Geometric differential equations on the universal spaces of the Jacobians of elliptic and hyperelliptic curves.

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Introduction

In the work by V. M. Buchstaber and D. V. Leykin, "Solution of the Problem of Differentiation of Abelian Functions over Parameters for Families of (n,s)-Curves" (2008), a method is described for constructing the Gauss-Manin connection on the universal space of Jacobians of (n,s)-curves.

It is based on the theory of multidimensional sigma-functions, which is developed in "Kleinian functions, hyperelliptic Jacobians and applications" by V. M. Buchstaber, V. Z. Enolski, D. V. Leykin (1997).

The case of hypergeometric curves, that is (2,2g+1)-curves, is studied in detail.

V. M. Buchstaber proposed the following problems:

1) Construction of cometrics in the parameter space, compatible with the Gauss-Manin connection.

2) Find differential equations describing such cometrics.

The later problem is related to the more general problem of constructing differential equations describing the geometry of universal spaces of Jacobians of elliptic and hyperelliptic curves.

We consider the universal bundle
of hyperelliptic genus g curves

$$\begin{array}{l} \mathcal{E}_g \\ \Big\downarrow V_\lambda = \{(x,y)\in\mathbb{C}^2 \mid y^2 = x^{2g+1} + \lambda_4 x^{2g-1} + \dots + \lambda_{4g+2}\} \\ \mathcal{B} = \{\lambda = (\lambda_4, \dots, \lambda_{4g+2}) \in \mathbb{C}^{2g} \mid \Delta \neq 0\} \end{array}$$

For $g = 1$ we have

$$\lambda_4 = -\frac{1}{4} g_2, \qquad \lambda_6 = -\frac{1}{4} g_3,$$

and $\Delta = g_2^3 - 27 g_3^2$.

Set $\deg \lambda_k = -2k$.

The vector fields

$$\begin{pmatrix} l_0 \\ l_2 \end{pmatrix} = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 \\ 6\lambda_6 & -\frac{1}{12}\lambda_4^2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \lambda_4} \\ \frac{\partial}{\partial \lambda_6} \end{pmatrix}, \quad g = 1,$$

$$\begin{pmatrix} l_0 \\ l_2 \\ l_4 \\ l_6 \end{pmatrix} = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\ 6\lambda_6 & 8\lambda_8 - \frac{12}{5}\lambda_4^2 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & -\frac{4}{5}\lambda_4\lambda_8 \\ 8\lambda_8 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & 4\lambda_4\lambda_8 - \frac{12}{5}\lambda_6^2 & 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8 \\ 10\lambda_{10} & -\frac{4}{5}\lambda_4\lambda_8 & 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8 & 4\lambda_6\lambda_{10} - \frac{8}{5}\lambda_8^2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \lambda_4} \\ \frac{\partial}{\partial \lambda_6} \\ \frac{\partial}{\partial \lambda_8} \\ \frac{\partial}{\partial \lambda_{10}} \end{pmatrix}$$

$$g = 2,$$

form a basis in the space of vector fields on $\mathcal{B}$.

The locally-trivial bundle $\Omega_g \to \mathcal{B}$ is associated to the bundle $\mathcal{E}_g$. Its fiber over the point λ is the linear $2g$-dimensional space of holomorphic 1-forms on the curve V_λ with a puncture at infinity.
The natural operation of transporting the fibers along curves in the base is defined on this bundle: it is called the Gauss-Manin connection in the bundle $\mathcal{E}_g$.
Let us choose the basis in the fiber of Ω_g as

$$D = \begin{pmatrix} \frac{xdx}{y} & \frac{dx}{y} \end{pmatrix}^\top, \qquad g = 1,$$

$$D = \begin{pmatrix} \frac{xdx}{y} & \frac{dx}{y} & \frac{x^2dx}{y} & \frac{(3x^3+\lambda_4 x)dx}{y} \end{pmatrix}^\top, \qquad g = 2.$$

The Christoffel coefficient Γ_k of the Gauss-Manin connection is defined by the condition that the holomorphic 1-form

$$l_k D + \Gamma_k D$$

is exact on the curve. We have $\nabla_{l_k} D = -\Gamma_k D$.

Define on Ω_g the symmetric cometric $G = (g^{i,j})$ such that it is compatible with the Gauss-Manin connection in $\mathcal{E}_g$, that is

$$\nabla_{l_k} g^{i,j} = 0, \qquad i,j = 1,2,\ldots,2g, \; k = 0,2,\ldots,4g-2.$$

Lemma

The symmetric cometric G is defined by the dynamical system

$$l_k G = -\Gamma_k G - G\Gamma_k^\top, \quad k = 0,2,\ldots,4g-2.$$

The Christoffel coefficients of the Gauss–Manin connection in the bundle $\mathcal{E}_1$, associated with the fields l_0, l_2, are equal to

$$\Gamma_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & \frac{g_2}{12} \\ -1 & 0 \end{pmatrix}.$$

Consider in $\mathcal{B}$ the algebraic transform

$$t = g_2^3 - 27 g_3^2, \qquad w = \frac{27 g_3^2}{g_2^3 - 27 g_3^2}.$$

The fields l_0 and l_2 take the form

$$l_0 = 12 t \frac{\partial}{\partial t}, \qquad l_2 = 2\sqrt{3} t^{1/6} (1 + w)^{2/3} w^{1/2} \frac{\partial}{\partial w}.$$

Set $g^{1,2} = f_{1,2}(w)$,

$$g^{1,1} = \frac{\sqrt{3}t^{1/6}(1+w)^{2/3}}{w^{1/2}} f_{1,1}(w), \qquad g^{2,2} = \frac{(1+w)^{1/3}}{\sqrt{3}t^{1/6}w^{1/2}} f_{2,2}(w).$$

Theorem

The connection ∇ is compatible with the cometric G, if

$$2 \det M \frac{d}{dw} F = MF,$$

where $$F = \begin{pmatrix} f_{1,1}(w) \\ f_{1,2}(w) \\ f_{2,2}(w) \end{pmatrix}, \quad M = \begin{pmatrix} (3-w) & -\frac{w}{6} & 0 \\ 3(1+w) & 0 & -\frac{1+w}{12} \\ 0 & 6w & (3+w) \end{pmatrix}.$$

Set $v = w + 1$,

$$f_{1,1} = -\sqrt{w}\left(6w(v)h''(v) + (9 + 11w)h'(v) + \frac{7}{6}h(v)\right),$$
$$f_{1,2} = \sqrt{w}h(v),$$
$$f_{2,2} = -6\sqrt{w}\left(36w(v)h''(v) + 6(9 + 13w)h'(v) + 13h(v)\right).$$

The system becomes

$$\partial_v\left(v\partial_v - \frac{1}{3}\right)\left(v\partial_v - \frac{2}{3}\right)h(v) = \left(v\partial_v + \frac{1}{2}\right)^3 h(v).$$

$$\partial_v \left(v\partial_v - \frac{1}{3} \right) \left(v\partial_v - \frac{2}{3} \right) h(v) = \left(v\partial_v + \frac{1}{2} \right)^3 h(v).$$

$$\partial_v \left(v\partial_v + \beta_1 - 1 \right) \left(v\partial_v + \beta_2 - 1 \right) h(v) = \\ = \left(v\partial_v + \alpha_1 \right) \left(v\partial_v + \alpha_2 \right) \left(v\partial_v + \alpha_3 \right) h(v).$$

$$v \left(v\partial_v - \alpha_1 + 1 \right) \left(v\partial_v - \alpha_2 + 1 \right) \left(v\partial_v - \alpha_3 + 1 \right) h(v) = \\ = \left(v\partial_v - \beta_1 \right) \left(v\partial_v - \beta_2 \right) \left(v\partial_v - \beta_3 \right) h(v).$$

Differential equations of the form

$$\partial_z (z\partial_z + \beta_1 - 1)(z\partial_z + \beta_2 - 1)\, h(z) = \\ = (z\partial_z + \alpha_1)(z\partial_z + \alpha_2)(z\partial_z + \alpha_3)\, h(z)$$

are solved in terms of generalized hypergeometric series

$${}_3F_2 \begin{bmatrix} \alpha_1, \alpha_2, \alpha_3; z \\ \beta_1, \ \beta_2 \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n (\alpha_3)_n}{(\beta_1)_n (\beta_2)_n} \frac{z^n}{n!},$$

where $(\alpha)_0 = 1$, $(\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1)$.
The series converges for $|z| < 1$.
Set $s = Re(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3)$.
For $s \leqslant 0$ the series does not converge for $|z| = 1$.

Theorem

In the region $|g_2^3| < |g_2^3 - 27g_3^2|$ *we have*

$$f_{1,2}(w) = A\sqrt{w}\,{}_3F_2\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2};\ w+1\\ \frac{1}{3},\ \frac{2}{3}\end{matrix}\right] +$$
$$+ B\sqrt{w}(w+1)^{1/3}{}_3F_2\left[\begin{matrix}\frac{5}{6},\ \frac{5}{6},\ \frac{5}{6};\ w+1\\ \frac{2}{3},\ \frac{4}{3}\end{matrix}\right] +$$
$$+\ C\sqrt{w}(w+1)^{2/3}{}_3F_2\left[\begin{matrix}\frac{7}{6},\ \frac{7}{6},\ \frac{7}{6};\ w+1\\ \frac{4}{3},\ \frac{5}{3}\end{matrix}\right],$$

A, B, C *are constant, and*

$$f_{1,1}(w) = -6w(1+w)f''_{1,2} - (3+5w)f'_{1,2} - \frac{1}{6}f_{1,2},$$
$$f_{2,2}(w) = -216w(1+w)f''_{1,2} - 36(3+7w)f'_{1,2} - 6f_{1,2}.$$

Differential equations of the form

$$z\,(z\partial_z - \alpha_1 + 1)\,(z\partial_z - \alpha_2 + 1)\,(z\partial_z - \alpha_3 + 1)\,h(z) = \\ = (z\partial_z - \beta_1)\,(z\partial_z - \beta_2)\,(z\partial_z - \beta_3)\,h(z)$$

have singularities at $z = 0$, $z = 1$ and $z = \infty$.
These equations can be solved in terms of Meier G-functions $G_{3,3}^{m,n}(-z)$, where $0 \leqslant m \leqslant 3$, $0 \leqslant n \leqslant 3$, $m + n$ is odd, and

$$G_{3,3}^{m,n}\left(z \,\middle|\, \begin{matrix}\alpha_1, \alpha_2, \alpha_3\\ \beta_1, \beta_2, \beta_3\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(\beta_j - s)\prod_{j=1}^{n}\Gamma(1 - \alpha_j + s)}{\prod_{j=m+1}^{3}\Gamma(1 - \beta_j + s)\prod_{j=n+1}^{3}\Gamma(\alpha_j - s)} z^s ds.$$

Here the empty product is 1 and the poles of $\Gamma(\beta_j - s)$, $j = 1, \ldots, m$, differ from the poles of $\Gamma(1 - \alpha_k + s)$, $k = 1, \ldots, n$.
The integral converges for $|\arg z| < (m + n - 3)\pi$, as well as for $0 < |z| < 1$ and $|z| > 1$.

Theorem

In the region $0 < |\arg \frac{g_2^3}{g_2^3 - 27g_3^2}| < 2\pi$ *we have* $g^{1,2} = f_{1,2}(w)$,

$$g^{1,1} = \frac{\sqrt{3}t^{1/6}(1+w)^{2/3}}{w^{1/2}} f_{1,1}(w), \qquad g^{2,2} = \frac{(1+w)^{1/3}}{\sqrt{3}t^{1/6}w^{1/2}} f_{2,2}(w),$$

$$h(v) = AG_{3,3}^{1,3}\left(-v \left| \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 0, \frac{1}{3}, \frac{2}{3} \end{matrix} \right.\right) + BG_{3,3}^{1,3}\left(-v \left| \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{2}{3}, 0, \frac{1}{3} \end{matrix} \right.\right) + \\ + CG_{3,3}^{1,3}\left(-v \left| \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{3}, \frac{2}{3}, 0 \end{matrix} \right.\right),$$

A, B, C *are constant*, $f_{1,2}(w) = \sqrt{w}h(w+1)$,

$$f_{1,1}(w) = -6w(1+w)f_{1,2}'' - (3+5w)f_{1,2}' - \frac{1}{6}f_{1,2},$$

$$f_{2,2}(w) = -216w(1+w)f_{1,2}'' - 36(3+7w)f_{1,2}' - 6f_{1,2}.$$

Corollary

For a symmetric cometric $G = (g^{i,j})$ *compatible with the Gauss–Manin connection, the function* $\det G$ *is constant on its domain of definition.*

Corollary

The function $\det G$ *is constant in the domain*

$$g_2^3 \neq 27 g_3^2, \quad g_3 \neq 0, \quad g_2 \neq 0.$$

Set $\deg \lambda_k = -2k$.

The vector fields

$$\begin{pmatrix} l_0 \\ l_2 \end{pmatrix} = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 \\ 6\lambda_6 & -\frac{1}{12}\lambda_4^2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial\lambda_4} \\ \frac{\partial}{\partial\lambda_6} \end{pmatrix}, \quad g = 1,$$

$$\begin{pmatrix} l_0 \\ l_2 \\ l_4 \\ l_6 \end{pmatrix} = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\ 6\lambda_6 & 8\lambda_8 - \frac{12}{5}\lambda_4^2 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & -\frac{4}{5}\lambda_4\lambda_8 \\ 8\lambda_8 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & 4\lambda_4\lambda_8 - \frac{12}{5}\lambda_6^2 & 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8 \\ 10\lambda_{10} & -\frac{4}{5}\lambda_4\lambda_8 & 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8 & 4\lambda_6\lambda_{10} - \frac{8}{5}\lambda_8^2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial\lambda_4} \\ \frac{\partial}{\partial\lambda_6} \\ \frac{\partial}{\partial\lambda_8} \\ \frac{\partial}{\partial\lambda_{10}} \end{pmatrix}$$

$$g = 2,$$

form a basis in the space of vector fields on $\mathcal{B}$.

Differential equations, determined by dynamical systems on the base of the universal bundle of algebraic curves.

Each of the vector fields l_k determines a polynomial dynamical system in $\mathbb{C}^{2g}$ with coordinates $(\lambda_4, \lambda_6, \ldots, \lambda_{4g+2})$.
We consider $\lambda_4, \lambda_6, \ldots, \lambda_{4g+2}$ as functions in x, where $l_k = \frac{\partial}{\partial x}$.
For each s the functions λ_s, λ_s', λ_s'', …, $\lambda_s^{(2g)}$
are polynomials in the $2g$ variables $\lambda_4, \lambda_6, \ldots, \lambda_{4g+2}$.
According to the transcendence basis theorem
there exists a polynomial P, such that

$$P(\lambda_s, \lambda_s', \lambda_s'', \ldots, \lambda_s^{(2g)}) = 0.$$

This relation defines a differential equation on λ_s
that is homogeneous with respect to the grading

$$\deg \lambda_s^{(m)} = -2s - 2km.$$

Examples of differential equations $P(\lambda_s, \lambda_s', \lambda_s'', \dots, \lambda_s^{(2g)}) = 0$.

For $g = 1$ set $l_2 = \frac{\partial}{\partial x}$.

$$g_2'' = 2g_2^2,$$

and we have $g_3 = \frac{1}{6} g_2'$.
Therefore for constant c_0, c_3

$$g_2 = 3\wp(x + c_0; 0, c_3), \quad g_3 = \frac{1}{2}\wp'(x + c_0; 0, c_3).$$

For the coefficients of G we obtain

$$g^{1,1} = \frac{1}{2} f'' + \frac{1}{12} g_2 f, \quad g^{1,2} = \frac{1}{2} f', \quad g^{2,2} = f,$$

where f is a solution of

$$f''' + \wp(x + c_0; 0, c_3)\ f' + \frac{1}{2}\wp'(x + c_0; 0, c_3)\ f = 0.$$

Geometric differential equations: genus 2 case.

$$\Gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & -\frac{4}{5}\lambda_4 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{3}{5}\lambda_4 & 0 & 0 & -1 \\ 0 & 3\lambda_8 - \frac{4}{5}\lambda_4^2 & \frac{4}{5}\lambda_4 & 0 \end{pmatrix},$$

$$\Gamma_4 = \begin{pmatrix} 0 & -\frac{6}{5}\lambda_6 & 0 & -1 \\ 0 & \lambda_4 & -1 & 0 \\ -\frac{2}{5}\lambda_6 & \lambda_8 & 0 & 0 \\ \lambda_8 & 6\lambda_{10} - \frac{6}{5}\lambda_4\lambda_6 & \frac{6}{5}\lambda_6 & -\lambda_4 \end{pmatrix},$$

$$\Gamma_6 = \begin{pmatrix} 0 & -\frac{3}{5}\lambda_8 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -\frac{1}{5}\lambda_8 & 2\lambda_{10} & 0 & 0 \\ 2\lambda_{10} & -\frac{3}{5}\lambda_4\lambda_8 & \frac{3}{5}\lambda_8 & 0 \end{pmatrix}.$$

Examples of differential equations $P(\lambda_s, \lambda_s', \lambda_s'', \ldots, \lambda_s^{(2g)}) = 0$.

For $g = 2$ set $l_2 = \frac{\partial}{\partial x}$.
The equation on $\lambda_4 = f(x)$ is

$$5f'''' + 248f''f + 208(f')^2 + 576f^3 = 0,$$

and we have

$$\lambda_6 = \frac{1}{6}f', \quad \lambda_8 = \frac{1}{48}f'' + \frac{3}{10}f^2, \quad \lambda_{10} = \frac{1}{480}f''' + \frac{13}{150}f'f.$$

For the coefficients of G we obtain

$$\frac{\partial}{\partial x}G = -M_2 G - G M_2^\top \quad \text{for} \quad M_2 = \begin{pmatrix} 0 & -4h & -1 & 0 \\ 1 & 0 & 0 & 0 \\ -3h & 0 & 0 & -1 \\ 0 & \frac{5}{16}h'' + \frac{1}{2}h^2 & 4h & 0 \end{pmatrix},$$

where h is a solution of

$$h'''' + 248hh'' + 208(h')^2 + 2880h^3 = 0.$$

Examples of differential equations $P(\lambda_s, \lambda_s', \lambda_s'', \ldots, \lambda_s^{(2g)}) = 0$.

Set $l_4 = \frac{\partial}{\partial x}$.
The equation on $\lambda_4 = f(x)$ is

$$(f'' - 4f'f)f'''' - \frac{1}{2}(f''')^2 - \frac{2}{5}(11f''f - 10(f')^2 - 84f'f^2)f''' -$$
$$-\frac{1}{5}(29f' + 48f^2)(f'')^2 + \frac{8}{5}f'\left(52f''f^3 - 5(f')^3 + 18(f')^2f^2 - 192f'f^4\right) = 0,$$

and we have

$$\lambda_6^2 = -\frac{5}{98}f'' + \frac{5}{24}f'f,$$
$$\lambda_8 = \frac{1}{8}f',$$
$$384\lambda_6\lambda_{10} = -f''' + \frac{4}{5}f''f + 4(f')^2 + \frac{64}{5}f'f^2.$$

Examples of differential equations $P(\lambda_s, \lambda_s', \lambda_s'', \dots, \lambda_s^{(2g)}) = 0$.

Set $l_6 = \frac{\partial}{\partial x}$.
The equation on $\lambda_4 = f(x)$ is the resultant of the polynomials

$$R_1(z) = 768z^4 + 128f''z^2 - 112ff'^2z - 5f'f''' + 5f''^2,$$
$$R_2(z) = 24576z^6 + 9472f''z^4 - 2432f'^2fz^3 + 896f''^2z^2 - \\ - 8(93ff'' + 70f'^2)f'^2z - 25f'^2f'''' + 25f''^3 - 336f^2f'^4.$$

Namely, the equation is of the form

$$(f'''')^4(f')^4 - \frac{4}{15}\left(13f'''f''(f')^3 + 2(f'')^3(f')^2 - 126(f')^6f^2\right)(f'''')^3 + \dots + \\ + \dots + \frac{2^{13}\cdot 3^4\cdot 7^4}{5^8}(f')^{12}f^8 + \frac{2^{12}\cdot 7^5}{5^5}(f')^{14}f^3 = 0.$$