical quotient:

$$\begin{tikzcd} X /\!\!/ H = \operatorname{Spec} \mathbb{C}[X]^H \arrow[d, hook] & X \arrow[l, "\pi = \pi_H"'] \\ \mathbb{C}^m \ni (f_1(x), \ldots, f_m(x)) & x \arrow[l, mapsto] \end{tikzcd}$$

Idea behind: H-invariant functions on X = those pulled back from $X /\!\!/ H$.

H reductive $\implies$ $X /\!\!/ H$ exists, π surjective, π separates closed orbits + other good properties. False in general!

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Example: a bad quotient

Example

$$H = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$$ acts on $X = \mathrm{Mat}_{3\times 3}(\mathbb{C})$ by left multiplication:

$$X \ni x = \begin{array}{|c|cc|} \hline * & * & * \\ \hline * & * & * \\ x_{31} & x_{32} & x_{33} \\ \hline \end{array} \begin{array}{l} \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \Delta_{23} \end{array} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \Delta$$

$$\mathbb{C}[X]^H = \mathbb{C}[x_{31}, x_{32}, x_{33}, \Delta_{12}, \Delta_{23}, \Delta_{31}, \Delta]$$

$$X /\!\!/ H = \{x_{31}\Delta_{23} + x_{32}\Delta_{31} + x_{33}\Delta_{12} = 0\} \subset \mathbb{C}^7$$

π is non-surjective:

$$(X /\!\!/ H) \setminus \pi(X) = \{x_{31} = x_{32} = x_{33} = 0,\ \exists \Delta_{ij} \neq 0 \text{ or } \Delta \neq 0\}$$

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Transfer Principle

Suppose $H \subseteq G$, a bigger reductive group acting on X.

Transfer Principle

$\mathbb{C}[X]^H \simeq \mathbb{C}[G/H \times X]^G \simeq (\mathbb{C}[G/H] \otimes \mathbb{C}[X])^G$
finitely generated whenever $\mathbb{C}[G/H] = \mathbb{C}[G]^H$ *is so.*

Geometrically: $\exists G /\!\!/ H \implies \exists X /\!\!/ H \simeq (G /\!\!/ H \times X) /\!\!/ G$

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Grosshans subgroups

Definition

$H \subseteq G$ is a *Grosshans subgroup* if:

- G/H is quasiaffine,
- $\mathbb{C}[G/H]$ is finitely generated.

Example

$H = U$, a maximal unipotent subgroup in G.

For instance: $G = GL_n(\mathbb{C})$, $U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$.

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Quotients by unipotent subgroups

Observation: X spherical $\implies$ $B/U = T \curvearrowright X /\!\!/ U$ toric $\implies$ finitely many orbits

Replace U with $H \subset U$, $H \lhd B$, H Grosshans.

Conjecture (V. L. Popov, K. Pommerening)

$H \subseteq U$ normalized by a maximal torus $T \subseteq B$ $\implies$ H Grosshans.

Question (D. I. Panyushev, 2012)

Does B/H have finitely many orbits on $X /\!\!/ H$?

Caution: $\pi : X \to X /\!\!/ H$ may be non-surjective.

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Particular cases

Easy cases: (Answer: yes)

- $B \subseteq P \subseteq G$, $H =$ unipotent radical of P;
- $X^G = \{0\}$, $\dim \pi^{-1}(\pi(0)) = \dim Hx$, $x \in X$ general point.

From now on assume: $H = U' = [U, U]$.

Example

$$G = GL_n(\mathbb{C}), \qquad U' = \left\{ \begin{pmatrix} 1 & 0 & * & & \ast \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & * \\ & & & 1 & 0 \\ 0 & & & & 1 \end{pmatrix} \right\}.$$

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X affine spherical $\implies$ B/U' has finitely many orbits on $X /\!\!/ U'$.

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Conjecture is true for any X (with G fixed) $\iff$ all simple factors of G are A_1, B_2, G_2.

Is the conjecture true under certain restrictions on X?

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Representation theory

- All (algebraic) representations of G are completely reducible.
- $\forall$ irreducible G-module V $\exists$ unique B-stable line $\langle v\rangle \subseteq V$.
- $b \cdot v = \lambda(b)v$, $\forall b \in B$, where $\lambda : B \to \mathbb{C} \setminus 0$ is the *highest weight*.
- *Weight lattice* $\Lambda \ni \lambda : B = U \rtimes T \to T \simeq (\mathbb{C} \setminus 0)^n \to \mathbb{C} \setminus 0$, $\lambda(tu) = t_1^{\ell_1} \cdots t_n^{\ell_n}$, $\ell_1, \dots, \ell_n \in \mathbb{Z}$.
- *Monoid of dominant weights* $\Lambda_+ = \Lambda \cap C$, $C \subseteq \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ positive Weyl chamber.
- $\{$dominant weights$\} = \{$highest weights of irreducible G-modules$\}$
- $V = V(\lambda)$ is uniquely determined by its highest weight $\lambda \in \Lambda_+$.
- May assume w.l.o.g.: $G = Z \times G'$, Z torus, G' semisimple simply connected $\implies$ $B = Z \times (B \cap G')$, $U \subseteq G'$, $T = Z \times (T \cap G')$, $\Lambda_+|_{B\cap G'} = \langle \omega_1, \dots, \omega_r \rangle_{\mathbb{Z}_+}$,

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Definition

Weight monoid $\Lambda_+(X) = \{\lambda \in \Lambda_+ \mid V(\lambda) \hookrightarrow \mathbb{C}[X]\}$

Theorem

Suppose $\Lambda_+(X)|_{B\cap G'} \subseteq \langle \omega_{i_1}, \dots, \omega_{i_m} \rangle$, so that i_k and i_l are not connected by a single edge in the Dynkin diagram, $\forall k, l$.

Then B/U' acts on $X /\!\!/ U'$ with finitely many orbits.

Example

$X = GL_{2n}(\mathbb{C})/Sp_{2n}(\mathbb{C})$, moduli space of symplectic structures on $\mathbb{C}^{2n}$.

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Main problem: non-surjectivity of $\pi_H : X \to X /\!\!/ H$.

Solution: use the Transfer Principle.

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