On hyperbolic 3-manifolds with geodesic boundary

Andrei Vesnin

Soobolev Institute of Mathematics, Novosibirsk, Russia

Algebraic Topology and Abelian Functions
Conference in honour of Victor Buchstaber
on occasion of his 70th birthday

Plan:

- Three-dimensional manifolds and complexity.
- Geometrization of 3-manifolds.
- Complexity of infinite families of closed hyperbolic 3-manifolds.
- Complexity of infinite families of hyperbolic 3-manifolds with boundary.

1. Three-dimensional manifolds and complexity

There are many ways to construct closed orientable 3-manifolds:

- Gluing from a polyhedron. [Seifert and Threlfall]: M is a manifold if and only if its Euler characteristic $\chi(M) = 0$.
- Heegaard splittings from two handlebodies. [Heegaard]: works for any manifold.
- Dehn surgeries on links in S^3. [Lickorish]: works for any manifold.
- As a branched (irregular) covering of S^3. [Alexander]: works for any manifold.

What is a good way to tabulate 3-manifolds?

The tabulation by a complexity.

There are many ways to construct closed orientable 3-manifolds:

- Gluing from a polyhedron. [Seifert and Threlfall]: M is a manifold if and only if its Euler characteristic $\chi(M) = 0$.
- Heegaard splittings from two handlebodies. [Heegaard]: works for any manifold.
- Dehn surgeries on links in S^3. [Lickorish]: works for any manifold.
- As a branched (irregular) covering of S^3. [Alexander]: works for any manifold.

What is a good way to tabulate 3-manifolds?

The tabulation by a complexity.

[H. Kneser, 1929]: Triangulation complexity $tc(M)$ of a closed 3-manifold M is the minimal number of tetrahedra needed to obtain M by gluing them together along faces.

Weakness: $tc(M)$ is not "good" since $tc(S^3) \neq 0$.
Thus an additivity will not hold: $tc(M \# S^3) < tc(M) + tc(S^3)$

Matveev complexity:
A subpolyhedron P in a 3-manifold M is a spine of M if
$\partial M \neq \emptyset$ and the complement $M \setminus P$ is homeomorphic to $\partial M \times (0, 1]$,
or
$\partial M = \emptyset$ and the complement $M \setminus P$ is an open ball.

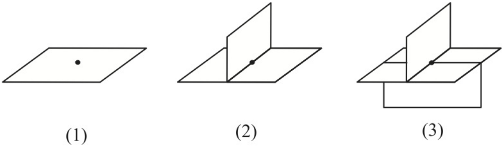
[H. Kneser, 1929]: Triangulation complexity $tc(M)$ of a closed 3-manifold M is the minimal number of tetrahedra needed to obtain M by gluing them together along faces.

Weakness: $tc(M)$ is not "good" since $tc(S^3) \neq 0$.
Thus an additivity will not hold: $tc(M\#S^3) < tc(M) + tc(S^3)$

Matveev complexity:
A subpolyhedron P in a 3-manifold M is a spine of M if
$\partial M \neq \emptyset$ and the complement $M \setminus P$ is homeomorphic to $\partial M \times (0, 1]$,
or
$\partial M = \emptyset$ and the complement $M \setminus P$ is an open ball.

A compact polyhedron P is said to be simple if for each vertex its link is homeomorphic to one of the following 1-dimensional polyhedra:

- circle; and the point is ordinary;
- circle with diameter; and the point is triple;
- circle with three radii (complete graph K_4); and the point is true vertex.



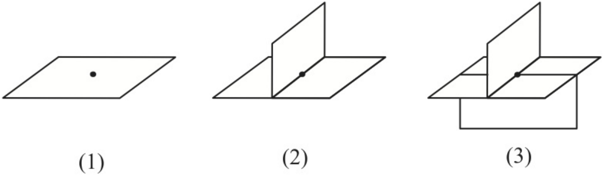
A compact polyhedron P is said to be almost simple if the link of any of its points can be embedded into a complete graph K_4. A true vertex of an almost simple polyhedron P is a point with the link K_4.

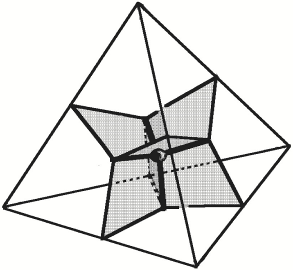
Definition of Matveev complexity: For a compact 3-manifold M $c(M) = k$ if M admits an almost simple spine with k true vertices and doesn't admit almost simple spines with fewer number of true vertices.

Finiteness property: For any k there is only finite number of closed irreducible manifolds of complexity $c(M) = k$.

Additivity property: If $M = M_1 \sharp M_2$ then $c(M) = c(M_1) + c(M_2)$.

Duality of triangulation and spine:





“Six-winged butterfly”

Closed 3-manifolds

Theorem [Matveev] Let M be a closed irreducible 3-manifold differ from $S^3, \mathbb{R}P^3, L_{3,1}$. Then $c(M) = tc(M)$.

Manifolds of small complexity:

- up to complexity 5 [Matveev, Savvateev 1974]: generated by computer, distinguished by hand;
- from complexity 6 to complexity 10 [Matveev, Ovchinnikov; Martelli, Petronio]: generated by computer, distinguished by computer;
- complexity 11, 12 [Matveev, Tarkaev, 2003].

0	1	2	3	4	5	6	7	8	9	10	11	12
3	2	4	7	14	31	74	175	436	1154	3078	8421	23452

2. Geometrization of 3-manifolds

Geometrization Conjecture

William Paul Thurston (30.10.1946 – 21.08.2012)

8 model geometries for 3-manifolds:

$\mathbb{S}^3, \mathbb{E}^3, \mathbb{H}^3$ – spherical, Euclidean, hyperbolic;

$\mathbb{S}^2 \times \mathbb{R}^1, \mathbb{H}^2 \times \mathbb{R}^1$ – product geometries;

$Nil, Sol, \widetilde{\mathrm{SL}_2(\mathbb{R})}$ – exotic geometries.

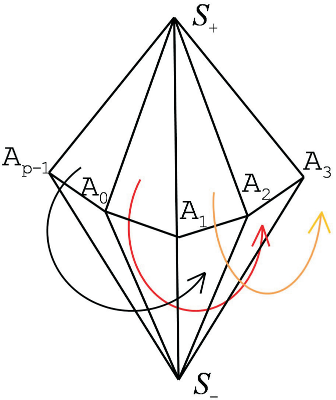
Complexity and geometry of closed orientable 3-manifolds

[Matveev — Tarkaev], [Burton]

Geo\\c	$\leqslant 5$	6	7	8	9	10	11	12	Total
$\mathbb{S}^3$	61	61	117	214	414	798	1582	3118	6365
$\mathbb{E}^3$	0	6	0	0	0	0	0	0	6
Nil	0	7	10	14	15	15	15	15	91
$\mathbb{H}^2 \times \mathbb{R}$	0	0	0	2	0	8	4	24	38
$\widetilde{\mathrm{SL}_2}(\mathbb{R})$	0	0	39	162	513	1416	3696	9324	15150
Sol	0	0	5	9	23	39	83	149	308
$\mathbb{H}^3$	0	0	0	0	4	25	120	461	610
non-Geo	0	0	4	35	185	777	2921	10361	14283
Total	61	74	175	436	1154	3078	8421	23452	36851

An example: Lens spaces $L_{p,q}$

Let $p \geqslant 3$, $0 < q < p$ and $(p, q) = 1$.
Consider p–gonal bipyramid
$S_+, A_0, A_1, \ldots, A_{p-1}, S_-$.
For every i we glue
$A_i S_+ A_{i+1} \to A_{i+q} S_- A_{i+q+1}$ (mod p).
Obtained manifold is a lens space $L_{p,q}$.
(Here $\chi = 2 - (1 + p) + p - 1 = 0$.)
Lens spaces admit spherical geometry.
Fundamental group $\pi_1(L_{p,q}) = \mathbb{Z}_p$.



Known results on complexity of spherical 3-manifolds

The initial table of manifolds:

complexity	manifolds
0	S^3, RP^3, $L_{3,1}$
1	$L_{4,1}$, $L_{5,2}$
2	$L_{5,1}$, $L_{7,2}$, $L_{8,3}$, S^3/Q_8
3	$L_{6,1}$, $L_{9,2}$, $L_{10,3}$, $L_{11,3}$, $L_{12,5}$, $L_{13,5}$, S^3/Q_{12}

Theorem [Jaco – Rubinstein – Tillmann, 2009]
For any integer $n \geqslant 2$ we have $c(L_{2n,1}) = 2n - 3$.

Theorem [Jaco – Rubinstein – Tillmann, 2011]
For any integer $n \geqslant 2$ we have $c(L_{4n,2n-1}) = n$ and $c(S^3/Q_{4n}) = n$.

3. Complexity of infinite families of closed hyperbolic 3-manifolds

Poincare model in the upper half-space

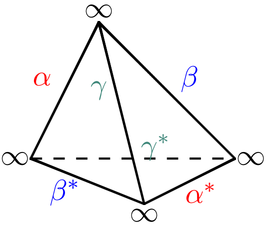
$\mathbb{H}^3 = \{(x, y, t) \in \mathbb{R}^3 \mid t > 0\}$ with $ds^2 = \dfrac{dx^2 + dy^2 + dt^2}{t^2}$.

$\mathrm{Iso}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$

Traditionally volumes of hyperbolic polyhedra and manifolds are calculating in terms of the Lobachevsky function:

$$\Lambda(\theta) = -\int_0^\theta \ln \mid 2 \sin x \mid dx$$

Tetrahedron with all vertices at infinity



Lemma $\alpha + \beta + \gamma = \pi$ and $\alpha^* = \alpha$, $\beta^* = \beta$, $\gamma^* = \gamma$.

Theorem [Milnor] Let $T(\alpha, \beta, \gamma)$ be a tetrahedron with all vertices at infinity. Then

$$\mathrm{vol}(T(\alpha, \beta, \gamma)) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma).$$

Corollary A tetrahedron with all vertices at infinity is of maximal volume if and only if it is $T(\pi/3, \pi/3, \pi/3)$. The maximal volume is

$$v_3 = \mathrm{vol}(T(\pi/3, \pi/3, \pi/3)) = 3\Lambda(\pi/3) = 1.0149426\ldots.$$

Complexity and volume

Observation: Let M be a closed orientable hyperbolic 3-manifold. Its complexity $c(M)$ and hyperbolic volume $\mathrm{vol}\,(M)$ satisfy the inequality

$$\frac{\mathrm{vol}\,(M)}{v_3} < c(M).$$

Realization: For cyclic coverings of S^3 branched along the figure-eight knot $\mathcal{F}$.

The figure-eight knot group

$$\Gamma = \pi_1(S^3 \setminus \mathcal{F}) = \langle a, b \,|\, ab^{-1}aba^{-1} = b^{-1}aba^{-1}b \rangle.$$

has a faithful representation θ into $\mathrm{PSL}_2\mathbb{C} = \mathrm{Iso}^+\mathbb{H}^3$:

$$\theta(a) \mapsto \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \qquad \theta(b) \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

where $\omega = -1/2 + \sqrt{-3}/2$. Moreover, $\Gamma \subset \mathrm{PSL}(\mathbb{Q}(\sqrt{-3}))$.

The figure-eight knot is a hyperbolic knot, i.e. its complement $S^3 \setminus \mathcal{F}$ is a hyperbolic 3-manifold.

Let M_n be the n-fold cyclic branched covering of S^3, branched along the figure-eight knot $\mathcal{F}$, $n \geq 2$.

$$\pi_1(M_n) = F(2, 2n) = \langle x_1, x_2, \dots, x_{2n} \,|\, x_i x_{i+1} = x_{i+2},\ i = 1, 2, \dots, 2n \rangle$$

that is a Fibonacci group introduced by Conway in 1965.

[Helling, Kim, Mennicke, 1998] For $n \geq 4$ manifold M_n is a closed orientable hyperbolic 3-manifold. It is called a Fibonacci manifold.

Theorem [Matveev – Petronio – Vesnin, 2009] There exists such N that for any $n \geq N$ complexity of the Fibonacci manifold M_n satisfies the following inequalities

$$2n \leq c(M_n) \leq 3n.$$

The upper bound holds for all $n \geq 4$ and is based on triangulations of fundamental polyhedra.

The lower bound is based on calculations of hyperbolic volumes of manifolds.

Theorem [Mednykh – Vesnin, 1994] For $n \geqslant 4$ volume $\mathrm{vol}\,(M_n)$ of a Fibonacci manifold equals

$$\mathrm{vol}(M_n) = 2n(\Lambda(a_n + \pi/n) + \Lambda(a_n - \pi/n)),$$

where $a_n = (1/2) \cdot \arccos(\cos(2\pi/n) - 1/2)$.

Analyzing the behavior of the volume function we get:

Corollary If n is big enough then

$$2n - \frac{34}{n} < \frac{\mathrm{vol}\,(M_n)}{v_3} < 2n.$$

Generalization for arbitrary 2-bridge knot

Theorem [Petronio – Vesnin, 2009]

Let $K(p,q)$, where $p \geq 3$, $0 < q < p$, $(p,q) = 1$, be a given two-bridge knot and let $\{M_n(p,q)\}_{n=2}^{\infty}$ be a sequence of n-fold cyclic branched coverings of S^3, branched along $K(p,q)$. Then:

$$c(M_n(p,q)) \leqslant n(p-1) \qquad \forall n.$$

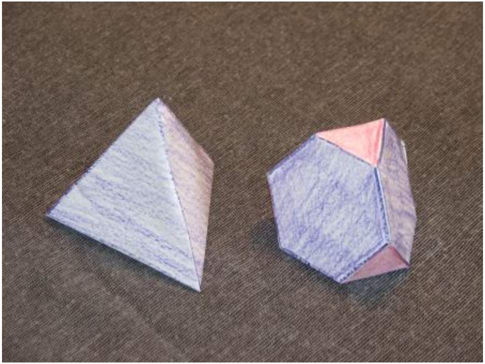
If in addition $K(p,q)$ is hyperbolic then for $n \geqslant 7$:

$$n \cdot \left(1 - \frac{4\pi^2}{n^2}\right)^{3/2} \cdot \max\{2, 2\ell(p,q) - 2.6668\} < c(M_n(p,q)).$$

Remark The figure-eight knot $\mathcal{F} = K(5,2)$.

4. Complexity of infinite families of hyperbolic 3-manifolds with boundary

Let Δ denote the standard tetrahedron, and let Δ^* be Δ minus open stars of its vertices.



Let M be a compact 3-manifold with $\partial M \neq \emptyset$. An ideal triangulation of M is a realization of M as a gluing of a finite number of copies of Δ^*, induced by a simplicial face-pairing of the corresponding Δ's.

Let $\mathcal{H}$ be the class of orientable compact 3-manifolds M having nonempty boundary ∂M and admitting a complete finite-volume hyperbolic metric with respect to which ∂M is totally geodesic.

Denote $\mathcal{H}_n = \{M : M \in \mathcal{H} \text{ and } c(M) = n\}$.

[Frigerio, Martelli, Petronio, 2004]
Enumeration: $\mathcal{H}_1 = \emptyset$; $|\mathcal{H}_2| = 8$; $|\mathcal{H}_3| = 74$; $|\mathcal{H}_4| = 5,002$.

Another subdivision of $\mathcal{H}$: subclasses $\mathcal{H}_{p,q}$.

$M \in \mathcal{H}_{p,q}$ if and only if

- number of connected components of ∂M equals q;
- M admits a special spine with p 2-components and doesn't admit with less number.

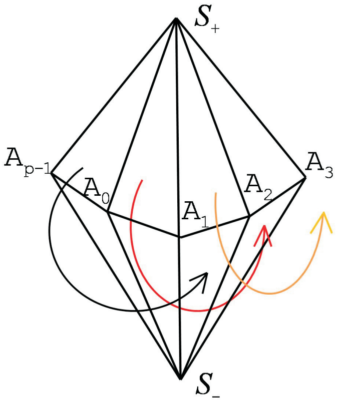
Lemma If $p, q \in \mathbb{N}$ are such that $p < q$, then $\mathcal{H}_{p,q} = \emptyset$.

Lemma If $M \in \mathcal{H}_{p,q}$, then $c(M) = p - \chi(M)$, where $\chi(M)$ is Euler characteristic.

[Frigerio, Martelli, Petronio]: manifolds from $\mathcal{H}_{q,q}$ for any $q \geqslant 1$.

An example: Paolluzi — Zimmermann manifolds

Let $n \geqslant 3$, $0 < k < n$, $(n, 2-k) = 1$.
Consider n–gonal bipyramid
$S_+, A_0, A_1, \ldots, A_{n-1}, S_-$.
For every i we glue
$A_i A_{i+1} S_+ \to S_- A_{i+k} A_{i+k+1}$ (mod n).
The result is a pseudo-manifold $M'_{n,k}$
with $\chi(M'_{n,k}) = n - 1$. Cutting a
neighborhood of the singular point we
get a manifold $M_{n,k}$ with boundary.

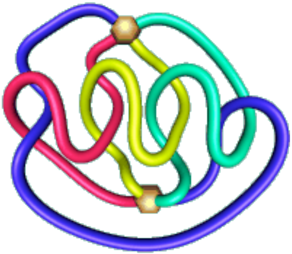


Properties of Paoluzzi — Zimmermann manifolds

- $M_{n,k}$ – hyperbolic manifold with boundary;
- $M_{3,1}$ – an example form Thurston lecture notes (1978), it has the smallest volume among all compact hyperbolic 3-manifolds with boundary, $\mathrm{vol}(M_{3,1}) \approx 6.451998$;
- A. Ushijima calculated volumes of manifolds $M_{n,k}$ for $n \leqslant 82$;
- fundamental group $\pi_1(M_{n,k})$ is isomorphic to $\langle x_0, \dots, x_{n-1} \mid \prod_{i=0}^{n-1} x_{i(2-k)} x_{i(2-k)+1}^{-1} x_{(i+1)(2-k)-1}^{-1} = 1 \rangle$;
- $\partial M_{n,k}$ is a closed orientable surface of genus $g = n - 1$;
- boundary $\partial M_{n,1}$ is embedded in $\mathbb{R}^3$ as a thickness of Suzuki graph θ_n.

Boundary of manifolds $M_{n,1}$

Suzuki's graph θ_n, $n \geqslant 3$ (see the case $n = 4$ in the figure).



θ_3 – the Kinoshita's graph
θ_2 – the trefoil knot

Generalizations

Analogous manifolds can be constructed in the case $(n, 2-k) = 2$.

[Fominykh — Vesnin, 2012]

- construction of manifolds $M_{n,k}$ for the case $(n, 2-k) = 2$;
- manifolds are hyperbolic with geodesic boundary;
- the volume formula is obtained;
- boundary of M_{n-1} is homeomorphic to a closed orientable surface of genus $n-1$.

Proposition

Suppose that integers $n \geqslant 4$ and k are such that $0 \leqslant k \leqslant n-1$ and $(n, 2-k) = 2$. Then

$$\begin{aligned} \operatorname{vol} M_{n,k} &= n\,[\Lambda(\pi/n+\theta) - \Lambda(\pi/n-\theta) + \Lambda(\pi/2+\alpha-\theta) \\ &\quad +\Lambda(\pi/2-\alpha-\theta) + \Lambda(2(\pi/n-\alpha)+\theta) \\ &\quad -\Lambda(2(\pi/n-\alpha)-\theta) + 2\Lambda(\pi/2-\theta)]\,, \end{aligned}$$

where

$$\theta = \operatorname{arctg} \frac{\sqrt{\cos^2\alpha - \sin^2(\pi/n)\sin^2(2(\pi/n-\alpha))}}{\cos(\pi/n)\cos(2(\pi/n-\alpha))} \in [0, \pi/2),$$

and $\alpha \in [0, \pi/n)$ is such that $t = \operatorname{tg}\alpha$ is a root of the equation

$$\begin{gathered} \cos\varphi\sin\varphi \cdot t^4 + (2\cos^2\varphi - \cos\varphi)\cdot t^3 + 3\sin\varphi\cdot t^2 + \\ +(2\cos^2\varphi + 3\cos\varphi)\cdot t - \cos\varphi\sin\varphi - \sin\varphi = 0, \end{gathered}$$

where $\varphi = 2\pi/n$.

Complexity of hyperbolic 3-manifolds with boundary

Theorem [Frigerio – Martelli – Petronio, 2003]
If $M \in \mathcal{H}$ has dual spine with n true vertices and one 2-cell, then $c(M) = n$. Moreover, $M \in \mathcal{H}_{1,1}$.

Theorem 1 [Fominykh – Vesnin, 2011]
Suppose $n \geqslant 4$ and $(n, 2-k) = 1$. Then $c(M_{n,k}) = n$. The minimal spine has two 2-cells, and $M_{n,k} \in \mathcal{H}_{2,1}$.

Remark Here $M_{3,1}$ is the Thurston's manifold, and $c(M_{3,1}) = 2$.

Theorem 2 [Fominykh – Vesnin, 2012]
Suppose $n \geqslant 6$ and $(n, 2-k) = 2$. Then $c(M_{n,k}) = n$. The minimal spine has three 2-cells, and $M_{n,k} \in \mathcal{H}_{3,1}$.

Sketch of the proof of theorem 1

- $M_{n,k}$ is hyperbolic manifold.
- $c(M_{n,k}) \leqslant n$; spine $P_{n,k}$ dual to the original subdivision of the bipiramid into n tetrahedra $\mathcal{T}_i = S_+ S_- A_i A_{i+1}$ has exactly n true vertices.
- Assume on the contrary that $P_{n,k}$ is not minimal.
- Let P be a minimal special spine.
- P has $n-1$ true vertices and one 2-cell, since $\chi(P) = \chi(M_{n,k}) = \chi(P_{n,k}) = 2 - n$.

Sketch of the proof of theorem 1

- [Matveev – Ovchinnikov – Sokolov] ε-invariant of 3-manifolds = homologically trivial part of Turaev – Viro invariant of 5-th order.
- Let $\mathcal{F}(P)$ be the set of all simple subpolyhedra of P, including the empty set and P.
- For each $Q \in \mathcal{F}(P)$ corresponds its weight
$$w_\varepsilon(Q) = (-1)^{V(Q)} \varepsilon^{\chi(Q) - V(Q)},$$
where $V(Q)$ – number of true vertices, $\chi(Q)$ – Euler characteristic, and $\varepsilon = (1+\sqrt{5})/2$ – a root of the equation $\varepsilon^2 = \varepsilon + 1$.
- ε-invariant $t(M)$ is defined as
$$t(M) = \sum_{Q \in \mathcal{F}(P)} w_\varepsilon(Q).$$

Sketch of the proof of theorem 1

- $t(M_{n,k}) = (-1)^n \varepsilon^{2-2n} + \varepsilon^{1-n} + 1$ – from spine $P_{n,k}$.
- $t(M_{n,k}) = (-1)^{n-1} \varepsilon^{3-2n} + 1$ – from assumed spine P with one 2-cell.
- equality $\varepsilon^{n-3} = (-1)^{n-1}$ doesn't hold if $n \geqslant 4$. Contradiction.

The proof of Theorem 2 is based on similar (but much more technical) ideas, involving properties of Fibonacci numbers.

Summary

Complexity is known for

- all closed orientable 3-manifolds up to complexity 13 (more than 60000 manifolds);
- few infinite families of spherical manifolds;
- few infinite families of hyperbolic manifolds with boundary.

Thank you!