# Equivariant cobordism of unitary toric manifolds

## —Joint work with Qiangbo Tan

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# Outline

- Notation–unitary toric manifold and quasitoric manifold
- Problem
- Main result 1–equivariant Chern numbers
- Two applications
- Main result 2–relation with real case

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# Notion–Unitary toric manifold

A **unitary toric manifold** $M^{2n}$ of dimension $2n$ is a smooth closed manifold with an effective $T^n$-action such that its tangent bundle admits a $T^n$-equivariant stably complex structure.

**Remark**

- The notion of unitary toric manifolds was introduced by Masuda in his paper [Tohoku Math. J. **51** (1999), 237–265].
- If the fixed point set of a unitary toric manifold $M^{2n}$ is nonempty, then it is 0-dimensional (i.e., it consists of some isolated points).
- More generally, a **unitary $T^k$-manifold** $M^{2n}$ means that $M^{2n}$ is a smooth closed manifold with an effective $T^k$-action such that its tangent bundle admits a $T^k$-equivariant stably complex structure.

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# Notion–Quasitoric manifolds as examples

- **A quasitoric mfd** $M^{2n}$ is a closed smooth manifold with an effective action of $T^n$ such that

  1) $M^{2n}$ is locally iso. to the standard $T^n$-repre. on $\mathbb{C}^n$;

  2) its orbit space $M^{2n}/T^n$ is a simple convex polytope.

# Two key points for Davis–Januszkiewicz theory of quasitoric mfds

$\pi : M^{2n} \longrightarrow P^n$: a quasitoric mfd over $P^n$.

**Algebraic topology**

- **Equivariant cohomology:** $H^*_{T^n}(M) \cong R(P^n;\mathbb{Z})$ where $R(P^n;\mathbb{Z})$ is the Stanley-Reisner face ring of $P^n$:

$$R(P^n;\mathbb{Z}) = \mathbb{Z}[F_1, ..., F_m]/I$$

  $I = (F_{i_1} \cdots F_{i_r} | F_{i_1} \cap \cdots \cap F_{i_r} = \emptyset)$ is an ideal, and each $F_i$ is a facet (ie., codim-one face) of $P^n$.
- **Betti numbers:** $(b_0, b_2, ..., b_{2n}) = (h_0, h_1, ..., h_n)$ where $(h_0, h_1, ..., h_n)$ is the $h$-vector of $P^n$
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## Geometric topology

- **Characteristic function:** Each $\pi : M^{2n} \longrightarrow P^n$ determines

$$\lambda : \mathcal{F}(P^n) \longrightarrow \mathbb{Z}^n$$

  mapping $n$ facets at each vertex to a basis of $\mathbb{Z}^n$, where $\mathcal{F}(P^n)$:=all facets of $P^n$.
- **Reconstruction:** $M^{2n}$ can be recovered by the pair $(P^n, \lambda)$.

# Notion–Quasitoric manifolds as examples

**Theorem (Buchstaber–Ray–Panov) (2010)**

Each omnioriented quasitoric manifold is a unitary toric manifold.

**RK.** An omniorientation of $M$ is a collection of orientations of facial submfds and $M$

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# Basic problem

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To classify unitary $T^k$-manifolds $M^{2n}$ up to equivariant cobordism.

# Two questions

**Case: $k = n$ (i.e., unitary toric manifolds)**

$\Omega_{2n}^{U,T^n}$: the group formed by equivariant cobordism classes of all $2n$-dim unitary toric manifolds.

**Question 1**

Which kinds of equivariant chern numbers completely determine a class of $\Omega_{2n}^{U,T^n}$?

**Question 2**

Whether dose each class of $\Omega_{2n}^{U,T^n}$ contain an omnioriented quasitoric manifold as its representative?

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# A note on Question 1–GGK Theorem

**Theorem (Guillemin–Ginzburg–Karshon) (2002)**

Let $M$ be a unitary $T^k$-manifold fixing **isolated points**.
Then $M \sim 0$ if and only if all equivariant Chern numbers of $M$ are zero.

**RK.** **Atiyah–Bott–Berline–Vergne localization theorem** (1984): Let $T^k \curvearrowright M^{2n}$: unitary $T^k$-mfld fixing isolated points. Then

$$< c_\omega^{T^k}, [M] >= \sum_{p \in M^T} \frac{c_\omega^{T^k}|_p}{\chi^T(p)} \in H^*(BT^k)$$

where $\omega = (i_1, ..., i_n)$ is a multi-index, $c_\omega^{T^n}|_p$ is the restriction of $c_\omega^{T^k}$ to $p$, and $\chi^T(p)$ is the equivariant Euler class of $\tau_p M \longrightarrow p$.

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# Two notes on Question 2

**Note 1:** Buchstaber and Ray gave an affirmative answer to Question 2 in non-equivariant case. Namely, they showed

**Theorem (Buchstaber–Ray) (1999)**

Each class of $\Omega_{2n}^U$ contains a $2n$-dim quasitoric manifold as its representative, where $\Omega_{2n}^U$ is the group formed by cobordism classes of all stably complex $2n$-manifolds.

# Two notes on Question 2

**Note 2: In the case of $(\mathbb{Z}_2)^n \curvearrowright M^n$ (called 2-torus manifold), we have obtained**

- Each class of $\mathfrak{M}_n$ contains an $n$-dim small cover as its representative, where $\mathfrak{M}_n$ is the group formed by equiv. cobordism classes of all 2-torus mfds.
- $\mathfrak{M}_* = \sum \mathfrak{M}_n$ is generated by classes of all generalized real Bott mfds (i.e., small covers over the products of simplices)
- In particular,

$$\dim_{\mathbb{Z}_2} \mathfrak{M}_3 = 13$$

$$\dim_{\mathbb{Z}_2} \mathfrak{M}_4 = 510$$

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# Main result 1–Equivariant Chern numbers

On Question 1, we have

**Theorem (Lü-Tan)**

Let $M$ be a unitary toric manifold. Then $M$ bounds equivariantly if and only if the equivariant Chern numbers

$$\langle (c_1^{T^n})^i (c_2^{T^n})^j, [M] \rangle = 0$$

for all $i, j \in \mathbb{N}$, where $[M]$ is the fundamental class of $M$ with respect to the given orientation.

Our result is a refinement of the following result

**Theorem (Guillemin–Ginzburg–Karshon)**

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# Main result 1–Equivariant Chern numbers

**Proof:**

**Atiyah–Bott–Berline–Vergne localization theorem**

Let $M^{2n}$ be a $(2n)$-dimensional unitary toric manifold. Then

$$< c_\omega^{T^n}, [M] >= \sum_{p \in M^{T^n}} \frac{\sigma_1(p)^{i_1} \cdots \sigma_n(p)^{i_n}}{\pm \sigma_n(p)}$$

where $\omega = (i_1, ..., i_n)$ is a multi-index.

For $p \in M^T$, $\sigma(p)$ means the collection of $\sigma_1(p), ..., \sigma_n(p)$.

**Key Lemma**

$$\sigma(p) = \sigma(q) \Longleftrightarrow \sigma_1(p) = \sigma_1(q) \text{ and } \sigma_2(p) = \sigma_2(q)$$

# Two applications–Application I

**Application I:** the lower bound of the number of isolated fixed points

**Theorem (Lü-Tan)**

Suppose that $M^{2n}$ is a $(2n)$-dimenional unitary toric manifold. If $M$ does not bound equivariantly, then the number of fixed points is at least $\lceil \frac{n}{2} \rceil + 1$, where $\lceil \frac{n}{2} \rceil$ denotes the minimal integer no less than $\frac{n}{2}$.

# Application I-A remark

**Remark.** This gives a supporting evidence on **Kosniowski conjecture**, saying that for a unitary $S^1$-manifold $M^{2n}$ with isolated fixed points, if $M^{2n}$ does not bound equivariantly then the number of fixed points is greater than $f(n)$, where $f(n)$ is some linear function.

As was noted by Kosniowski, the most likely function is $f(n) = \frac{n}{2}$, so the number of fixed points of $M^{2n}$ is at least $[\frac{n}{2}] + 1$.

**Kosniowski conjecture is still open!!!**

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# Two applications–Application II

## Application II to Buchstaber-Panov-Ray conjecture

**Buchstaber-Panov-Ray conjecture**

Let $M^{2n}$ be a specially omnioriented quasitoric manifold. Then $M^{2n}$ represents 0 in $\Omega^U_{2n}$.

**Remark.** Buchstaber-Panov-Ray have showed that when $n < 5$, the conjecture holds.

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# Application II to Buchstaber-Panov-Ray conjecture

## A partial answer to Buchstaber-Panov-Ray conjecture

**Theorem (Lü-Tan)**

Let $M^{2n}$ be a specially omnioriented quasitoric manifold. If $n$ is odd, then $M^{2n}$ represents 0 in $\Omega^U_{2n}$.
Namely, if $n$ is odd, then Buchstaber-Panov-Ray conjecture holds.

# Main result 2

Now let us discuss Question 2. Recall

**Question 2**

Whether dose each class of $\Omega_{2n}^{U,T^n}$ contain an omnioriented quasitoric manifold as its representative?

- $\Omega_{2n}^{U,T^n}$: the group formed by equivariant cobordism classes of all $2n$-dim unitary toric manifolds.
- $\mathfrak{M}_n$: the group formed by equivariant cobordism classes of all $n$-dim 2-torus manifolds $(\mathbb{Z}_2)^n \curvearrowright M^n$.

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# A homomorphism

Define a homomorphism $\Phi_n : \Omega_{2n}^{U,T^n} \longrightarrow \mathfrak{M}_n$ as follows:

- Given a class $\{M^{2n}\}$ in $\Omega_{2n}^{U,T^n}$, GKM theory tells us that $M^{2n}$ gives a GKM graph $(\Gamma_M, \alpha)$, where $\alpha : E(\Gamma_M) \longrightarrow \mathrm{Hom}(T^n, S^1) \cong H^2(BT^n)$ is an axial function with certain condition.
- Then we may obtain a mod 2 GKM graph $(\Gamma_M, \widetilde{\alpha})$ where $\widetilde{\alpha} : E(\Gamma_M) \longrightarrow \mathrm{Hom}((\mathbb{Z}_2)^n, \mathbb{Z}_2) \cong H^1(B(\mathbb{Z}_2)^n; \mathbb{Z}_2)$.
- Lü-Tan showed that such a mod 2 GKM graph $(\Gamma_M, \widetilde{\alpha})$ uniquely determines a class $\{M^n\}$ in $\mathfrak{M}_n$.
- Finally, define

$$\Phi_n(\{M^{2n}\}) = \{M^n\}.$$

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# A homomorphism

**$\Phi_n : \Omega_{2n}^{U,T^n} \longrightarrow \mathfrak{M}_n$ is well-defined.**

- It suffices to show that if $\{M^{2n}\} = 0$ in $\Omega_{2n}^{U,T^n}$, then $\Phi(\{M^{2n}\}) = \{M^n\} = 0$ in $\mathfrak{M}_n$.
- By ABBV localization theorem, take $\omega = (0, \ldots, 0, 2)$, we have

$$< c_\omega^{T^n}, [M] >= \sum_{p\in V(\Gamma_M)} (\pm \prod_{e\in E_p} \alpha(e)) = 0 \text{ in } H^*(BT^n)$$

- Further, we have

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## $\Phi_n : \Omega_{2n}^{U,T^n} \longrightarrow \mathfrak{M}_n$ is well-defined.

- It suffices to show that if $\{M^{2n}\} = 0$ in $\Omega_{2n}^{U,T^n}$, then $\Phi(\{M^{2n}\}) = \{M^n\} = 0$ in $\mathfrak{M}_n$.
- By ABBV localization theorem, take $\omega = (0, ..., 0, 2)$, we have

$$< c_\omega^{T^n}, [M] >= \sum_{p \in V(\Gamma_M)} \big( \pm \prod_{e \in E_p} \alpha(e) \big) = 0 \text{ in } H^*(BT^n)$$

- Further, we have

$$\sum_{p \in V(\Gamma_M)} \prod_{e \in E_p} \widetilde{\alpha}(e) = 0 \text{ in } H^*(B(\mathbb{Z}_2)^n; \mathbb{Z}_2)$$

- By a result of Stong, $\Phi(\{M^{2n}\}) = \{M^n\} = 0$ in $\mathfrak{M}_n$.

# Main result 2

**Theorem (Lü-Tan)**

The homomorphism $\Phi_n : \Omega_{2n}^{U,T^n} \longrightarrow \mathfrak{M}_n$ is onto.

**Remark.** We only obtain that there is at least one class in $\Phi_n^{-1}(\alpha) \subset \Omega_{2n}^{U,T^n}$, containing an omnioriented quasitoric mfd as its a representative.

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# Proof outline of Main result 2

$\mathcal{Q}_{2n}^{T^n}$: consists of equivariant cobordism classes of all $2n$-dim quasitoric manifolds.

$$\mathcal{Q}_{2n}^{T^n} \subseteq \Omega_{2n}^{U,T^n}$$

**Recall: Lü-Tan's result**

$\mathfrak{M}_n$ is generated by those classes of all small covers over the products of simplices (i.e., generalized real Bott manifolds).

**Key Lemma**

The natural homomorphism $\Psi_n : \mathcal{Q}_{2n}^{T^n} \longrightarrow \mathfrak{M}_n$ is onto.

# Proof outline of Main result 2

$$\begin{array}{ccc} & & \mathcal{Q}_{2n}^{T^n} \\ & \overset{\subseteq}{\nearrow} & \Big\downarrow \Phi_n \\ \mathcal{Q}_{2n}^{T^n} & \xrightarrow[\Psi_n]{} & \mathfrak{M}_n \end{array}$$

# Further problem

**Problem 1**

Is the number $[\frac{n}{2}] + 1$ the best possible lower bound of the number of fixed points for $(2n)$-dimensional nonbounding unitary toric manifolds?

**Problem 2**

$$\mathcal{Q}_{2n}^{T^n} = \Omega_{2n}^{U,T^n}?$$

In particular, if this is true, to find which kinds of omnioriented quasitoric manifolds can be used as generators of $\mathcal{Q}_{2n}^{T^n}$?

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# Thank You!