Sokolov system and integrable elasticae via discriminant separability and Buchstaber's two-valued groups

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Algebraic Topology and Abelian Functions
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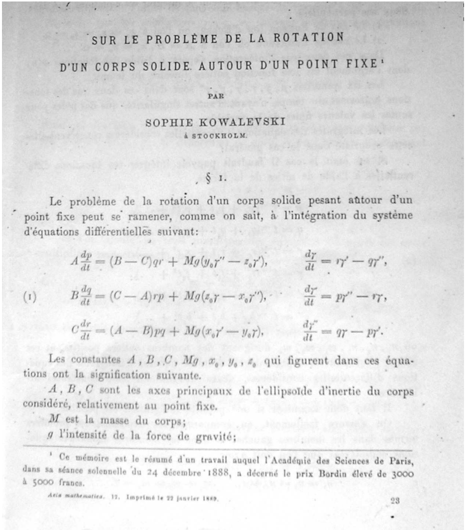
Dedicated to V. M. Buchstaber

Contents

Sophie's world: the story of the history of integrability



SUR LE PROBLÈME DE LA ROTATION

D'UN CORPS SOLIDE AUTOUR D'UN POINT FIXE[1]

PAR

SOPHIE KOWALEVSKI

À STOCKHOLM.

§ I.

Le problème de la rotation d'un corps solide pesant autour d'un point fixe peut se ramener, comme on sait, à l'intégration du système d'équations différentielles suivant:

$$
(1)\qquad
\begin{aligned}
A\frac{dp}{dt} &= (B-C)qr + Mg(y_0\gamma'' - z_0\gamma'), & \frac{d\gamma}{dt} &= r\gamma' - q\gamma'',\\
B\frac{dq}{dt} &= (C-A)rp + Mg(z_0\gamma - x_0\gamma''), & \frac{d\gamma'}{dt} &= p\gamma'' - r\gamma,\\
C\frac{dr}{dt} &= (A-B)pq + Mg(x_0\gamma' - y_0\gamma), & \frac{d\gamma''}{dt} &= q\gamma - p\gamma'.
\end{aligned}
$$

Les constantes $A, B, C, Mg, x_0, y_0, z_0$ qui figurent dans ces équations ont la signification suivante.

A, B, C sont les axes principaux de l'ellipsoïde d'inertie du corps considéré, relativement au point fixe.

M est la masse du corps;

g l'intensité de la force de gravité;

[1] Ce mémoire est le résumé d'un travail auquel l'Académie des Sciences de Paris, dans sa séance solennelle du 24 décembre 1888, a décerné le prix Bordin élevé de 3000 à 5000 francs.

Acta Mathematica, 1889

Sophie's world or Too much happiness

Alice Munro: Too Much Happiness
Man Booker International Prize 2009

Kowalevski case

1888

Kowalevski problem

Describe parameters (A, B, C, x_0, y_0, z_0) for which the Euler-Poisson equations have general solution in form of uniform (meromorphic) functions having only moving poles as singularities.

Kowalevski top

Defined by

$$A = B = 2C, \quad \chi = (x_0, 0, 0).$$

Additional, the Kowalevski integral is of fourth order in impulses

$$F_4 = (\Omega_1^2 - \Omega_2^2 + \frac{x_0}{I_3}\Gamma_1)^2 + (2\Omega_1\Omega_2 + \frac{x_0}{I_3}\Gamma_2)^2.$$

Euler-Poisson equations, I group

$$
\begin{aligned}
2\dot{p} &= qr \\
2\dot{q} &= -pr - c\gamma'' \\
\dot{r} &= c\gamma'
\end{aligned}
$$

c is a constant.

Integral surface

Common level set of the four integrals

$$
\begin{aligned}
6h = H &:= 2(p^2 + q^2) + r^2 - 2c\gamma, \\
2l = L &:= 2(p\gamma + q\gamma') + r\gamma'', \\
k^2 = K &:= (p^2 - q^2 + c\gamma)^2 + (2pq + c\gamma')^2,
\end{aligned}
$$

with the condition $(\Gamma, \Gamma) = 1$,is two-dimensional.

Kowalevski polynomials

Kowalevski coordinates

$$
\begin{aligned}
x_1 &= p + qi \\
x_2 &= p - qi \\
e_1 &= x_1^2 + (g + ig_1) \\
e_2 &= x_2^2 + (g - ig_1)
\end{aligned}
$$

First integrals

$$
\begin{aligned}
k^2 &= e_1 e_2 \\
r^2 &= E + e_1 + e_2 \\
rg_3 &= F - x_1 e_1 - x_2 e_2 \\
g_3^2 &= G + x_2^2 e_1 + x_1^2 e_2
\end{aligned}
$$

$$
\begin{aligned}
E &= 6l_1 - (x_1 + x_2)^2 \\
F &= 2l + x_1 x_2 (x_1 + x_2) \\
G &= 1 - k^2 - x_1^2 x_2^2
\end{aligned}
$$

$II \cdot IV - III^2 = 0$

$$
e_1[x_2^2 E + 2x_2 F + G] + e_2[x_1^2 E + 2x_1 F + G] + EG - F^2 + k^2(x_1 - x_2)^2 = 0.
$$

Kowalevski procedure

$$
\begin{aligned}
P(x_2) &= x_2^2 E + 2x_2 F + G \\
R(x_1, x_2) &= E x_1 x_2 + F(x_1 + x_2) + G \\
R_1(x_1, x_2) &= EF - G^2
\end{aligned}
$$

$$
\begin{gathered}
P(x_2)e_1 + P(x_1)e_2 = -R_1 - k^2(x_1 - x_2)^2 \\
e_1 e_2 = k^2
\end{gathered}
$$

$$
(\sqrt{P(x_2)e_1} \pm \sqrt{P(x_1)e_2})^2 = -R_1 - k^2(x_1 - x_2)^2 \pm 2\sqrt{P(x_1)P(x_2)}k
$$

$$\left(\frac{\sqrt{P(x_2)e_1}}{x_1 - x_2} \pm \frac{\sqrt{P(x_1)e_2}}{x_1 - x_2} \right)^2 = (s_1 \pm k)(s_2 \mp k)$$

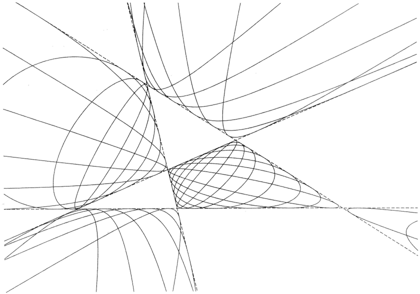
$$s^2 - \frac{2R}{(x_1 - x_2)^2} s - \frac{R_1}{(x_1 - x_2)^2} = 0.$$

$$Q(s, x_1, x_2) = (x_1 - x_2)^2 s^2 - 2R(x_1, x_2)s - R_1(x_1, x_2).$$

Pencil of conics

Two conics and tangential pencil

$$C_1 \; : \; a_0 w_1^2 + a_2 w_2^2 + a_4 w_3^2 + 2a_3 w_2 w_3 + 2a_5 w_1 w_3 + 2a_1 w_1 w_2 = 0$$
$$C_2 \; : \; w_2^2 - 4 w_1 w_3 = 0$$



Darboux coordinates

$$t_{C_2}(\ell_0) : z_1\ell_0^2 - 2z_2\ell_0 + z_3 = 0$$

$$\hat{z}_1\ell^2 - 2\hat{z}_2\ell + \hat{z}_3 = 0$$

$$\hat{z}_1 = 1, \quad \hat{z}_2 = \frac{x_1 + x_2}{2}, \quad \hat{z}_3 = x_1x_2$$

Pencil equation in the Darboux coordinates

$$F(s, x_1, x_2) = L(x_1, x_2)s^2 + K(x_1, x_2)s + H(x_1, x_2).$$

Theorem [V. D. (2010)]

(i) There exists a polynomial $P = P(x)$ such that the discriminant of the polynomial F in s as a polynomial in variables x_1 and x_2 separates the variables:

$$\mathcal{D}_s(F)(x_1, x_2) = P(x_1)P(x_2). \tag{1}$$

(ii) There exists a polynomial $J = J(s)$ such that the discriminant of the polynomial F in x_2 as a polynomial in variables x_1 and s separates the variables:

$$\mathcal{D}_{x_2}(F)(s, x_1) = J(s)P(x_1). \tag{2}$$

Due to the symmetry between x_1 and x_2 the last statement remains valid after exchanging the places of x_1 and x_2.

Gauge equivalence

Gauge transformations

$$x \mapsto \frac{a_1 x + b_1}{c_1 x + d_1}$$
$$y \mapsto \frac{a_2 y + b_2}{c_2 y + d_2}$$
$$z \mapsto \frac{a_3 z + b_3}{c_3 z + d_3}$$

Discriminantly separable polynomials – definition

For a polynomial $F(x_1, \ldots, x_n)$ we say that it is discriminantly separable if there exist polynomials $f_i(x_i)$ such that for every $i = 1, \ldots, n$

$$\mathcal{D}_{x_i} F(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \prod_{j \neq i} f_j(x_j).$$

It is symmetrically discriminantly separable if

$$f_2 = f_3 = \cdots = f_n,$$

while it is strongly discriminatly separable if

$$f_1 = f_2 = f_3 = \cdots = f_n.$$

It is weakly discriminantly separable if there exist polynomials $f_i^j(x_i)$ such that for every $i = 1, \ldots, n$

$$\mathcal{D}_{x_i} F(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \prod_{j \neq i} f_j^i(x_j).$$

Geometric interpretation of the Kowalevski fundamental equation

$$Q(w,x_1,x_2) := (x_1-x_2)^2 w^2 - 2R(x_1,x_2)w - R_1(x_1,x_2) = 0$$

$$\begin{aligned} R(x_1,x_2) = & -x_1^2x_2^2 + 6\ell_1 x_1 x_2 + 2\ell c(x_1+x_2) + c^2 - k^2 \\ R_1(x_1,x_2) = & -6\ell_1 x_1^2 x_2^2 - (c^2-k^2)(x_1+x_2)^2 - 4c\ell x_1 x_2 (x_1+x_2) \\ & + 6\ell_1(c^2-k^2) - 4c^2\ell^2 \end{aligned}$$

$$\begin{aligned} a_0 = & -2 \qquad a_1 = 0 \qquad a_5 = 0 \\ a_2 = & \ 3\ell_1 \qquad a_3 = -2c\ell \qquad a_4 = 2(c^2-k^2) \end{aligned}$$

Geometric interpretation of the Kowalevski fundamental equation

Theorem [V. D. (2010)]

The Kowalevski fundamental equation represents a point pencil of conics given by their tangential equations

$$\hat{C}_1 : -2w_1^2 + 3l_1 w_2^2 + 2(c^2 - k^2)w_3^2 - 4clw_2 w_3 = 0;$$
$$C_2 : w_2^2 - 4w_1 w_3 = 0.$$

The Kowalevski variables w, x_1, x_2 in this geometric settings are the pencil parameter, and the Darboux coordinates with respect to the conic C_2 respectively.

n-valued Buchstaber-Novikov groups

n-valued groups, local (formal) - Buchstaber-Novikov 1970

$$m \; : \; X \times X \to (X)^n, \qquad m(x,y) = x * y = [z_1, \dots, z_n]$$

$(X)^n$ — symmetric n-th power of X

Associativity

Equality of two n^2-sets:

$$[x * (y * z)_1, \dots, x * (y * z)_n] \quad \text{and} \quad [(x * y)_1 * z, \dots, (x * y)_n * z]$$

for every triplet $(x, y, z) \in X^3$.

Unity e

$e * x = x * e = [x, \dots, x]$ for each $x \in X$.

Inverse $\mathrm{inv} : X \to X$

$e \in \mathrm{inv}(x) * x$, $e \in x * \mathrm{inv}(x)$ for each $x \in X$.

n-valued Buchstaber-Novikov groups

Action of n-valued group X on the set Y

$$\phi \ : \ X \times Y \to (Y)^n$$
$$\phi(x, y) = x \circ y$$

Two n^2-multi-subsets in Y:

$$x_1 \circ (x_2 \circ y) \quad \text{and} \quad (x_1 * x_2) \circ y$$

are equal for every triplet $x_1, x_2 \in X$, $y \in Y$.
Additionally , we assume:

$$e \circ y = [y, \dots, y]$$

for each $y \in Y$.

Two-valued group on $\mathbf{CP}^1$-V. M Buchstaber 1990

The equation of a pencil

$F(s, x_1, x_2) = 0$

Isomorphic elliptic curves

$$\Gamma_1 \;:\; y^2 = P(x) \qquad \deg P = 4$$
$$\Gamma_2 \;:\; t^2 = J(s) \qquad \deg J = 3$$

Canonical equation of the curve Γ_2

$$\Gamma_2 \;:\; t^2 = J'(s) = 4s^3 - g_2 s - g_3$$

Birational morphism of curves $\psi \;:\; \Gamma_2 \to \Gamma_1$

Induced by fractional-linear mapping $\hat{\psi}$ which maps zeros of the polynomial J' and ∞ to the four zeros of the polynomial P.

Two-valued group on $\mathbf{CP}^1$ - V. M. Buchstaber 1990

There is a group structure on the cubic Γ_2. Together with its subgroup $\mathbf{Z}_2$, it defines the standard two-valued group structure on $\mathbf{CP}^1$:

$$s_1 *_c s_2 = \left[-s_1 - s_2 + \left(\frac{t_1 - t_2}{2(s_1 - s_2)} \right)^2, -s_1 - s_2 + \left(\frac{t_1 + t_2}{2(s_1 - s_2)} \right)^2 \right],$$

where $t_i = J'(s_i)$, $i = 1, 2$.

Theorem [V. D. (2010)]

The general pencil equation after fractional-linear transformations

$$F(s, \hat{\psi}^{-1}(x_1), \hat{\psi}^{-1}(x_2)) = 0$$

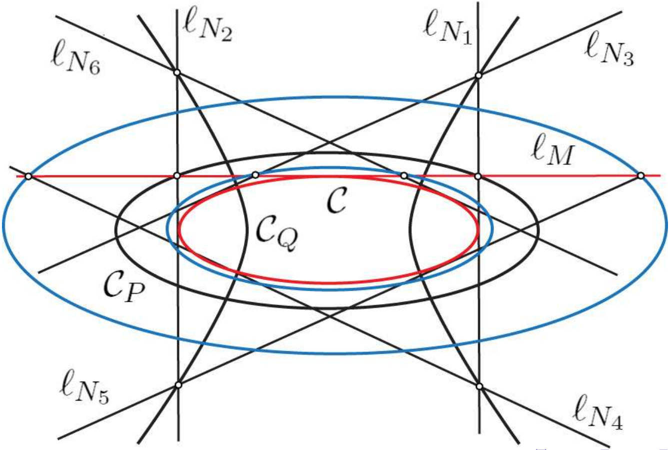
defines the two valued coset group structure $(\Gamma_2, \mathbb{Z}_2)$.

$$
\begin{tikzcd}
\mathbb{C}^4 \arrow[r, "i_{\Gamma_1}\times i_{\Gamma_1}\times m"] \arrow[d, "i_{\Gamma_1}\times i_{\Gamma_1}\times id\times id"'] \arrow[dr, "i_a\times i_a\times m"] & \Gamma_1 \times \Gamma_1 \times \mathbb{C} \arrow[r, "\psi^{-1}\times\psi^{-1}\times id"] \arrow[d, "p_1\times p_1\times id"] & \Gamma_2 \times \Gamma_2 \times \mathbb{C} \arrow[ddl, "p_1\times p_1\times id"] \\
\Gamma_1 \times \Gamma_1 \times \mathbb{C} \times \mathbb{C} \arrow[d, "\varphi_1\times\varphi_2"'] & \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{C} \arrow[d, "\hat{\psi}^{-1}\times\hat{\psi}^{-1}\times id"'] & \\
\mathbb{C} \times \mathbb{C} \arrow[d, "m_2"'] & \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{C} \arrow[d, "m_c\times\tau_c"'] & \\
\mathbb{CP}^2 & \mathbb{CP}^2 \times \mathbb{C}/\sim \arrow[l, "f"'] &
\end{tikzcd}
$$

Two-valued group $\mathbf{CP}^1$

Theorem [V. D. (2009/2010)]

Associativity conditions for the group structure of the two-valued coset group $(\Gamma_2, \mathbb{Z}_2)$ and for its action on $\mathbf{CP}^1$ are equivalent to the great Poncelet theorem for a triangle.



Systems of the Kowalevski type

Given a system and coordinates x_1, x_2, e_1, e_2, r, γ_3, such that

$$\begin{aligned} \dot{x}_1 &= -if_1 \\ \dot{x}_2 &= if_2 \\ \dot{e}_1 &= -me_1 \\ \dot{e}_2 &= me_2 \end{aligned} \tag{3}$$

where

$$\begin{aligned} f_1^2 &= P(x_1) + e_1 A(x_1, x_2) \\ f_2^2 &= P(x_2) + e_2 A(x_1, x_2). \end{aligned} \tag{4}$$

Additional assumption is that the first integrals and invariant relations can be rewritten in a form

$$P(x_2)e_1 + P(x_1)e_2 + e_1 e_2 A(x_1, x_2) - C(x_1, x_2) = 0. \tag{5}$$

which is equivalent to

$$-2\dot{x}_1\dot{x}_2 = B(x_1, x_2). \tag{6}$$

Additional assumption is that the first integrals and invariant relations can be rewritten in a form

$$P(x_2)e_1 + P(x_1)e_2 + e_1 e_2 A(x_1, x_2) - C(x_1, x_2) = 0. \tag{5}$$

which is equivalent to

$$-2\dot{x}_1\dot{x}_2 = B(x_1, x_2). \tag{6}$$

Theorem [V. D.-K. Kukić, 2011]

Given a system of the Kowalevski type. Such a system can be integrated by solving the system

$$\frac{ds_1}{\sqrt{\Phi(s_1)}} + \frac{ds_2}{\sqrt{\Phi(s_2)}} = 0$$
$$\frac{s_1\,ds_1}{\sqrt{\Phi(s_1)}} + \frac{s_2\,ds_2}{\sqrt{\Phi(s_2)}} = -\imath dt,$$

where

$$\Phi(s) = J(s)(s-k)(s+k),$$

and s_1, s_2 are the solutions of the equation $\mathcal{F}(x_1, x_2, s) = 0$ and k is a constant such that

$$e_1 e_2 = k^2.$$

The system is linearized on $\Gamma : \ y^2 = \Phi(s)$.

Sokolov system [Theor. Math. Phys. (2001)]

Consider the Hamiltonian

$$\hat{H} = M_1^2 + M_2^2 + 2M_3^2 + 2c_1\gamma_1 + 2c_2(\gamma_2 M_3 - \gamma_3 M_2) \tag{7}$$

on $e(3)$ with the Lie-Poisson structure:

$$\{M_i, M_j\} = \epsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = \epsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0 \tag{8}$$

It holds

$$\begin{aligned} \gamma_1^2 + \gamma_2^2 + \gamma_3^2 &= a, \\ \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3 &= b. \end{aligned}$$

New variables:

$$
\begin{aligned}
z_1 &= M_1 + iM_2, \\
z_2 &= M_1 - iM_2, \\
e_1 &= z_1^2 - 2c_1(\gamma_1 + i\gamma_2) - c_2^2 a \\
&\quad - c_2(2\gamma_2 M_3 - 2\gamma_3 M_2 + 2i(\gamma_3 M_1 - \gamma_1 M_3)), \\
e_2 &= z_2^2 - 2c_1(\gamma_1 - i\gamma_2) - c_2^2 a \\
&\quad - c_2(2\gamma_2 M_3 - 2\gamma_3 M_2 + 2i(\gamma_1 M_3 - \gamma_3 M_1)).
\end{aligned}
$$

Additional first integral can be written in the form:

$$
e_1 e_2 = k^2.
$$

The equations of motion in the new variables

$$\begin{aligned}
\dot{e}_1 &= -4iM_3 e_1, \\
\dot{e}_2 &= 4iM_3 e_2, \\
-\dot{z}_1{}^2 &= P(z_1) + e_1(z_1 - z_2)^2, \\
-\dot{z}_2{}^2 &= P(z_2) + e_2(z_1 - z_2)^2
\end{aligned}$$

where P is a polynomial of degree four

$$P(z) = -z^4 + 2Hz^2 - 8c_1 bz - k^2 + 4ac_1^2 - 2c_2^2(2b^2 - Ha) + c_2^4 a. \quad (9)$$

It holds

$$\dot{z}_1 \cdot \dot{z}_2 = -\left(F(z_1, z_2) + (H + c_2^2 a)(z_1 - z_2)^2\right),$$

where $F(z_1, z_2)$ is a biquadratic polynomial given with:

$$F(z_1, z_2) = -\frac{1}{2}\left(P(z_1) + P(z_2) + (z_1^2 - z_2^2)^2\right). \quad (10)$$

The identity is satisfied

$$(z_1 - z_2)^2[2F(z_1, z_2)(H + c_2^2 a) + (z_1 - z_2)^2(H + c_2^2 a)^2 - P(z_1)e_2 - P(z_2)e_1 - e_1 e_2(z_1 - z_2)^2] + F^2(z_1, z_2) - P(z_1)P(z_2) = 0.$$

Let $C(z_1, z_2)$ denotes a biquadratic polynomial such that

$$F^2(z_1, z_2) - P(z_1)P(z_2) = (z_1 - z_2)^2 C(z_1, z_2),$$

then we have

$$P(z_1)e_2 + P(z_2)e_1 = \tilde{C}(z_1, z_2) - e_1 e_2(z_1 - z_2)^2, \qquad (11)$$

$$\tilde{C}(z_1, z_2) = C(z_1, z_2) + 2F(z_1, z_2)(H + c_2^2 a) + (H + c_2^2 a)^2(z_1 - z_2)^2.$$

The polynomial equation $\tilde{F}(z_1, z_2, s) = 0$ with

$$\tilde{F}(z_1, z_2, s) = (z_1 - z_2)^2 s^2 + \tilde{B}(z_1, z_2)s + \tilde{C}(z_1, z_2)$$

with the coefficients

$$\tilde{B}(z_1, z_2) = F(z_1, z_2) + (H + c_2^2 a)(z_1 - z_2)^2$$

has the discriminants: $\mathcal{D}_s(\tilde{F})(z_1, z_2) = P(z_1)P(z_2)$

$$\mathcal{D}_{z_1}(\tilde{F})(s, z_2) = J(s)P(z_2),\ \mathcal{D}_{z_2}(Q)(s, z_1) = J(s)P(z_1)$$

where J is a polynomial of degree three

$$J = -8s^3 + 4(H + 3ac_2^2)s^2 + (8c_2^2 b^2 + 2k^2 - 8ac_1^2 - 8c_2^4 a^2 - 8c_2^2 Ha)s$$
$$- 8c_1^2 b^2 - 4c_2^4 ab_4^2 c_1^2 a^2 c_2^2 - k^2 c_2^2 a - Hk^2 + 2aH^2 c_2^2 - 4Hb^2 c_2^2$$
$$+ 4Hc_1^2 a + 4c_2^4 Ha^2 + 2c_2^6 a^3.$$

It plays the role of the Kowalevski fundamental equation.

With $\tilde{s}_1$, $\tilde{s}_2$ denote the roots of $\mathcal{F}(z_1, z_2, s) = 0$ as a quadratic equation in s

$$\tilde{s}_{1,2} = \frac{-\tilde{B}(z_1, z_2) \pm \sqrt{\tilde{B}^2(z_1, z_2) - 4(z_1 - z_2)^2 \tilde{C}(z_1, z_2)}}{2(z_1 - z_2)^2}. \qquad (12)$$

Then

$$\begin{aligned} \frac{dz_1}{\sqrt{P(z_1)}} + \frac{dz_2}{\sqrt{P(z_2)}} &= \frac{d\tilde{s}_1}{\sqrt{J(\tilde{s}_1)}} \\ -\frac{dz_1}{\sqrt{P(z_1)}} + \frac{dz_2}{\sqrt{P(z_2)}} &= \frac{d\tilde{s}_2}{\sqrt{J(\tilde{s}_2)}}. \end{aligned} \qquad (13)$$

The Sokolov system can be integrated in the theta functions of genus two:

$$\begin{aligned} \frac{d\tilde{s}_1}{\sqrt{\Phi(\tilde{s}_1)}} + \frac{d\tilde{s}_2}{\sqrt{\Phi(\tilde{s}_2)}} &= 0 \\ \frac{\tilde{s}_1\, d\tilde{s}_1}{\sqrt{\Phi(\tilde{s}_1)}} + \frac{\tilde{s}_2\, d\tilde{s}_2}{\sqrt{\Phi(\tilde{s}_2)}} &= dt, \end{aligned} \qquad (14)$$

where $\Phi(s) = -4J(s)(s-k)(s+k)$.

Remark:

Komarov, Sokolov, Tsiganov [J.Phys. A (2003)] introduced:

$$s_{1,2} = \frac{F(z_1, z_2) \pm \sqrt{P(z_1)P(z_2)}}{2(z_1 - z_2)^2}, \tag{15}$$

$$\dot{s}_1 = \frac{\sqrt{P_5(s_1)}}{s_1 - s_2}, \quad \dot{s}_2 = \frac{\sqrt{P_5(s_2)}}{s_2 - s_1}, \quad P_5(s) = P_3(s)P_2(s)$$

$$\begin{aligned} P_3(s) &= s(4s^2 + 4sH + H^2 - k^2 + 4c_1^2 a + 2c_2^2(Ha - 2b^2) \\ &\quad + c_2^4 a^2) + 4c_1^2 b^2, \\ P_2(s) &= 4s^2 + 4(H + c_2^2 a)s + H^2 - k^2 + 2c_2^2 ha + c_2^4 a^2. \end{aligned}$$

Our variables $\tilde{s}_i$ and their s_i are connected by:

$$\tilde{s}_i = s_i + \frac{H + c_2^2 a}{2}.$$

Deformation

Deformation of the Kowalevski top

$$F(x_1, x_2, s) := s^2 A + sB + C = 0.$$

A gauge transformation

$$s \mapsto t + \alpha.$$

$$F_\alpha(x_1, x_2, t) = t^2 A + t(B + 2\alpha A) + (C + \alpha B + \alpha^2 A) = 0.$$

$$C = F^2 - EG, A = (x_1 - x_2)^2$$

$$
\begin{aligned}
A_\alpha &= A \\
B_\alpha &= B + 2\alpha A \\
C_\alpha &= C + \alpha B + \alpha^2 A
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
F_\alpha &= F + \alpha F_1 \\
E_\alpha &= E + \alpha E_1 \\
G_\alpha &= G + \alpha G_1
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
B &= 2FF_1 - E_1 G - EG_1 \\
A &= F_1^2 - E_1 G_1
\end{aligned}
\tag{18}
$$

From

$$B = -2(Ex_1x_2 + F(x_1 + x_2) + G)$$

we get

$$\begin{aligned} F_1 &= -(x_1 + x_2) \\ G_1 &= 2x_1x_2 \\ E_1 &= 2 \end{aligned} \tag{19}$$

One easily checks

$$F_1^2 - E_1G_1 = A.$$

$$
\begin{aligned}
E_\alpha &= 6l_1 - (x_1 + x_2)^2 + 2\alpha \\
F_\alpha &= 2cl + x_1 x_2 (x_1 + x_2) - \alpha (x_1 + x_2) \\
G_\alpha &= c^2 - k^2 - x_1^2 x_2^2 + 2\alpha x_1 x_2
\end{aligned}
\tag{20}
$$

Elastic deformation

Jurdjevic considered a deformation of the Kowalevski case associated to a Kirchhoff elastic problem. The systems are defined by the Hamiltonians

$$H = M_1^2 + M_2^2 + 2M_3^2 + \gamma_1$$

where deformed Poisson structures $\{\cdot,\cdot\}_\tau$ are defined by

$$\{M_i, M_j\}_\tau = \epsilon_{ijk} M_k, \quad \{M_i, \gamma_j\}_\tau = \epsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\}_\tau = \tau\epsilon_{ijk} M_k,$$

the deformation parameter takes values $\tau = 0, 1, -1$. The classical Kowalevski case corresponds to the case $\tau = 0$.

Denote

$$e_1 = x_1^2 - (\gamma_1 + i\gamma_2) + \tau$$
$$e_2 = x_2^2 - (\gamma_1 - i\gamma_2) + \tau,$$

where

$$x_{1,2} = \frac{M_1 \pm iM_2}{2}.$$

Integrals of motion

The integrals of motion

$$I_1 = e_1 e_2$$
$$I_2 = H$$
$$I_3 = \gamma_1 M_1 + \gamma_2 M_2 + \gamma_3 M_3$$
$$I_4 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \tau(M_1^2 + M_2^2 + M_3^2)$$

may be rewritten in the form:

$$k^2 = I_1 = e_1 \cdot e_2$$
$$M_3^2 = e_1 + e_2 + \hat{E}(x_1, x_2)$$
$$M_3\gamma_3 = -x_2 e_1 - x_1 e_2 + \hat{F}(x_1, x_2)$$
$$\gamma_3^2 = x_2^2 e_1 + x_1^2 e_2 + \hat{G}(x_1, x_2),$$

where

$$\hat{G}(x_1, x_2) = -x_1^2 x_2^2 - 2\tau x_1 x_2 - 2\tau(I_1 - \tau) + \tau^2 - I_2$$
$$\hat{F}(x_1, x_2) = (x_1 x_2 + \tau)(x_1 + x_2) + I_3$$
$$\hat{E}(x_1, x_2) = -(x_1 + x_2)^2 + 2(I_1 - \tau).$$

Theorem

A gauge transformation

$$s \mapsto t + \alpha$$

transforms the Kowalevski top to Jurdjevic elasticae according to the formulae

$$\begin{aligned} \tau &= -\alpha \\ l_1 &= 3l_1 \\ l_3 &= 2cl \\ l_2 &= c^2 - k^2 + 2\alpha(3l_1 + \alpha) + \alpha^2 \end{aligned}$$

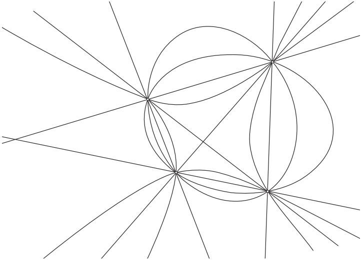
Classification of the strongly discriminantly separable polynomials

Natural question: to classify discriminantly separable polynomials of degree two in each of three variables, up to gauge transformations.

Theorem (V. D. - K. Kukić, 2010)

All strongly discrimiantly separable polynomials in three variables of degree two in each variable, with polynomial P with four simple roots, are gauge equivalent to the two valued group defined by the equation:

$$(x + y + z + \frac{g_2}{4}xyz)^2 - (4 + g_3 xyz)(xy + yz + zx) = 0.$$



Experimental Math

Other n-valued groups

$$p_2 = s_1^2 - 2^2 s_2, \quad Dp_2 = xy.$$

$$p_3 = s_1^3 - 3^3 s_3$$

$$Dp_3 = y^2 x^2 (x - y)^2.$$

$$p_4 = s_1^4 - 2^3 s_1^2 s_2 + 2^4 s_2^2 - 2^7 s_1 s_3$$

$$Dp_4 = y^3 x^3 (x - y)^2 (y + 4x)^2 (4y + x)^2.$$

$$p_5 = s_1^5 - 5^4 s_1^2 s_3 + 5^5 s_2 s_3$$

$$Dp_5 = y^4 x^4 (x - y)^4 (-y^2 - 11xy + x^2)^2 (-y^2 + 11xy + x^2)^2.$$

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