Deformations and Contractions of Algebraic Structures

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1 Introduction

Deformations of analytic and algebraic objects is an old problem both in mathematics and physics. The theory of deformations originated with the problem of classifying all possible pairwise non-isomorphic complex structures on a given differentiable real manifold. The fundamental idea, which should be credited to Riemann, was to introduce an analytic structure therein.

In mathematics one of the main tools to study a given object is by deforming it to families with "similar" structure. In physics we can develop from known theories new ones, like quantum groups, deformations of Hopf algebras, q-deformed physics, deformation quantization, deformed geometry and gravitation, quantum field theory etc.

In this talk I restrict myself to the case of Lie algebras – one of the most important categories in physics.

2 Basic notions

Let $\mathcal{L}$ be a Lie algebra with Lie bracket μ_0 over a field $\mathbb{K}$.

a) *Intuitive definition of deformation*

A deformation of $\mathcal{L}$ is a one-parameter family $\mathcal{L}_t$ of Lie algebras with the bracket (possibly infinite series)

$$\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \dots$$

where φ_i are $\mathcal{L}$-valued 2-cochains, i.e. elements of $\mathrm{Hom}_{\mathbb{K}}(\Lambda^2\mathcal{L}, \mathcal{L}) = C^2(\mathcal{L}; \mathcal{L})$, and $\mathcal{L}_t$ is a Lie algebra for each $t \in \mathbb{K}$. Two deformations, $\mathcal{L}_t$ and $\mathcal{L}'_t$ are equivalent if there exists a linear automorphism $\widehat{\psi}_t = \mathrm{id} + \psi_1 t + \psi_2 t^2 + \dots$ of $\mathcal{L}$ where ψ_i are linear maps over $\mathbb{K}$, i.e. elements of $C'(\mathcal{L}; \mathcal{L})$ such that

$$\mu'_t(x, y) = \widehat{\psi}_t^{-1}\big(\mu_t(\widehat{\psi}_t(x), \widehat{\psi}_t(y))\big) \quad \text{for} \quad x, y \in \mathcal{L}.$$

The Jacobi identity for the algebras $\mathcal{L}_t$ implies that the 2-cochain φ_1 is indeed a cocycle. If μ'_t is an equivalent deformation with cochains φ'_i, then

$$\varphi'_1 - \varphi_1 = d_1\psi_1$$

hence every equivalence class of deformations defines uniquely an element of $H^2(\mathcal{L};\mathcal{L})$.

We call a deformation infinitesimal if it is a deformation up to order 1. It follows that nonequivalent infinitesimal deformations are in one-to-one correspondence with the 2-dimensional chomology classes.

The problem with the classical theory is that it is not satisfactory to describe all nonequivalent deformations of a given object. For that purpose one has to introduce deformations with base.

b) *General definition*

Consider a deformation $\mathcal{L}_t$ and as a family of Lie algebras, but as a Lie algebra over the algebra $\mathbb{K}[[t]]$. The natural generalization is to allow more parameters, or to take in general a commutative algebra over $\mathbb{K}$ with identity as base of a deformation. Let us fix an augmentation $\varepsilon : A \to \mathbb{K}$, $\varepsilon(1) = 1$ and set $\operatorname{Ker} \varepsilon = m$, which is a maximal ideal.

Definition. A deformation λ of $\mathcal{L}$ with base (A, m) is a Lie A-algebra structure on $A \otimes_{\mathbb{K}} \mathcal{L}$ with bracket $[\ ,\ ]_\lambda$ such that

$$\varepsilon \otimes \mathrm{id} : A \otimes \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

Two deformations of a Lie algebra $\mathcal{L}$ with the same base A are called equivalent if there exists a Lie algebra isomorphism between the two copies of $A \otimes \mathcal{L}$ with the two Lie algebra structure, compatible with $\varepsilon \otimes \mathrm{id}$.

c) ***Formal deformations***

Let A be a complete local algebra. A formal deformation of $\mathcal{L}$ with base A is a Lie A-algebra structure on the completed tensor product $A \widehat{\otimes} \mathcal{L} = \varprojlim_{n \to \infty} \big((A/m^n) \otimes \mathcal{L}\big)$ s.t.

$$\varepsilon \widehat{\otimes} \mathrm{id} : A \widehat{\otimes} \mathcal{L} \longrightarrow \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

d) ***Miniversal formal deformations***

It is known that in the category of algebraic varieties the quotient by a group action does not always exist. Specifically, there is no universal deformation in general of a Lie algebra $\mathcal{L}$ with a commutative algebra base B with the property that for any other deformation of $\mathcal{L}$ with base A there exists a unique homomorphism $f : B \to A$ that induces an equivalent deformation. If such a homomorphism exists (but not unique), we call the deformation of $\mathcal{L}$ with base B *versal*.

Definition. A formal deformation η of a Lie algebra $\mathcal{L}$ with a complete local algebra base B is called *miniversal*, if

i) for any formal deformation λ of $\mathcal{L}$ with any complete local base A there exists a homomorphism $f : B \to A$

s.t. the deformation λ is equivalent to the push-out of η by f;

ii) if A satisfies $m^2 = 0$, then f is unique.

Using Schlessinger's general set-up, I was able to prove that for formal deformations, under some minor restriction, there exists a miniversal deformation.

Theorem. *Assume that the space $H^2(\mathcal{L};\mathcal{L})$ is finite-dimensional. Then there exists a miniversal formal deformation of $\mathcal{L}$, and the base of this deformation is formally embedded into $H^2(\mathcal{L};\mathcal{L})$, i.e. it can be described in $H^2(\mathcal{L};\mathcal{L})$ by a finite system of equations.*

Construction of miniversal deformation

1. Universal infinitesimal deformation

Let $A = \mathbb{K} \oplus H^2(L;L)'$. Fix a homomorphism

$$\mu : H^2(L;L) \to C^2(L;L) = \operatorname{Hom}\bigl(H^2(L;L), L\bigr).$$

Define a Lie A-algebra structure on the space

$$A \otimes L = \mathbb{K} \otimes L \oplus H^2(L;L)' \otimes L = L \oplus \operatorname{Hom}\bigl(H^2(L;L), L\bigr)$$

with the bracket

$$[\ell_1 \otimes \phi_1, \ell_2 \otimes \phi_2] = \bigl([\ell_1, \ell_2], \psi\bigr), \quad \text{where}$$
$$\psi(\alpha) = \mu(x)(\ell_1, \ell_2) + \bigl[\phi_1(\alpha), \ell_2\bigr] + \bigl[\ell_1, \phi_2(\alpha)\bigr]$$
$$\ell_1, \ell_2 \in L, \quad \phi_1, \phi_2 \in \operatorname{Hom}\bigl(H^2(L;L), L\bigr), \quad \alpha \in H^2(L;L).$$

This is an infinitesimal deformation with base A.

Proposition. *The deformation L_A constructed in this way is universal.*

2. The construction follows a recursive procedure. In order to extend the infinitesimal deformation to the next order, we need that all the Massey 2-products are trivial. The Massey products are elements of $H^3(L;L)$, and the base of a miniversal deformation will be the factor of the algebra of formal power series $\mathbb{K}\big[[H^2(L;L)]'\big]$ by some ideal. The ideal is generated by the nontrivial Massey products, corresponding to the elements of $H^3(L;L)$.

e) *Different definitions of miniversal deformation*

The analytic definition of miniversal deformation is different and much earlier. In the Kodaira–Spencer theory a deformation is versal if the Kodaira–Spencer map is surjective, miniversal if the reduced Kodaira–Spencer map is isomorphism.

Remark. The algebraic formulation of these definitions coincides with ours.

Arnold in his article in 1971 studies the analytic space of $n \times n$ matrices. He calls matrix family an $A : \Lambda \to \mathbb{C}^{n^2}$ holomorphic map, where Λ is a neighborhood of the origin in the parameter space $\mathbb{C}^\ell$. A deformation of the matrix $A(0)$ is the germ of the family A in $\mathbb{C}$. Two deformations are equivalent if there exists a similarity transformation between them.

A deformation A of the matrix A_0 is versal if any deformation A' of A_0 is equivalent to the deformation induced from A.

Remark. This definition is the same as ours.

In Arnold's set-up, a versal deformation is miniversal if the dimension of the parameter space of the versal deformation is the least possible.

Proposition. *The notion of miniversal deformation in analytic structures is equivalent to the definition in algebraic set-up.*

Proof. The parameter space of the base can not be less than the dimension of $H^2(L; L)$, because if it were, then we could not get those deformations, the infinitesimal part of which correspond to the left-out cohomology classes.

On the other hand, we constructed a versal deformation, for which the dimension of its parameter space is the dimension of $H^2(L; L)$, and we proved that this is miniversal in algebraic sense. $\square$

Remark. The notion of miniversal deformation in analytic structures is transparent but it is hard to deal with for a given structure. The translation to algebraic structures has the advantage that we can get cohomology computations involved, for which we have known methods.

3 Contractions

A contraction is a procedure somewhat opposite to deformation. Contractions typically transform a Lie algebra into a "more abelian" Lie algebra, and deformations, which are understood in a strict sense, lead to Lie algebra with more intricate Lie brackets.

Contractions are important in physics because they explain in terms of Lie algebra why some theories arise as a limit regime of more "exact" theories. Motivated by the need to relate the symmetries underlying Einstein's mechanics and Newtonian mechanics, Inönü and Wigner introduced the concept of contractions. It consists in multiplying the generators of the symmetry by "contraction parameters" such that when these parameters reach some singularity point, one obtains a non-isomorphic Lie algebra with the same dimension. The method has been generalized a few years later by Saletan. Contractions were used by Lévy-Leblond to emphasize that the condition of largely timelike intervals is just as crucial as the infinite velocity of light, in order to contract the Poincaré algebra to the Galilei algebra. Another physical example is the contraction of the de Sitter algebras to the Poincaré algebra, in the limit of large radius.

The mathematics literature contains various concepts similar to contractions: "degeneration", "orbit closure", "perturbation" etc. Orbit closures arise in many areas of mathematics where algebraic or topological transformation groups are considered, such as invariant theory, representation theory, theory of singularities etc. For algebraic structures on a fixed finite-dimensional vector space, degeneration means that the orbits under the action of the general linear group are the isomorphism classes, and so orbit closure coincides with the closure of these classes.

When computing a contraction, one has a particular Lie algebra in mind, and wants to know all Lie algebras which can "jump" to the one we have in mind.

What does it mean: jump? It is possible that a formal deformation $\mathcal{L}_t$ is well-defined as a Lie algebra for small values of t, for which $\mathcal{L}_0 = \mathcal{L}$, but for which $\mathcal{L}_t \sim \mathcal{L}'$ for all values of t except $t = 0$, where $\mathcal{L}'$ is not equivalent to $\mathcal{L}$. This kind of a deformation family is called a *jump deformation*.

The other type of deformation family $\mathcal{L}_t$ is the following. If $\mathcal{L}_t$ runs along a family of nonequivalent Lie algebras, it is called a *smooth deformation* family.

Definition. Let g_t be a family of automorphisms of V, defined in a punctured neighborhood of zero. If $\lim_{t\to 0}\big(g_t^*(\mathcal{L}')\big) = \mathcal{L}$ and $\mathcal{L}$ is not equivalent to $\mathcal{L}'$, then the Lie algebra $\mathcal{L}$ is a contraction of $\mathcal{L}'$.

The commutation relations of a contracted Lie algebra $\mathcal{L}'$ of $\mathcal{L}$ are given by the limit

$$[x, y]' = \lim_{t\to t_0} g_t^{-1}\big([g_t(x), g_t(y)]\big),$$

where g_t is a non-singular linear transformation of $\mathcal{L}$ with t_0 being a singularity point of its inverse g_t^{-1}. In mathematical terms, the orbits under the action of the general linear group are the Lie algebra isomorphism classes, and the bracket $[\ ,\ ]'$ is a contraction of $[\cdot,\cdot]$ if it is in the Zariski closure of the orbit of $[\cdot,\cdot]$.

To determine all possible contractions of a Lie algebra $\mathcal{L}'$ by finding all automorphisms g_t such that $\lim_{t\to 0}\big(g_t^*(\mathcal{L}')\big) = \mathcal{L}$ exists can be a daunting task. What can be done?

A non-trivial contraction always induces a non-trivial (inverse) jump deformation. The converse is not always true: there are deformations which do not admit an inverse contraction. For example, one can never have a contraction inside a parameterized family of Lie algebras, but deformations within a family exist. Also, nothing can be contracted to the parameterized family, whereas there can be many nontrivial deformations, as we will see on an example later. We should emphasize that the irreversibility occurs only when we have a family of smooth deformations. In other words, there is one-to-one correspondence between contractions and jump deformations.

From constructing a miniversal deformation of a Lie algebra, one can determine all jump deformations. So one can say that the miniversal deformations contain all the information about contractions as well.

The point of view of deformation theory is a bit different from the point of view of contractions. When computing a contraction, one has a particular Lie algebra in mind, and wants to know all Lie algebras which can jump to the one you have in mind. This is quite different from the perspective of deformation theory, where one is interested in seeing what the object of question deforms to. Both perspectives give valuable insights.

In my talk, I present some examples and point out some of the advantages and disadvantages of these two approaches. Some contractions can be computed by use of diagonal matrices.

Theorem (Weimar-Woods, 2000)**.** *If there is a contraction from $\mathcal{L}'$ to $\mathcal{L}$, where $\mathcal{L}$ and $\mathcal{L}'$ are Lie algebra structures on a finite-dimensional space V, then there is a basis of V and an automorphism g_t of V, which has diagonal matrix form* $\operatorname{diag}\left(t^{\lambda_1}, ..., t^{\lambda_n}\right)$*, where λ_i are integers, such that*

$g_t^*(\mathcal{L}')$ *is equivalent to* $\mathcal{L}$.

This Theorem makes it possible to determine all Lie algebras $\mathcal{L}$ which arise as contractions of $\mathcal{L}'$, even if the classification of Lie algebra structures on V is not known. However, it is not true that one can compute all contractions given a fixed basis using diagonal matrices.

4 Examples

a) 3-*dimensional complex Lie algebras*

$\mathbb{C}^3$:	$[x_i, x_j] = 0, \qquad i, j = 1, 2, 3$
$\mathfrak{n}_3(\mathbb{C})$:	$[x_1, x_2] = x_3$
$\mathfrak{r}_3(\mathbb{C})$:	$[x_1, x_2] = x_2, [x_1, x_3] = x_2 + x_3$
$\mathfrak{r}_{3,\lambda}(\mathbb{C}), (\lambda \in \mathbb{C}^*, \lvert\lambda\rvert \leq 1)$:	$[x_1, x_2] = x_2, [x_1, x_3] = \lambda x_3$
$\mathfrak{sl}_2(\mathbb{C})$:	$[x_1, x_2] = x_3, [x_2, x_3] = x_1, [x_3, x_1] = x_2$

Table 1: Three-dimensional complex Lie algebras

The results of contractions and deformations of three-dimensional complex Lie algebras are displayed in Fig. 1. The lines and arrows should be interpreted as follows: an arrow points toward the deformation, whereas a simple line connect Lie algebras related by both deformation and contraction, with the deformed Lie algebra lying upward. The left-pointing arrow symbol over $\mathfrak{r}_{3,\lambda\neq\pm1}(\mathbb{C})$ means that it deforms inside the family.

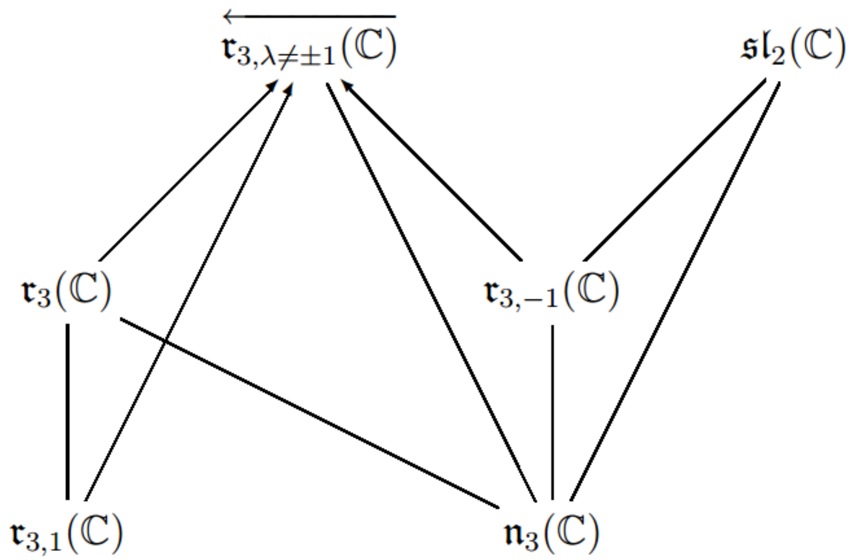

Fig.1 Contractions and deformations of the three-dimensional complex Lie algebras

The family of Lie algebras $\mathfrak{r}_{3,\lambda\neq\pm1}(\mathbb{C})$ has a non-trivial deformation into itself. Note that the two Lie algebras $\mathfrak{r}_{3,1}(\mathbb{C})$ and $\mathfrak{r}_{3,-1}(\mathbb{C})$ are singled out for two reasons. First, $\mathfrak{r}_{3,1}(\mathbb{C})$ can be deformed into $\mathfrak{r}_3(\mathbb{C})$ whereas $\mathfrak{r}_{3,\lambda\neq1}(\mathbb{C})$ cannot, and $\mathfrak{r}_{3,-1}(\mathbb{C})$ can deform into $\mathfrak{sl}_2(\mathbb{C})$, whereas $\mathfrak{r}_{3,\lambda\neq-1}(\mathbb{C})$ cannot. Second, $\mathfrak{r}_{3,1}(\mathbb{C})$ is special because it cannot be contracted to $\mathfrak{n}_3(\mathbb{C})$, unlike $\mathfrak{r}_{3,\lambda\neq1}(\mathbb{C})$.

b) ***Infinite dimensional Lie algebras***

In this section, we discuss deformations and contractions of some infinite dimensional Lie algebras. The physics literature about applications of infinite dimensional Lie algebras, namely to conformal field theory is enormous. Their interest stems from critical phenomena in two dimensions. Whereas the Witt and Virasoro algebras describe local invariance of conformal field theories on the (zero genus) Riemann sphere, the Lie algebras of Krichever–Novikov type discussed hereafter corresponds to higher genus. The physical interpretation of the contraction parameters introduced within these Lie algebras remain to be further explored. The difficulty encountered when deforming the known Lie algebras is that formal deformations are no longer sufficient to describe general deformations. The examples discussed below are formally rigid, so that they admit no non-trivial formal deformations. Nevertheless, there exist very interesting non-trivial global deformations. In the global deformation theory, we no longer have the tool of computing cohomology in order to get deformations, so the picture is much more difficult, and there are very few results so far.

This is where a combination of the contractions and deformations proves really fruitful, since it leads to new infinite dimensional objects, as we will show hereafter. In a domain where so few objects are known explicitly, each new object should be of interest both in mathematics and in physics, particularly, in conformal field theory. The deformations we consider here are over affine varieties, which are very special global deformations.

Witt, Virasoro, and Krichever–Novikov algebras

First, let us consider the Witt algebra $\mathfrak{W}$

$$[l_n, l_m] = (m - n)l_{n+m}, \qquad n, m \in \mathbb{Z}.$$

Its only one-dimensional central extension is the Virasoro algebra $\mathfrak{V}$.

Krichever and Novikov invented the algebras of Virasoro type.

It was shown recently that these infinite-dimensional Lie algebras can be interpreted as global deformations of the Witt or Virasoro algebra.

For simplicity, we will consider deformations of the Witt algebra, but this can be generalized is a natural way to the Virasoro algebra.

Despite its infinitesimal and formal rigidity, which prevents any non-trivial *formal* deformation, the Witt algebra $\mathfrak{W}$ can be non-trivially *globally* deformed into Krichever–Novikov type algebras $\mathfrak{KN}$. Such a phenomenon does not appear with Lie algebras of finite dimension.

An example of Krichever–Novikov algebras $\mathfrak{KN}$ is a two-dimensional family of Lie algebras parameterized over $\mathbb{C}^2$. The generators are given by the fields:

$$
\begin{aligned}
V_{2n+1} &= (X-e_1)^n Y \tfrac{d}{dX}, \\
V_{2n} &= 2(X-e_1)^{n-1}(X-e_2)(X+e_1+e_2)\tfrac{d}{dX},
\end{aligned}
$$

which satisfy the following Lie brackets:

$$
[V_n, V_m] = \begin{cases}
(m-n)V_{n+m}, & n,m \text{ odd}, \\
(m-n)(V_{n+m} + 3\ e_1\ V_{n+m-2}+ & \\
\qquad +(e_1-e_2)(2e_1+e_2)V_{n+m-4}), & n,m \text{ even}, \\
(m-n)V_{n+m} + (m-n-1)\ 3e_1\ V_{n+m-2}+ & \\
\qquad +(n-m-2)(e_1-e_2)(2e_1+e_2)V_{n+m-4}, & n \text{ odd}, m \text{ even}.
\end{cases}
$$

Note that for $e_1 = 0$, $e_2 = 0$, and $Y = X$, we recover the Witt algebra, and the fields reduce to

$$
l_n = X^{n+1}\frac{d}{dX}.
$$

A different $\mathfrak{KN}$ algebra can be obtained as a one-parameter global deformation, by taking the following field basis:

$$
V_{2n} \equiv X(X-\alpha)^n(X+\alpha)^n\frac{d}{dX}, \qquad V_{2n+1} \equiv (X-\alpha)^{n+1}(X+\alpha)^{n+1}\frac{d}{dX}.
$$

One calculates the Lie brackets:

(1)

$$
[V_n, V_m] = \begin{cases}
(m-n)V_{n+m}, & n,m \text{ odd}, \\
(m-n)(V_{n+m} + \alpha^2 V_{n+m-2}), & n,m \text{ even}, \\
(m-n)V_{n+m} + (m-n-1)\alpha^2 V_{n+m-2}, & n \text{ odd}, m \text{ even}.
\end{cases}
$$

There are many other ways as well, if we specify the base of the deformation being different affine lines in $\mathbb{C}^2$.

Clearly, they can be contracted back to the Witt algebra. Let us show it on the second type, equation (1), by utilizing a very elegant and simple Weimar-Woods contraction: if we define

$$l_n \equiv \varepsilon^n V_n, \qquad \text{for all } n \in \mathbb{Z}, \tag{2}$$

then equation (1) becomes

$$\begin{aligned} [l_n, l_m]_\varepsilon &= \varepsilon^{n+m}[V_n, V_m] \\ &= \begin{cases} (m-n)l_{n+m}, & n, m \text{ odd}, \\ (m-n)(l_{n+m} + \varepsilon^2\alpha^2 l_{n+m-2}), & n, m \text{ even}, \\ (m-n)l_{n+m} + (m-n-1)\varepsilon^2\alpha^2 l_{n+m-2}, & n \text{ odd}, m \text{ even}. \end{cases} \end{aligned}$$

Then, it is clear that, in the limit where ε approaches zero, we retrieve the commutation relations of $\mathfrak{W}$. Therefore, the operations of deformation and contraction are mutually reversible in this case.

In addition to retrieving the Witt algebra $\mathfrak{W}$, one may contract $\mathfrak{KN}$ to other, so far unknown, Lie algebras. Let us discuss an example of such a contraction of $\mathfrak{KN}$ which turns out to be a deformation of the respective contraction of $\mathfrak{W}$. Moreover, this exotic contration of $\mathfrak{KN}$ may be contracted back to the corresponding contraction of $\mathfrak{W}$ by utilizing equation (2). In order to do so, let us define $\mathcal{U}_\varepsilon$ *à la* Weimar-Woods:

$$\mathcal{U}_\varepsilon \equiv \varepsilon^{n_0}\mathrm{id}_{\mathfrak{g}_0} + \varepsilon^{n_1}\mathrm{id}_{\mathfrak{g}_1}, \tag{3}$$

where 0 and 1 denote the even and odd sectors of the powers of $\mathfrak{KN}$, respectively. Then, if we take the Lie brackets (1) as a specific example, we obtain the modified brackets:
(4)

$$[V_n, V_m]_\varepsilon = \begin{cases} \varepsilon^{2n_1-n_0}(m-n)V_{n+m}, & n, m \text{ odd}, \\ \varepsilon^{n_0}(m-n)(V_{n+m}+\alpha^2 V_{n+m-2}), & n, m \text{ even}, \\ \varepsilon^{n_0}[(m-n)V_{n+m}+(m-n-1)\alpha^2 V_{n+m-2}], & n \text{ odd}, m \text{ even}. \end{cases}$$

Clearly, we must have non-negative values of n_0 and $2n_1-n_0$. We obtain the trivial abelian Lie algebra when these expressions take on positive values. Another trivial contraction is given by $n_0 = n_1 = 0$; then it leaves the commutators of equation (1) unchanged.

Two more contractions may be obtained with the splitting of equation (3). One is the Inönü–Wigner contraction, given by $n_0 = 0$ and $n_1 = 1$ (or any n_1 positive); then the contracted commutation relations read:
(5)

$$[V_n, V_m] = \begin{cases} 0, & n, m \text{ odd}, \\ (m-n)(V_{n+m}+\alpha^2 V_{n+m-2}), & n, m \text{ even}, \\ (m-n)V_{n+m}+(m-n-1)\alpha^2 V_{n+m-2}, & n \text{ odd}, m \text{ even}. \end{cases}$$

This defines a new family of infinite dimensional Lie algebras. In the spirit of sequences of contractions, if we further contract these Lie algebras with $\mathcal{U}_\varepsilon$ such as defined in equation (2), then we obtain

$$[l_n, l_m] = \begin{cases} 0, & n, m \text{ odd}, \\ (m-n) l_{n+m}, & n \text{ or } m \text{ even}. \end{cases}$$

This algebra is clearly non-isomorphic to $\mathfrak{W}$. Indeed, it is a contraction of $\mathfrak{W}$, obtained with equation (3), utilized this time with 0 being the even sector, and 1 the odd sector of $\mathfrak{W}$.

The second contraction of the $\mathfrak{KN}$ algebra of equation (1) is constructed by choosing $n_0 > 0$ and $2n_1 - n_0 = 0$ in equation (4):

$$[V_n, V_m] = \begin{cases} (m-n) V_{n+m}, & n, m \text{ odd}, \\ 0, & n, m \text{ even}, \\ 0, & n \text{ odd}, m \text{ even}. \end{cases}$$

Again, this algebra is not isomorphic to the Witt algebra $\mathfrak{W}$.