Almost complex quasitoric manifolds.

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Outline

We sketch the solution of the following problem:

Find out whether a T^n-invariant almost complex structure exists on a given quasitoric manifold.

We start with necessary definitions, then provide precise formulations, main ideas of the proof and several examples.

Quasitoric manifolds

M is a topological manifold of dimension $2n$, equipped with a continious action of compact torus T^n.

1) The action is locally standart: like $T^n \circlearrowleft \mathbb{C}^n$.

The condition 1 implies that orbit space M/T^n has the structure of manifold with corners.

2) M/T^n is isomorphic, as manifold with corners, to a simple polytope P.

M is said to be *quasitoric* if it satisfies conditions 1 and 2 [D-J].

Examples

1) $M = \mathbb{C}P^n$, $P = \Delta^n$.

2) M is any projective toric variety, P is a Delzant polytope.

3) Non-toric example: $M = \mathbb{C}P^2 \# \mathbb{C}P^2$, $P = I \times I$.

There may exist multiple quasitoric M's for given P!

Combinatorial data

Denote by $p : M \to P$ the projection map. The submanifolds of the form $p^{-1}(F_j)$ where $F_j \subset P$, $\dim F_j = n-1$, $j = 1 \ldots m$, are called *characteristic submanifolds*.

We assume that M and all of $p^{-1}(F_j)$ are oriented. Then every 1-dim subgroup $\mathrm{Stab}(p^{-1}(F_j))$ determines a vector $\Lambda_j \subset \mathbb{Z}^n$. The vectors Λ_j, $j = 1 \ldots m$, form an integer-valued matrix $\Lambda = (n \times m)$.

Theorem ([D-J],[B-P-R]). M is uniquely determined by combinatorial data (P, Λ).

So one can try to express topological properties of M in purely combinatorial terms.

Examples

1) $M = \mathbb{C}P^n$, $P = \Delta^n$, $\Lambda =$

$$\begin{pmatrix} 1 & 0 & \dots & -1 \\ 0 & 1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & -1 \end{pmatrix},$$

2) M is any projective toric variety, P is a Delzant polytope, $\Lambda =$ (1-dim cones of a normal fan),

3) $M = \mathbb{C}P^2 \# \mathbb{C}P^2$, $P = I \times I$, $\Lambda =$

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix}.$$

Remarks

1) M admits a canonical T^n-invariant smooth structure ([D-J],[B-P-R]).
2) M admits a canonical T^n-invariant stably complex structure J_{can} ([D-J],[B-P-R]).

Almost and stably complex structures

Complex structure J on a real vector bundle ξ is $J \in \mathrm{End}(\xi)$ s.t. $J^2 = -id$.

If $\xi = \tau(M)$, M is *almost complex*. If $\xi = \tau(M) \oplus \mathbb{R}^l$, M is said to be *stably complex*.

J is called T^n*-invariant* if it commutes with dt for any $t \in T^n$.

Non-invariant case: Thomas theorem (1967)

Let J_{st} be a stably complex structure on M. Then J_{st} is equivalent to some almost complex structure if and only if

$$\big(c_n(J_{st}), [M]\big) = \chi(M)$$

– necessary condition is also the sufficient!

The problem

Problem 7.6 from [D-J]:

Find a combinatorial criteria in terms of Λ *for existence of* T^n*-invariant almost complex structure on* M.

Related work

Mikiya Masuda. Unitary toric manifolds, multi-fans and equivariant index (1999).

A very well-known paper containing, besides many other results, description of the case of 4-dimensional smooth almost complex toric manifolds.

Related work

Natalia Dobrinskaya (unpublished, 2000s) – handling the case of $M^{2n}, n \leqslant 7$, by means of obstruction theory on M^{2n}.

Related work

M. Poddar, S. Ganguli. Almost complex structures, blowdowns and McKay correspondence in quasitoric orbifolds (2012) – extending this proof to quasitoric orbifolds.

The answer

The canonical structure J_{can} on M is equivalent to some T^n-invariant almost complex structure if and only if

$$\det \Lambda_v = +1$$

for every vertex v of polytope P.

Here by Λ_v we denote corresponding square matrix – rows remain the same and columns correspond to facets adjacent to v.

Example 1

$M = \mathbb{C}P^1 \times \mathbb{C}P^1$, $P = I \times I$, $\Lambda =$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

– a toric example. All signs $\det \Lambda_v$ are $+1$:

$$\det\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \det\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \det\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \det\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = +1.$$

Example 2

$M = \mathbb{C}P^2 \# \mathbb{C}P^2$, $P = I \times I$, $\Lambda =$

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix}.$$

– no almost complex structure, because $\det\left(\begin{smallmatrix} -1 & 1 \\ 2 & -1 \end{smallmatrix}\right) = -1$.

Example 3

$M = \mathbb{C}P^2 \# \mathbb{C}P^2 \# \mathbb{C}P^2$, P is a pentagon, $\Lambda =$

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 \end{pmatrix}.$$

All signs are $+1$.

M admits an T^n-inv. almost complex structure – but NOT a symplectic structure, by a result of Taubes.

Therefore M is not a projective toric variety!

Proof of existence

We utilize the fact that P is a convex polytope – and therefore a cellular complex.

Hence, its cellular cohomology groups with any coefficients are all zero in positive dimensions.

Proof: defining an obstruction cochain

It is possible to construct a cellular obstruction cochain

$$\sigma^i_J \in H^i(P^n, \pi_{i-1}(SO/U))$$

that controls extension of structure J from $p^{-1}(sk_{i-1}(P^n))$ to $p^{-1}(sk_i(P^n))$.

Main job is to show that σ^i_J is well-defined.

Proof: where should signs appear?

In the very beginning,

when we construct T^n-invariant almost complex structure on $TM^{2n}|_{fixed\ points}$ and then extend it to $TM^{2n}|_{p^{-1}(sk_1(P^n))}$.

This is possible only when all signs are $+1$!

The set of invariant structures

Let S be the set of all T^n-invariant almost complex structures J inducing given omniorientation.

Structures are considered up to T^n-equivariant homotopy.

Assume S is non-empty.

1. S is «affine equivalent» to $\mathbb{Z}^{f_1(P^n)-f_0(P^n)+1}$.
2. Any two structures from S are non-equivariantly homotopic!

The set of positive omniorientations: upper bound

Theorem: the number of positive omniorientations (that is, number of T^n-invariant structures up to non-equivariant homotopy) does not exceed 2^n.

We have a dimension-based upper bound.

Happy birthday, Victor Matveevich!

References

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