On Gromov's macroscopic dimension conjecture

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Scalar curvature

- The scalar curvature Sc_v of a Riemannian manifold V^n at a point $v \in V^n$ is the number defined by

$$Vol\ B_V(\epsilon, v) = Vol\ B_{\mathbb{R}^n}(\epsilon, 0)(1 - \frac{Sc_v}{6n}\epsilon^2 + o(\epsilon^2))$$

  where $B_V(\epsilon, v)$ is the ϵ-ball centered at $v \in V^n$.
- Sc_v = the sum of sectional curvatures over all 2-planes $e_i \wedge e_j$ in the tangent space to v, where $e_1, \dots, e_n$ is the orthonormal basis.

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Properties

- There is the product formula:

$$Sc_{(v_1,v_2)} = Sc_{v_1} + Sc_{v_2}$$

  for $(V_1 \times V_2, \mathcal{G}_1 \oplus \mathcal{G}_2)$.
- Thus, for every closed manifold M, the product $M \times S^2$ admits a metric with $Sc > 0$
- Take the S^2 factor to be ϵ-small !
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Gromov's Conjecture

GROMOV PSC CONJECTURE.

$dim_{mc}\widetilde{M}^n \leq n-2$ for every closed n-manifold with $Sc(M) > 0$.

EXAMPLE: $N^n = M^{n-2} \times S^2$ admits a metric with $Sc > 0$. The universal cover $\widetilde{N^n} = \widetilde{M^{n-2} \times S^2}$ looks at most $(n-2)$-dimensional on a large scale.

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- The conjecture is from Gelfand-80 book
  M. Gromov, *Positive curvature, macroscopic dimension, spectral gaps and higher signatures*, Functional analysis on the eve of the 21st century. Vol II, Birhauser, Boston, 1996.
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Macroscopic dimension

- For a metric space X, the *macroscopic dimension*

$$dim_{mc} X \leq k$$

  iff there is a uniformly cobounded map $\phi : X \to N^k$ to a k-dimensional simplicial complex.
- A map $\phi : X \to N$ is *uniformly cobounded* if there is $b > 0$ such that $diam(\phi^{-1}(y)) \leq b$ for all $y \in N$.

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Properties of macroscopic dimension

- $dim_{mc} X \leq dim X$
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- $dim_{mc} \mathbb{R}^n = n$ and
- generally, $dim_{mc} V^n = n$ for every uniformly contractible manifold with proper metric.

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Gromov vs Gromov-Lawson

- Gromov's Conjecture implies the Gromov-Lawson:

Gromov-Lawson Conjecture

A closed aspherical manifold M cannot carry a metric with $Sc > 0$.

- *Proof.* The result follows from the facts that $\widetilde{M}$ is uniformly contractible and $dim_{mc} V^n = n$ for all uniformly contractible manifolds.
- The Gromov-Lawson is a Novikov type conjecture.

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Gromov modulo Novikov

- Gromov's conjecture for a group π = Gromov's conjecture for manifolds with the fundamental group π.
- For which groups π Gromov's conjecture does hold?

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Finite index subgroups

- PROPOSITION. *Let $\pi' \subset \pi$ be a subgroup of finite index, $[\pi : \pi'] < \infty$. If Gromov's conjecture holds for manifolds with fundamental group π', then it holds for manifolds with the fundamental group π.*
- Proof. Let $\pi_1(M) = \pi$ and let M have a PSC metric. Then M' corresponding to π' has a PSC metric. Then $dim_{mc}\tilde{M}' \leq n - 2$ by Gromov's Conjecture for π'. Note that $\tilde{M}' = \tilde{M}$.

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Previous Results

Let $ko = KO\langle 0\rangle$ be the connective cover of the real K-theory.

Theorem (Bolotov-Dr. 2010)

Suppose that a discrete group π has the following properties:

- The Strong Novikov Conjecture holds for π.
- The natural map $per : ko_n(B\pi) \to KO_n(B\pi)$ is injective.

Then the Gromov PSC Conjecture holds true for spin n-manifolds M with the fundamental group $\pi_1(M) = \pi$.

Corollary.

The Gromov conjecture holds for spin n-manifolds M with the fundamental group $\pi_1(M)$ equal the product of free groups $F_1 \times \cdots \times F_n$. In particular, it holds for free abelian groups.

New Results

Theorem 1

Suppose that π is a virtual duality group that satisfies the Strong Novikov conjecture. Then the Gromov PSC Conjecture holds true for spin n-manifolds M with the fundamental group $\pi_1(M) = \pi$.

Theorem 2

Suppose that π is a virtual duality group that satisfies the Coarse Baum-Connes conjecture. Then the Gromov PSC Conjecture holds true for almost spin n-manifolds M with the fundamental group $\pi_1(M) = \pi$.

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Duality groups

- We recall that a group π is called a *duality group* if there is a π-module D such that

$$H^i(\pi, M) \cong H_{m-i}(\pi, M \otimes D)$$

for all π-modules M and all i where $m = cd(\pi)$.

- A group π is called a *virtual duality group* if it contains a duality group as a finite index subgroup.
- Examples of virtual duality groups include the fundamental groups of aspherical manifolds, free groups, polycyclic groups, arithmetic groups, knot groups, mapping class groups, their products, etc.

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Strong Novikov Conjecture

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The real analytic assembly map

$$\alpha : KO_*(B\pi) \to KO_*(C_r^*(\pi))$$

is a monomorphism for torsion free groups π.

Here $C_r^*(\pi)$ is the completion of $\mathbb{R}\pi$ in the operator norm where $\mathbb{R}\pi$ acts on $\ell^2(\pi)$ by multiplication on the left.

Rosenberg's Theorem

Let $f : M \to B\pi$ be a classifying map for the universal cover of M.

Rosenberg's Theorem

Let M be a closed connected spin manifold with $Sc > 0$. Then $\alpha \circ f_*([M]_{KO}) = 0$.

This result led to the Gromov-Lawson-Rosenberg Conjecture which is an extension of the Gromov-Lawson conjecture to general manifolds. (disproved by T. Schick)

GLR-conjecture

Let M be a closed connected spin manifold. Then M admits a metric with $Sc > 0$ iff $\alpha \circ f_*([M]_{KO}) = 0$.

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Coarse Baum-Connes conjecture

- The coarse Baum-Connes conjecture for a metric space X:
  The coarse assembly map
  $$\mathcal{A}_\infty : KX_*(X) \to K_*(C^*_{Roe}(X))$$
  is an isomorphism.
- The Baum-Connes conjecture for a torsion free group π:
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Coarse K-homology

- $\mathcal{A}_\infty$ is the direct limit of coarse index maps

$$\mathcal{A}_* : K_*^{lf}(N_i) \to K_*(C^*_{Roe}(N_i)) \cong K_*(C^*_{Roe}(X))$$

for an Anti-Čech approximation $\{N_i\}$ of X.

- By the definition the coarse K-homology of X is

$$KX_*(X) = \lim_{\to} K_*^{lf}(N_i).$$

- When $X = \pi$,

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Macrscopically small manifolds

- There is a *coarsening map* $c_Y : H_*^{lf}(Y) \to HX_*(Y)$.
- DEFINITION(Gong - G. Yu) An open manifold N is *macroscopically small* if $c_N([N]) = 0$ for integral coefficients.
- THEOREM 3. *The universal cover $\widetilde{M}$ of a closed n-manifold is macroscopically small if and only if $\dim_{mc} \widetilde{M} < n$.*

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Roe's Theorem

Theorem (Roe)

Suppose that a closed almost spin manifold M has positive scalar curvature. Then the coarse index map

$$\mathcal{A}_* : K_*^{lf}(\widetilde{M}) \to K_*(C^*_{Roe}(\pi))$$

takes the K-theory fundamental class $[\widetilde{M}]_K$ to zero.

Open problem

- Roe's theorem implies that $\widetilde{M}$ is macroscopically small over $\mathbb{Q}$.
- Is every $\mathbb{Q}$-macroscopically small manifold macroscopically small?
- 'Yes' implies the first part of Gromov's PSC conjecture:

$$\dim_{mc} \widetilde{M} \leq n - 1$$

for a n-manifold M with PSC.

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First part of Gromov's conjecture

Theorem 4

Suppose that the natural transformation $ku_*(E\pi) \to KU_*(E\pi)$ is a monomorphism for a group π that satisfies the coarse Baum-Connes conjecture. Then the first part of Gromov's conjecture holds for manifolds with the fundamental group π: $\dim_{mc} \widetilde{M} \leq n-1$ for every n-manifold with PSC and $\pi_1(M) = \pi$.

Second part of Gromov's conjecture

The second part of Gromov's conjecture (where it's proven) is based on the fact that the suspension of the Hopf map $h : S^n \to S^{n-1}$ induces a non-zero homomorphism for the real K-theory:

$$h_* : KO_n(S^n) \to KO_n(S^{n-1}).$$

Thank you!!!