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Classification of Complex Projective Towers up to Dimension 8 and Cohomological Rigidity

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$\boxed{\text{Def}}$ Complex projective tower ($\mathbb{C}P$ - tower) of height m (or m - stage $\mathbb{C}P$ - tower)

$$C_m \xrightarrow{\pi_m} C_{m-1} \xrightarrow{\pi_{m-1}} \cdots\cdots \quad \cdot \quad \cdots \longrightarrow C_1 \longrightarrow C_0 = \{pt\}$$

where $C_i = P(\xi_{i-1})$: projectivization of ξ_{i-1}

& $\xi_{i-1} \longrightarrow C_{i-1}$: $\mathbb{C}$ - vector bundle over C_{i-1}.

Each C_i is called the ith stage of the tower

In particular, $C_1 = \mathbb{C}P^{n_1}$ for some $n_1 \in \mathbb{N}$.

$\boxed{\text{Examples}}$

(1). Hirzebruch Surface

$$H_p = P(\gamma^p \oplus \underline{\mathbb{C}}) \longrightarrow \mathbb{C}P^1$$

where $\gamma \longrightarrow \mathbb{C}P^1$ is the tautological line bundle

& $\gamma^p = \underbrace{\gamma \otimes \cdots \otimes \gamma}_{p\text{-times}}$

Facts (a). $H_p \cong H_q$ as algebraic varieties

$\iff |p| = |q|$

(b) $H_p \cong H_q$ (diffeomorphic)

$\iff p \equiv q \pmod 2$

$\iff H^*(H_p : \mathbb{Z}) \cong H^*(H_q : \mathbb{Z})$ as graded rings

(2) Bott towers of height m

$$B_m \longrightarrow B_{m-1} \longrightarrow \cdots\cdots \longrightarrow B_1 \longrightarrow B_0 = \{pt\}$$

$$B_1 = \mathbb{C}P^1$$

where $\quad B_i = P(\gamma_{i-1} \oplus \underline{\mathbb{C}})$

& $\quad \gamma_{i-1} \longrightarrow B_{i-1}$: $\mathbb{C}$-line bundle over B_{i-1}.

In particular, Hirzebruch surfaces are Bott towers of height 2.

(3) Generalized Bott towers of height m

$$B_m \to B_{m-1} \to \cdots\cdots\cdots \to B_1 \to B_0 = \{pt\}$$

$$B_1 = \mathbb{C}P^{n_1}.$$

where $\quad B_i = P(\eta_{i-1})$

& $\quad \eta_{i-1} = \bigoplus \mathbb{C}$-line bundles over B_{i-1}

(Generalized) Bott towers have toric manifold structures, namely they are nonsingular normal varieties with the half dimensional complex torus action having a dense orbit.

(4) Flag manifolds

$$Fl(\mathbb{C}^{n+1}) = \{ 0 \subset V_1 \subset \cdots\cdots \subset V_n \subset \mathbb{C}^{n+1} \mid \dim V_k = k \}$$

$$\cong U(n+1)/T^{n+1}, \quad T^{n+1} = S^1 \times \cdots \times S^1$$

Note that $\mathbb{C}P^n \cong U(n+1)/U(1)\times U(n)$

$$\& \quad M/H = M \times_K (K/H).$$

$$\begin{tikzcd}
U(n+1) \times_{U(1)\times U(n)} (U(n) \times_{U(1)\times U(n-1)} (U(n-1) \times_{U(1)\times U(n-2)} \cdots\cdots (U(3) \times_{U(1)\times U(2)} \mathbb{C}P^1) \cdots\cdots) \arrow[d] \arrow[r, phantom, "="] & C_n \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
U(n+1) \times_{U(1)\times U(n)} (U(n) \times_{U(1)\times U(n-1)} \mathbb{C}P^{n-2}) \arrow[d] \arrow[r, phantom, "="] & C_3 \arrow[d] \\
U(n+1) \times_{U(1)\times U(n)} \mathbb{C}P^{n-1} \arrow[d] \arrow[r, phantom, "="] & C_2 \arrow[d] \\
\mathbb{C}P^n \arrow[r, phantom, "="] & C_1
\end{tikzcd}$$

- Cohomological Rigidity Problem

$\boxed{\text{Def}}$ A class of smooth manifolds (resp. topological manifolds) $\mathcal{F}$ is said to be cohomologically rigid if $\forall\, M, N \in \mathcal{F}$

M and N are diffeomorphic (resp. homeomorphic)

$\Leftrightarrow$ $H^*(M;\mathbb{Z}) \cong H^*(N;\mathbb{Z})$ as graded rings.

$\boxed{\text{Cohomological Rigidity Question for toric manifolds}}$

Is the class of toric manifolds cohomologically rigid?

So far there is no negative answer to the question, but some positive partial answers for

(generalized) Bott towers

[Masuda-Panov] [Choi-Masuda-S]

$B_m :=$ m-stage generalized Bott tower.

If $H^*(B_m ; \mathbb{Z}) \cong H^*(\prod_{i=1}^{m} \mathbb{C}P^{n_i} ; \mathbb{Z})$,

then $B_m \cong \prod^{m} \mathbb{C}P^{n_i}$ (diffeomorphic)

[Choi-Masuda-S], [Choi]

The class of m-stage Bott towers is cohomologically rigid for $m \leq 4$.

[Choi-Masuda-S]

The class of 2-stage gen. Bott towers is coh. rig. $\square$

[Choi-S]

The class of Bott towers with one-twist is coh. rig. $\square$

A generalized Bott tower B_m is Q-trivial if $H^*(B_m : \mathbb{Q}) \cong H^*(\prod_{i=1}^m \mathbb{C}P^{n_i} : \mathbb{Q})$.

[Park-S]

Any Q-trivial gen. Bott tower with $n_i \geq 2$ $\forall_{i=1,\cdots,m}$ is diffeomorphic to $\prod_{i=1}^m \mathbb{C}P^{n_i}$.

For real (generalized) Bott towers.

Real (generalized) Bott tower is defined similarly to the complex case except that all involved bundles are sum of real line bundles.

[Kamishima - Masuda]

Two real Bott towers B_m and B_m' are diffeomorphic $\Longleftrightarrow$ $H^*(B_m ; \mathbb{Z}_2) \cong H^*(B_m' : \mathbb{Z}_2)$

[Masuda]

$\exists$ two generalized Bott towers B_m and B_m' with $H^*(B_m ; \mathbb{Z}_2) \cong H^*(B_m' ; \mathbb{Z}_2)$, but $B_m \not\cong B_m'$.

Cohomological Rigidity Question for other classes of manifolds.

- For quasitoric manifolds:
  There are some positive partial answers but no negative example yet.

- For torus manifold

[Kuroki] $M = \mathbb{C}P^1 \times S^{2\ell}$, $M' = S^3 \times_{S^1} (\mathbb{C} \oplus \mathbb{R}^{2\ell-1})$

M & M' admits $T^{\ell+1}$-action with fixed points.
$\therefore$ They are torus manifolds.
$H^*(M;\mathbb{Z}) \cong H^*(M';\mathbb{Z})$ but $M \not\cong M'$.

The construction of $\mathbb{C}P$ towers resembles that of generalized Bott towers, but $\mathbb{C}P$ towers are not toric manifolds in general.

$\boxed{\text{Question}}$ Is the class of $\mathbb{C}P$-towers coh. rig.?

$\boxed{\text{Answer}}$

(I) Yes, for $\mathbb{C}P$ towers of dimension ≤ 6.
(II) No, for $\mathbb{C}P$ towers of dimension $= 8$.

Indeed, there are two $\mathbb{C}P$ towers of $\dim = 8$

$$C_2 \longrightarrow C_1 = \mathbb{C}P^3 \longrightarrow C_0$$
$$C_2' \longrightarrow C_1' = \mathbb{C}P^3 \longrightarrow C_0$$

s.t. $C_2 \not\cong C_2'$ (because $\pi_6(C_2) \not\cong \pi_6(C_2')$)

but $H^*(C_2; \mathbb{Z}) \cong H^*(C_2'; \mathbb{Z})$.

Note that the above C_2 and C_2' are not 2-stage generalized Bott towers because we already know that 2-stage generalized Bott manifolds are cohomologically rigid.

For (I), we classify all $\mathbb{C}P$ towers of $\dim \leq 6$ up to diffeomorphism, and show that their cohology rings are all distinct.

§ Some Preliminaries

[Borel-Hirzebruch]

X : topological space

ξ : $\mathbb{C}$-vector bundle of rank n over X.

$\Rightarrow$

$$H^*(P(\xi):\mathbb{Z}) \cong H^*(X:\mathbb{Z})[x] \Big/ \left\langle x^{n+1} + \sum_{i=1}^{n} (-1)^i c_i(\pi^*\xi)\, x^{n+1-i} \right\rangle$$

where $\pi^*\xi$ is the pull back of ξ along $\pi : P(\xi) \to X$,

x = 1st Chern class of the tautological line bundle over $P(\xi)$

Apply the Borel - Hirzebruch formula to a $\mathbb{C}P$-tower

$$C_m \longrightarrow C_{m-1} \longrightarrow \cdots \cdots \longrightarrow C_1 \longrightarrow C_0$$

With $C_i = P(\xi_{i-1})$, to get

$$H^*(C_m : \mathbb{Z}) \cong H^*(C_{m-1} : \mathbb{Z})[x_m] \Big/ \left< x_m^{n_m+1} + \sum_{i=1}^{n_m} (-1)^i c_i(\xi_{m-1})\, x_m^{n_m+1-i} \right>$$

$$\cong \cdots\cdots\cdots$$

$$\cong \mathbb{Z}[x_1, \cdots, x_m] \Big/ \left< x_k^{n_k+1} + \sum_{i=1}^{n_k} (-1)^i c_i(\xi_{k-1})\, x_k^{n_k+1-i} \;\middle|\; k = 1, \cdots, m \right>$$

Proposition 1

ξ : $\mathbb{C}$-vector bundle over X.
γ : $\mathbb{C}$-line bundle over X.
$\Rightarrow$ $P(\xi)$ is diffeomorphic to $P(\xi \otimes \gamma)$.

Proposition 2

ξ : $\mathbb{C}$-vector bundle of rk 2 over X
γ : $\mathbb{C}$-line bundle over X.

$$\Rightarrow \quad c_1(\xi \otimes \gamma) = c_1(\xi) + 2c_1(\gamma)$$
$$c_2(\xi \otimes \gamma) = c_1(\gamma)^2 + c_1(\gamma)c_1(\xi) + c_2(\xi).$$

Proof of Proposition 2

Consider

$$\begin{tikzcd} \pi^*\xi \otimes \pi^*(\gamma) \arrow[r] \arrow[d] & \xi\otimes\gamma \arrow[d] \\ P(\xi\otimes\gamma) \arrow[r, "\pi"] \arrow[dr, "\varphi"', "\cong"] \arrow[r, phantom, shift right=3, "\circlearrowright"] & X \\ & P(\xi) \arrow[u, "\pi_\xi"'] \end{tikzcd}$$

Let γ_ξ : tautological line bundle in $\pi_\xi^*(\xi)$

$\Rightarrow \quad \pi_\xi^*(\xi) = \gamma_\xi \oplus \gamma_\xi^\perp$ (Note: $\mathrm{rk}\,\gamma_\xi^\perp = 1$)

$$\begin{aligned}
\pi^*(c(\xi\otimes\gamma)) &= c(\pi^*(\xi\otimes\gamma)) \\
&= c(\pi^*\xi \otimes \pi^*\gamma) \\
&= c(\varphi^*\pi_\xi^*\xi \otimes \pi^*\gamma) \\
&= c(\varphi^*(\gamma_\xi \oplus \gamma_\xi^\perp) \otimes \pi^*\gamma) \\
&= c(\varphi^*\gamma_\xi \otimes \pi^*\gamma)\, c(\varphi^*(\gamma_\xi^\perp \otimes \pi^*\gamma)) \\
&= (1+\varphi^* c_1(\gamma_\xi) + \pi^* c_1(\gamma))(1+\varphi^* c_1(\gamma_\xi^\perp) + \pi^* c_1(\gamma))
\end{aligned}$$

$$\therefore\ \pi^* c_1(\xi \otimes \gamma) = \varphi^* c_1(\gamma_\xi) + \pi^* c_1(\gamma) + \varphi^* c_1(\gamma_\xi^\perp) + \pi^* c_1(\gamma)$$

$$\pi^* c_2(\xi \otimes \gamma) = (\varphi^* c_1(\gamma_\xi) + \pi^* c_1(\xi))(\varphi^* c_1(\gamma_\xi^\perp) + \pi^* c_1(\xi))$$

On the other hand

$$\pi^* c_1(\xi) = \varphi^* c_1(\pi_\xi^* \xi) = \varphi^* c_1(\gamma_\xi \oplus \gamma_\xi^\perp)$$

$$= \varphi^* c_1(\gamma_\xi) + \varphi^* c_1(\gamma_\xi^\perp)$$

$$\pi^* c_2(\xi) = \varphi^* c_2(\pi_\xi^* \xi) = \varphi^* c_1(\gamma_\xi)\, c_1(\gamma_\xi^\perp)$$

$$\therefore\ \pi^* c_1(\xi \otimes \gamma) = \pi^* c_1(\xi) + 2\pi^* c_1(\gamma).$$

$$\pi^* c_2(\xi \otimes \gamma) = \pi^* c_2(\xi) + \pi^* c_1(\xi)\pi^* c_1(\gamma) + \pi^* c_1(\gamma)^2$$

Since $\pi^* : H^*(X) \longrightarrow H^*(P(X))$ is injective, we have the desired equalities. $\square$

§ Classification of $\mathbb{C}P$ towers of dim ≤ 6.

I. dim 2 $\mathbb{C}P$-tower $= \mathbb{C}P^1$

II. dim 4 $\mathbb{C}P$-towers

(1) height 1 : $\mathbb{C}P^2$

(2) height 2 : Hirzebruch surfaces

$$H_0 = \mathbb{C}P^1 \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1 \longrightarrow \{pt\}$$

$$H_1 = P(\gamma \oplus \mathbb{C}) \longrightarrow \mathbb{C}P^1 \longrightarrow \{pt\}$$

$$\cong \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$$

III. dim 6 $\mathbb{C}P$- towers

(1) height 1 : $\mathbb{C}P^3$

(2) height 2 : $C_2 \longrightarrow C_1 \longrightarrow \{pt\}$

There are two cases (a) $C_1 = \mathbb{C}P^1$ case
& (b) $C_1 = \mathbb{C}P^2$ case.

(a) If $C_1 = \mathbb{C}P^1$, then $C_2 = P(\xi_1)$

where ξ_1 is a rk 3 $\mathbb{C}$-vector bundle over $\mathbb{C}P^1$.

By the dimension reason,

$\xi_1 = \gamma_1 \oplus \gamma_2 \oplus \gamma_3$: sum of line bundles

$\Rightarrow$ C_2 is a 2-stage Bott tower.

In [Choi-Masuda-S] such 2-stage Bott towers are classified up to diffeomorphisms:

$$C_2 \cong P(\gamma^k \oplus 2\mathbb{C}), \quad k = 0, 1, 2$$

where $\gamma \to \mathbb{C}P^1$ is the tautological line bundle.

(b) If $C_1 = \mathbb{C}P^2$, then $C_2 = P(\xi_1)$ where $\xi_1 \longrightarrow \mathbb{C}P^2$ is a rk 2 complex vector bundle

Theorem [Schwarzenberger 1961]

There is a bijection

$$\begin{aligned} \mathrm{Vect}^2(\mathbb{C}P^2) &\xrightarrow{\phi} H^2(\mathbb{C}P^2) \oplus H^4(\mathbb{C}P^2) \\ \eta &\longmapsto (c_1(\eta), c_2(\eta)) \end{aligned}$$

$\square$

By Proposition 1 & 2, in order to classify $C_2 = P(\xi_1)$ up to diffeomorphism, we may assume that

$$c_1(\xi_1) \equiv 0 \text{ or } 1 \in \mathbb{Z}_2$$

$\therefore$ All 6-dimensional height 2 $\mathbb{C}P$-towers are

$$C_2 = P(\gamma^k \oplus 2\underline{\mathbb{C}}) \longrightarrow C_1 = \mathbb{C}P^1 \longrightarrow \{pt\}$$

for $k = 0, 1, 2$

and

$$C_2 = P(\eta_{(s,\alpha)}) \longrightarrow C_1 = \mathbb{C}P^2 \longrightarrow \{pt\}$$

where $c_1(\eta_{(s,\alpha)}) = s$

$c_2(\eta_{(s,\alpha)}) = \alpha$

for any $s = 0$ or 1 and $\alpha \in \mathbb{Z}$

We have the following classification result of 6-dim. $\mathbb{C}P$-towers of height 2.

Let CPT^n_k = the class of n-dim $\mathbb{C}P$-towers with height k. $/\underset{\text{diffeom}}{\cong}$

$\boxed{\text{Theorem 1}}$

CPT^6_2 consists of the following distinct $\mathbb{C}P$-towers

$$P(\gamma \oplus 2\underline{\mathbb{C}}) \longrightarrow \mathbb{C}P^1$$
$$P(\gamma^2 \oplus 2\underline{\mathbb{C}}) \longrightarrow \mathbb{C}P^1$$
$$P(\eta_{(0,\alpha)}) \longrightarrow \mathbb{C}P^2 \quad \text{for } \alpha \in \mathbb{Z}$$
$$P(\eta_{(1,\beta)}) \longrightarrow \mathbb{C}P^2 \quad \text{for } \beta \in \mathbb{Z}.$$

Moreover the cohomology rings of such manifolds are mutually non-isomorphic. $\square$

(3) height 3 : $C_3 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow pt$

In this case C_2 = Hirzebruch surface H_k, $k=0,1$ and $C_3 = P(\xi_2)$ where ξ_2 is a rk 2 complex vector bundle over H_k.

We have a similar results on rk 2 complex vector bundles on H_k to [Schwarzenberger 1961].

Proposition 3 There is a bijection

$$\begin{aligned} \mathrm{Vect}^2(\mathbb{C}P^2) &\xrightarrow{\phi} H^2(H_k) \oplus H^4(H_k) \\ \eta &\longmapsto (c_1(\eta), c_2(\eta)) \end{aligned}$$

for $H_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$
$H_1 = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

Sketch Proof of Proposition 3

Since $\dim_{\mathbb{R}} H_i = 4$, we can see that

$$\mathrm{Vect}^2_{\mathbb{C}}(H_k) \longrightarrow \tilde{K}(H_k), \quad \xi \longmapsto [\xi]$$

is bijective. It is enough to show that

$$c' : \tilde{K}(H_k) \longrightarrow H^2(H_k) \oplus H^4(H_k),$$
$$[\xi] \longmapsto (c_1(\xi), c_2(\xi))$$

is bijective

Consider $\mathbb{C}P^1 \vee \mathbb{C}P^1 \longrightarrow H_k \longrightarrow \underbrace{H_k / \mathbb{C}P^1 \vee \mathbb{C}P^1}_{\cong S^4}$

base space fiber

$$\Rightarrow \quad \begin{tikzcd} & \tilde{K}(S^4) \arrow[r] \arrow[d, "c_2"', "\cong"] \arrow[dr, phantom, "\circlearrowright"] & \tilde{K}(H_k) \arrow[r] \arrow[d, "c'"] \arrow[dr, phantom, "\circlearrowright"] & \tilde{K}(\mathbb{C}P^1 \vee \mathbb{C}P^1) \arrow[d, "(c_1, c_1)"] & \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & H^2(H_k) \oplus H^4(H_k) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & 0 \\ & & (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z} & & \end{tikzcd}$$

If we prove that c' is surjective, then from the exactness of horizontal sequence and the commutativity of the diagram we can have the desired bijectivity.

Showing the surjectivity of c' is not difficult. $\square$

All 6-dimensional $\mathbb{C}P$-towers of height 3 are

$$C_3 = P(\eta_{(p,q,\alpha)}) \longrightarrow C_2 = H_0 \longrightarrow C_1 = \mathbb{C}P^1 \longrightarrow \{pt\}$$

and

$$C_3 = P(\xi_{(r,s,\beta)}) \longrightarrow C_2 = H_1 \longrightarrow C_1 = \mathbb{C}P^1 \longrightarrow \{pt\}$$

where $c_1(\eta_{(p,q,\alpha)}) = (p,q) \in H^2(H_0) \cong \mathbb{Z} \oplus \mathbb{Z}$

$c_2(\eta_{(p,q,\alpha)}) = \alpha \in H^4(H_0) \cong \mathbb{Z}$.

and similarly for $c_i(\xi_{(r,s,\beta)}) \in H^{2i}(H_1)$.

Again by Proposition 1 & 2, we may assume that

$$p, q, r, s \in \mathbb{Z}_2$$

Theorem 2 CPT^6_3 consists of the following distinct $\mathbb{C}P$-towers

$P(\eta_{(0,0,\alpha)}) \longrightarrow H_0$ for $\alpha \in \mathbb{Z}_{\geq 0}$

$P(\eta_{(1,0,\alpha)}) \longrightarrow H_0$ for $\alpha \in \mathbb{Z}_{\geq 0}$

$P(\eta_{(1,1,\alpha)}) \longrightarrow H_0$ for $\alpha \in \mathbb{N}$

$P(\xi_{(0,0,\beta)}) \longrightarrow H_1$ for $\beta \in \mathbb{N}$

$P(\xi_{(1,0,\beta)}) \longrightarrow H_1$ for $\beta \in \mathbb{Z}_{\geq 0}$

$P(\xi_{(0,1,\beta)}) \longrightarrow H_1$ for $\beta \in \mathbb{Z}$.

Moreover the cohomology rings of these manifolds are mutually non-isomorphic. $\square$

Remark We have the following diffeomorphisms.

(1) $P(\eta_{(1,0,\alpha)}) \cong P(\eta_{(0,1,\alpha)})$

(2) $P(\eta_{(0,0,\alpha)}) \cong P(\eta_{(0,0,-\alpha)})$

(3) $P(\eta_{(1,0,\alpha)}) \cong P(\eta_{(1,0,-\alpha)})$

(4) $P(\eta_{(1,1,\alpha)}) \cong P(\eta_{(1,1,-\alpha+1)})$

(5) $P(\xi_{(0,0,\beta)}) \cong P(\xi_{(0,0,-\beta)})$

(6) $P(\xi_{(1,0,\beta)}) \cong P(\xi_{(1,0,-\beta)})$

(7) $P(\xi_{(0,1,\beta)}) \cong P(\xi_{(1,1,-\beta)})$

(8) $P(\eta_{(1,0,0)}) \cong P(\xi_{(0,0,0)})$

Proof of Remark

(1) $P(\eta_{(1,0,\alpha)}) \cong P(\eta_{(0,1,\alpha)})$ follows from the self diffeomorphism

$$\phi : H_0 = \mathbb{C}P^1 \times \mathbb{C}P^1 \circlearrowleft \qquad (x, y) \longmapsto (y, x).$$

(2) $P(\eta_{(0,0,\alpha)}) \cong P(\eta_{(0,0,-\alpha)})$

(3) $P(\eta_{(1,0,\alpha)}) \cong P(\eta_{(1,0,-\alpha)})$

(4) $P(\eta_{(1,1,\alpha)}) \cong P(\eta_{(1,1,-\alpha+1)})$

follows from the following argument.

Consider the self diffeomorphism

$$f : H_0 = \mathbb{C}P^1 \times \mathbb{C}P^1 \circlearrowleft \qquad (x, y) \longmapsto (x, \bar{y}).$$

Then f is an orientation reversing map.

$$\therefore \quad e(f^*(\eta_{(p,q,\alpha)})) = c_2(f^*(\eta_{(p,q,\alpha)})) = -c_2(\eta_{(p,q,r)}) = -\alpha$$

$$H^*(\mathbb{C}P^1 \times \mathbb{C}P^1) \cong \mathbb{Z}[x, y] / \langle x^2, y^2 \rangle$$

$\Rightarrow c_1(f^*\eta_{(p,q,\alpha)}) = f^*(px+qy) = px - qy.$

$\therefore f^*(\eta_{(p,0,\alpha)}) \cong \eta_{(p,0,-\alpha)}$

which proves the isomorphisms (2) & (3).

Let γ_2 be the line bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$ with $c_1(\gamma_2) = y$.

$$\Rightarrow c_1(f^*(\eta_{(1,1,\alpha)}) \otimes \gamma_2) = c_1(f^*(\eta_{(1,1,\alpha)}) + 2c_1(\gamma_2)$$
$$= x - y + 2y = x + y$$
$$c_2(f^*(\eta_{(1,1,\alpha)}) \otimes \gamma_2) = c_1(\gamma_2)^2 + c_1(\gamma)\, c_1(f^*(\eta_{(1,1,\alpha)})$$
$$+ c_2(f^*(\eta_{(1,1,\alpha)})$$
$$= y^2 + y(x-y) - \alpha xy$$
$$= (1-\alpha)xy$$

$\therefore f^*(\eta_{(1,1,\alpha)} \otimes \gamma) \cong \eta_{(1,1,1-\alpha)}$

which induces the isomorphism (4).

(5) $P(\xi_{(0,0,\beta)}) \cong P(\xi_{(0,0,-\beta)})$

(6) $P(\xi_{(1,0,\beta)}) \cong P(\xi_{(1,0,-\beta)})$

(7) $P(\xi_{(0,1,\beta)}) \cong P(\xi_{(1,1,-\beta)})$

follows from the following argument.

Note that

$$H_1 \cong S^3 \times_{S^1} P(t^1 \oplus \mathbb{C})$$

where $t^k = \mathbb{C}$ is the irred. rep of S^1, $g \cdot z = g^k \cdot z$,

and

$$H^*(H_1) \cong \mathbb{Z}[x,y] / \langle x^2,\ y^2 + xy \rangle$$

Now consider the self diffeomorphism on H_1

$$\begin{tikzcd} f : S^3 \times_{S^1} P(t^1 \oplus \mathbb{C}) \arrow[r, "g"] \arrow[d, equal] & S^3 \times_{S^1} P(t^{-1} \oplus \mathbb{C}) \arrow[r, "h"] \arrow[d, equal] & S^3 \times_{S^1} P(t^1 \oplus \mathbb{C}) \arrow[d, equal] \\ P(\gamma \oplus \mathbb{C}) & P(\gamma^{-1} \oplus \mathbb{C}) & P(\gamma \oplus \mathbb{C}) \end{tikzcd}$$

ori. reversing of the fiber

$\otimes \gamma$

$\Rightarrow$ $f^*: H^*(H_1) \circlearrowleft$ maps $f^*(x) = x$ & $f^*(y) = -x-y$

Then we have the bundle isomorphisms

$$f^*\xi_{(0,0,\beta)} \cong \xi_{(0,0,-\beta)}$$

$$f^*\xi_{(1,0,\beta)} \cong \xi_{(1,0,-\beta)}$$

which proves the isomorphisms (5) & (6), and

$$f^*\xi_{(0,1,\beta)} \cong \xi_{(-1,-1,\beta)}.$$

Let γ_{x+y} the line bundle on H_1 with $c_1(\gamma_{x+y}) = x+y$

$\Rightarrow$

$$\gamma_{x+y} \otimes \xi_{(-1,-1,\beta)} \cong \xi_{(1,1,-\beta)}$$

which proves the isomorphism (7)

(8) $P(\eta_{(1,0,0)}) \cong P(\xi_{(0,0,0)})$ is obvious.

$\square$

$\boxed{\text{Corollary}}$ [Cohomological Rigidity of $\mathbb{C}P$ towers of dim ≤ 6]

The class of $\mathbb{C}P$ towers of dimension ≤ 6 is cohomologically rigid.

Method of Proof

We use Borel-Hirzebruch formula for cohomology rings $\mathbb{C}P$-towers, plus long computations on truncated polynomial rings.

§ Cohomological non-rigidity for $\mathbb{C}P$ towers of dim $= 8$.

Consider 8 dim $\mathbb{C}P$-towers of heigh 2 with $C_1 = \mathbb{C}P^3$, i.e.

$$C_2 = P(\eta) \longrightarrow C_1 = \mathbb{C}P^3 \longrightarrow \{pt\}$$

where η is a rk 2 complex vector bundle over $\mathbb{C}P^3$.

$\boxed{\text{Theorem}}$ [Atiyah – Rees 1976]

$\exists$ injective map

$$\begin{aligned} \phi : \mathrm{Vect}_2(\mathbb{C}P^3) &\longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \\ \xi &\longmapsto (\alpha(\xi), c_1(\xi), c_2(\xi)) \end{aligned}$$

where $\alpha(\xi) = 0$ if $c_1(\xi) \equiv 1 \pmod 2$.

$\square$

$\therefore$ All 8-dim $\mathbb{C}P$-tower of height 2 with $c_1 = \mathbb{C}P^3$ are

(1) $M_0(u) = P(\eta_{(0,0,u)})$

(2) $M_1(u) = P(\eta_{(1,0,u)})$

(3) $N(u) = P(\eta_{(0,1,u)})$

(Again we may assume that $c_1(\eta) = 0$ or 1)

$\boxed{\text{Theorem 3}}$ TFAE

(1) $N(u)$ is diffeomorphic to $N(u')$

(2) $H^*(N(u);\mathbb{Z}) \cong H^*(N(u');\mathbb{Z})$

(3) $u = u' \in \mathbb{Z}$.

$\square$

$\boxed{\text{Theorem 4}}$ Assume $u(u+1)/12 \in \mathbb{Z}$

(1) $M_\alpha(u) \overset{\text{diff}}{\cong} M_\beta(u') \iff (\alpha, u) = (\beta, u') \in \mathbb{Z}_2 \times \mathbb{Z}$.

(2) $H^*(M_\alpha(u)) \cong H^*(M_\beta(u)) \iff u = u' \in \mathbb{Z}$ $\square$

$\boxed{\text{Corollary}}$

The class of $\mathbb{C}P$ towers of dim 8 is not cohomologically rigid.

§ Sketch of Proof of Theorem 4 (1)

Consider the pull-back diagram

$$\begin{tikzcd}
 & \mathbb{C}P^1 \arrow[r, "\cong"] \arrow[d] & \mathbb{C}P^1 \arrow[r, "\cong"] \arrow[d] & \mathbb{C}P^1 \arrow[d] \\
 & P(\xi_{\alpha,\nu}) := p^*(M_\alpha(\nu)) \arrow[r] \arrow[d] & M_\alpha(\nu) \arrow[r] \arrow[d] & EU(2) \times_{U(2)} \mathbb{C}P^1 \arrow[d] \\
S^1 \arrow[r] & S^7 \arrow[r, "p"] & \mathbb{C}P^3 \arrow[r, "\mu_{\alpha,\nu}"] & BU(2)
\end{tikzcd}$$

From the homotopy exact sequences of the fibrations, we can see that

$$\pi_*(P(\xi_{\alpha,\nu})) \xrightarrow{\cong} \pi_*(M_\alpha(\nu)) \quad \text{for } * \geq 3 .$$

Theorem 4 (1) follows from the following.

$\boxed{\text{Proposition}}$ Assume $u(u+1)/12 \in \mathbb{Z}$.
Then we have the following isomorphisms

(1) $\pi_6(P(\xi_{\alpha,u})) \cong \mathbb{Z}_{12}$ if $\alpha \equiv u(u+1)/12 \pmod 2$
(2) $\pi_6(P(\xi_{\alpha,u})) \cong \mathbb{Z}_6$ if $\alpha \not\equiv u(u+1)/12 \pmod 2$

.

Proof.

(1) Assume $\alpha \equiv u(u+1)/12 \pmod 2$
In this case, by [Atiyah-Rees]
$\exists$ rk 2 complex vector bundle $\tilde{\mu}_{\alpha,u} \longrightarrow \mathbb{C}P^4$
so that the following is a pull-back diagram:

$$
\begin{tikzcd}
\xi_{\alpha,u} \arrow[r] \arrow[d] & \eta_{(\alpha,0,u)} \arrow[r] \arrow[d] & \tilde{\mu}_{\alpha,u} \arrow[r] \arrow[d] & EU(2) \times_{U(2)} \mathbb{C}^2 \arrow[d] \\
S^7 \arrow[r, "p"] & \mathbb{C}P^3 \arrow[r] & \mathbb{C}P^4 \arrow[r] & BU(2)
\end{tikzcd}
$$

Since $\pi_7(\mathbb{C}P^4) \cong \pi_7(S^9) = \{0\}$,

$\Rightarrow$ $\xi_{\alpha,u} \longrightarrow S^7$ is a trivial bundle,

hence $p(\xi_{\alpha,u}) \cong S^7 \times \mathbb{C}P^1$

$\therefore$ $\pi_6(p(\xi_{\alpha,u})) = \pi_6(S^7) \times \pi_6(\mathbb{C}P^1) \cong \mathbb{Z}_{12}$

(2) Assume $\beta \not\equiv u(u+1)/12 \pmod 2$

(and let $\alpha \equiv u(u+1)/12 \pmod 2$)

Let $\mu_{\alpha,u} : \mathbb{C}P^3 \longrightarrow BU(2)$ be the classifying map of the bundle $\eta_{(\alpha,0,u)}$.

Then the classifying map $\mu_{\beta,u} : \mathbb{C}P^3 \longrightarrow BU(2)$ of the bundle $\xi_{(\beta,0,u)}$ factors through

$$\mu_{\beta,u} : \mathbb{C}P^3 \xrightarrow{\rho} \mathbb{C}P^3 \vee S^6 \xrightarrow{\mu_{\alpha,u} \vee \kappa} BU(2)$$

pinching ∂D^6 to a point where D^6 is a disk nbd of $x \in \mathbb{C}P^3$

Here $[\kappa] \in \pi_6(BU(2)) \cong \mathbb{Z}_2$ is the generator.

$\therefore$ We have the following pull-back diagram of $\mathbb{C}P^1$ - fibrations:

$$\begin{tikzcd}
\mathbb{C}P^1 \arrow[r, "\cong"] \arrow[d] & \mathbb{C}P^1 \arrow[r] \arrow[d] & \mathbb{C}P^1 \arrow[d] & \\
P(\xi_{\beta,u}) \arrow[r] \arrow[d] & M_\beta(u) \arrow[r] \arrow[d] & EU(2) \times_{U(2)} \mathbb{C}P^1 \arrow[d] & \cong EU(2) \times_{U(2)} \left(U(2)/T^2 \right) \cong BT^2 \\
S^1 \arrow[r] & S^7 \arrow[r, "p"] & \mathbb{C}P^3 \arrow[r, "\mu_{\beta,u}"] \arrow[d, "\rho"] & BU(2) \\
& & \mathbb{C}P^3 \vee S^6 \arrow[ur, "\nu_\alpha"'] &
\end{tikzcd}$$

From the homotopy exact sequences of the $\mathbb{C}P^1$-fibrations $\mathbb{C}P^1 \longrightarrow P(\xi_{\alpha,u}) \longrightarrow S^7$ and $\mathbb{C}P^1 \longrightarrow EU(2) \times_{U(2)} \mathbb{C}P^1 \longrightarrow BU(2)$, we have

$$\begin{tikzcd}
\pi_7(S^7) \cong \mathbb{Z} \arrow[r] \arrow[d, "(\mu_{\beta,u} \circ p)_*"'] & \pi_6(\mathbb{C}P^1) \arrow[r] \arrow[d, "\cong"] & \pi_6(P(\xi_{\beta,u})) \arrow[r] \arrow[d] & \pi_6(S^7) = 0 \arrow[d] \\
\pi_7(BU(2)) \cong \mathbb{Z}_{12} \arrow[r, "\cong"] & \pi_6(\mathbb{C}P^1) \arrow[r] & \pi_6(BT^2) = 0 \arrow[r] & \pi_6(BU(2)) \cong \mathbb{Z}_2
\end{tikzcd}$$

$\therefore$ We have the following exact sequence :

$$\begin{array}{ccccccc} \pi_7(S^7) & \xrightarrow{(\mu_{\beta,u}\circ p)_*} & \pi_7(BU(2)) & \longrightarrow & \pi_6(P(\xi_{\beta,u})) & \longrightarrow & 0 \\ \wr\| & & \wr\| & & & & \\ \mathbb{Z} & & \mathbb{Z}_{12} & & & & \end{array}$$

It is enough to show that $(\mu_{\beta,u}\circ p)_*(1) = 6 \in \mathbb{Z}_{12}$ to conclude $\pi_6(P(\xi_{\beta,u})) \cong \pi_6(M_\beta(u)) \cong \mathbb{Z}_6$.

But $\mu_{\beta,u}\circ p = (\mu_{\alpha,u}\vee x)\circ \rho\circ p$

$$= \underbrace{(\mu_{\alpha,u}\circ\rho\circ p)}_{\simeq *}\vee(x\circ\rho\circ p)$$

is homotopic to the following composition

$$S^7 \xrightarrow{p} \mathbb{C}P^3 \xrightarrow{\rho} \mathbb{C}P^3 \vee S^6 \xrightarrow{\pi} S^6 \xrightarrow{\kappa} BU(2)$$

which induces

$$\begin{array}{ccccccc}
\pi_7(S^7) & \xrightarrow{(\rho\circ p)_*} & \pi_7(S^6) & \xrightarrow{\pi_*} & \pi_7(S^6) & \xrightarrow{\kappa_*} & \pi_7(BU(2)) \\
\cong & & \cong & & \cong & & \cong \\
\mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}_2 & & \mathbb{Z}/12 \\
\psi & & \psi^{2} & & \psi & & \psi \\
1 & \longmapsto & 1 & \longmapsto & 1 & \longmapsto & 6
\end{array}$$

this follows because $\kappa \in \pi_6(BU(2)) \cong \mathbb{Z}_2$ is the generator.

$\square$