Characterization of simplicial complexes with Buchstaber number two

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Toric topology

Canonical correspondence

simplicial complex K		moment-angle complex $\mathcal{Z}_K$
$\dim K = n - 1$	$\longrightarrow$	$\dim \mathcal{Z}_K = m + n$
number of vertices $= m$		canonical T^m-action

Combinatorics of K $\longleftrightarrow$ Topology of $\mathcal{Z}_K$

K-power

$A \subset X$ – a pair of topological spaces.

$$(X, A)^K = \bigcup_{\sigma \in K} X^\sigma \times A^{[m]\setminus\sigma} \subset X^m,$$

where $X^\sigma \times A^{[m]\setminus\sigma} = X_1 \times \cdots \times X_m$, $X_i = \begin{cases} X, & i \in \sigma \\ A, & i \in [m] \setminus \sigma \end{cases}$

Special cases

$D^2 = \{\boldsymbol{z} \in \mathbb{C} \colon |\boldsymbol{z}| \leqslant 1\}$, $S^1 = \{\boldsymbol{z} \in \mathbb{C} \colon |\boldsymbol{z}| = 1\}$,

$(D^2, S^1)^K$ – a moment-angle complex $\mathcal{Z}_K$.

$D^1 = \{x \in \mathbb{R} \colon |x| \leqslant 1\}$, $S^0 = \{\pm 1\}$

$(D^1, S^1)^K$ – a real moment-angle complex $\mathbb{R}\mathcal{Z}_K \subset \mathcal{Z}_K$.

There are canonical coordinate actions of $T^m = (S^1)^m$ on $\mathcal{Z}_K$, and $(S^0)^m \simeq \mathbb{Z}_2^m$ on $\mathbb{R}\mathcal{Z}_K$.

Definition

A Buchstaber invariant $s(K)$ is the maximal dimension r of toric subgroups $H \subset T^m$, $H \simeq T^r$, that act freely on $\mathcal{Z}_K$.
A real Buchstaber invariant $s_{\mathbb{R}}(K)$ is the maximal dimension s of subgroups $H_2 \subset \mathbb{Z}_2^m$ that act freely on $\mathbb{R}\mathcal{Z}_K$.

$$s(\Delta^{n-1}) = s_{\mathbb{R}}(\Delta^{n-1}) = 0,$$

$$1 \leqslant s(K) \leqslant s_{\mathbb{R}}(K) \leqslant m - n, \qquad K \neq \Delta^{n-1}$$

Buchstaber problem

Problem (V. M. Buchstaber, 02)

To find an EFFECTIVE combinatorial description of $s(K)$.

Modifications (V. M. Buchstaber, 12)

Problem’

For any n to calculate $s(K)$ for all simplicial complexes with $\dim K = n - 1$.

Problem”

For any r to characterize combinatorially simplicial complexes K with $s(K) = r$.

Two descriptions of a toric subgroup $\Rightarrow$

(S) $s(K)$ is the maximal r that admits a matrix $S \in \mathbb{Z}^{m \times r}$ such that for any $\sigma \in K$, the rows $\{S^i : i \in [m] \setminus \sigma\}$ span $\mathbb{Z}^r$;

(Λ) $s(K)$ is the maximal r that admits a mapping $\Lambda : [m] \to \mathbb{Z}^{m-r}$ such that for any simplex $\sigma \in K$ the vectors $\Lambda(\sigma)$ form part of a basis in $\mathbb{Z}^{m-r}$.

Two descriptions of a linear subspace in $\mathbb{Z}_2^m \Rightarrow$

(S_2) $s_{\mathbb{R}}(K)$ is the maximal r that admits a matrix $S \in \mathbb{Z}_2^{m \times r}$ such that for any $\sigma \in K$, the rows $\{S^i : i \in [m] \setminus \sigma\}$ span $\mathbb{Z}_2^r$;

(Λ_2) $s_{\mathbb{R}}(K)$ is the maximal r that admits a mapping $\Lambda : [m] \to \mathbb{Z}_2^{m-r}$ such that for any simplex $\sigma \in K$ the vectors $\Lambda(\sigma)$ are linearly independent.

Generalized chromatic number

Classical chromatic number $\gamma(K)$ – minimal r such that there exists a non-degenerate simplicial mapping $K \to \Delta^{r-1}$.

Simplicial complex $U(r)$

vertices $\longleftrightarrow$ vectors in $\mathbb{Z}^r \setminus \{0\}$

simplices $\longleftrightarrow$ parts of bases in $\mathbb{Z}^r$

$s(K)$ is the maximal r such that there exists a non-degenerate simplicial mapping $K \to U(m-r)$.

Lemma

1. $s(K) \geqslant r \iff s_{\mathbb{R}}(K) \geqslant r$ *for* $r = 1, 2, 3$.
2. $s(K) = s_{\mathbb{R}}(K)$ *for* $\dim K = 0, 1, 2$.

The proof is based on the following fact

Lemma

For a matrix $A \in \{0,1\}^{r \times r}$, $r = 1, 2, 3$, *the equality* $\det A = 1 \mod 2$ *implies* $\det A = \pm 1$. *For* $r \geqslant 4$ *this is not true.*

Some known results

$s(K)$ has been studied since 2001 by I. Izmestiev, A. Ayzenberg, M. Masuda and Y. Fukukawa, E and others.

Proposition (Izmestiev, 01)

$s(K) \geqslant m - \gamma(K)$;

Proposition (E, 08)

$s(K) \geqslant m - \gamma(K) + s(\Delta_{n-1}^{\gamma-1})$.

M. Masuda and Y. Fukukawa (09) obtained nice results for the real Buchstaber invariant of skeleta of simplices.

Simple polytopes

- Let $K_P = \partial P^{\Delta}$, P – simple convex n-polytope with m facets.
- Then $\mathcal{Z}_P$ is a smooth manifold (V. M. Buchstaber, T. E. Panov) such that $\mathcal{Z}_P / T^m = P$.
- $s(\mathrm{pt}) = 0$. We have $1 \leqslant s(P) \leqslant m - n$ for $\dim P > 0$.
- If $s(P) = m - n$, then $\mathcal{Z}_P / T^{m-n}$ is a quasitoric manifold – topological generalization of an algebraic toric manifold.
- The 4-colors theorem $\Longrightarrow s(P) = m - n$ for $n = 3$.
- I. Izmestiev (2001) found the lower bound in terms of the Joswig group of projectivities of P.
- There are polytopes P and Q with equal numbers of faces and chromatic numbers but $s(P) \neq s(Q)$ (E, 08).

Minimal non-simplices

Definition

The set $\omega \subset [m]$ is called a non-simplex, if $\omega \notin K$. Non-simplex ω is minimal, if it's any proper subset belongs to K.

Denote by $N(K)$ the set of all minimal non-simplices.

$$\sigma \in K \Leftrightarrow \nexists \omega \in N(K) \colon \omega \subset \sigma$$

Thus, $N(K)$ determines K in a unique way.
$K = \Delta^n \Leftrightarrow N(K) = \varnothing$;

Theorem (E, 09)

1. *Let* $[m] = \omega_1 \sqcup \dots \sqcup \omega_r$, *where* $\omega_i \notin K$. *Then* $s(K) \geqslant r$.
2. *Let* P *be a simple polytope with* $m - n = 3$. *Then* $s(P) = 3 \Leftrightarrow |N(K_P)| \leqslant 7$.

Problem': Polytopes

- $s(P^0) = 0$;
- $s(P^1) = 1$;
- $s(P^2) = m - 2$;
- $s(P^3) = m - 3$ due to the 4-colors theorem;

Problem': Simplicial complexes

- If $\dim K = 0$ then $s(K) = m - 1$.

Theorem (Ayzenberg, 09)

If $\dim K = 1$, *then* $s(K) = m - \lceil \log_2(\gamma(K) + 1) \rceil$.
In general case $s(K) \leqslant m - \lceil \log_2(\gamma(K) + 1) \rceil$.

Proposition (E, 13)

If $\dim K = 2$, *then*

$$m - 1 - \lceil \log_2(\gamma(K)) \rceil \leqslant s(K) \leqslant m - \lceil \log_2(\gamma(K) + 1) \rceil.$$

In particular, if $\gamma(K) = 2^k$, *then* $s(K) = m - k - 1$.

Problem": Polytopes

Proposition

1. $s(P) = 1$ *if and only if* $P = \Delta^n$, $n > 0$ ($\Longleftrightarrow m - n = 1$).
2. *For any* $k \geqslant 2$ *there exists* P *with* $m - n = k$ *and* $s(P) = 2$;
3. *If* $s(P) = 2$, *then* $2 \leqslant m - n \leqslant 2 + \left[\frac{n}{2}\right]$. *In this case either* $P = I \times \Delta^n$, *or any two facets of* P *intersect. Moreover any* $(m - n - 2)$ *facets of* P *intersect. We have:*
   - *if* $m - n = 2$, *then* $P = \Delta^i \times \Delta^j$;
   - *if* $m - n = 3$, *then* $|N(K_P)| \leqslant 7$;
   - *if* $n = 2$, *then* $P = I \times I$;
   - *if* $n = 3$, *then* $P = I \times \Delta^2$.

Problem”: Simplicial complexes

Proposition

$s(K) \geqslant 1$ if and only if $N(K) \neq \varnothing$, i.e. $K \neq \Delta^n$.

Proposition

$s(K) \geqslant 2$ if and only if $N(K)$ contains one of the subsets:

1. $\{\tau_1, \tau_2, \tau_3\}$: $\tau_1 \cap \tau_2 \cap \tau_3 = \varnothing$;
2. $\{\tau_1, \tau_2\}$: $\tau_1 \cap \tau_2 = \varnothing$.

Theorem

$s(K) \geqslant 3$ if and only if $N(K)$ contains one of the subsets:

1. $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7\}$: $\tau_1 \cap \tau_2 \cap \tau_4 = \varnothing$; $\tau_1 \cap \tau_3 \cap \tau_5 = \varnothing$; $\tau_1 \cap \tau_6 \cap \tau_7 = \varnothing$; $\tau_2 \cap \tau_3 \cap \tau_6 = \varnothing$; $\tau_2 \cap \tau_5 \cap \tau_7 = \varnothing$; $\tau_3 \cap \tau_4 \cap \tau_7 = \varnothing$; $\tau_4 \cap \tau_5 \cap \tau_6 = \varnothing$;
2. $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6\}$: $\tau_1 \cap \tau_3 = \varnothing$; $\tau_1 \cap \tau_2 \cap \tau_4 = \varnothing$; $\tau_1 \cap \tau_2 \cap \tau_5 = \varnothing$; $\tau_1 \cap \tau_4 \cap \tau_6 = \varnothing$; $\tau_1 \cap \tau_5 \cap \tau_6 = \varnothing$; $\tau_2 \cap \tau_3 \cap \tau_6 = \varnothing$; $\tau_3 \cap \tau_4 \cap \tau_5 = \varnothing$;
3. $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$: $\tau_1 \cap \tau_2 = \varnothing$; $\tau_1 \cap \tau_5 = \varnothing$; $\tau_1 \cap \tau_3 \cap \tau_4 = \varnothing$; $\tau_2 \cap \tau_3 \cap \tau_5 = \varnothing$; $\tau_2 \cap \tau_4 \cap \tau_5 = \varnothing$;
4. $\{\tau_1, \tau_2, \tau_3, \tau_4\}$: $\tau_1 \cap (\tau_2 \cup \tau_3 \cup \tau_4) = \varnothing$; $\tau_2 \cap \tau_3 \cap \tau_4 = \varnothing$;
5. $\{\tau_1, \tau_2, \tau_3\}$: $\tau_1 \cap \tau_2 = \tau_1 \cap \tau_3 = \tau_2 \cap \tau_3 = \varnothing$.

Proposition

Condition (S_2) *is equivalent to the following condition* (S_2^*)*: for any nonzero vector* $\boldsymbol{a} \in \mathbb{Z}_2^k$ *there exists* $\omega(\boldsymbol{a}) \in N(K)$ *such that* $\langle \boldsymbol{a}, S^i \rangle = 1$ *in* $\mathbb{Z}_2$ *for all* $i \in \omega(\boldsymbol{a})$.

Proposition

We have $s_{\mathbb{R}}(K) \geqslant r$ *if and only if there exists a mapping* $\xi\colon\ \mathbb{Z}_2^r \setminus \{0\} \to N(K)$ *such that* $\xi(\boldsymbol{a}_1) \cap \cdots \cap \xi(\boldsymbol{a}_{2l+1}) = \varnothing$ *for any minimal linear dependence* $\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_{2l+1} = 0$.

Corollary

Condition $s_{\mathbb{R}}(K) \geqslant 3$ *is equivalent to the existence of the mapping* $\xi\colon\ \mathbb{Z}_2^3 \setminus \{0\} \to N(K)$ *such that* $\xi(\boldsymbol{a}) \cap \xi(\boldsymbol{b}) \cap \xi(\boldsymbol{c}) = \varnothing$ *for any triple of pairwise distinct vectors* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *with* $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = 0$.

Problem

To classify all simplicial complexes K with $s(K) = 2$ and all polytopes P with $s(P) = 2$.

Minimal non-simplices are closely related to other combinatorial characteristics of simplicial complexes such as bigraded Betti numbers

$$\beta^{-i,2j}(K) = \mathrm{rank}\,\mathrm{Tor}^{-i,2j}_{\mathbb{Z}[\mathbf{v}_1,\dots,\mathbf{v}_m]}(\mathbb{Z}[K],\mathbb{Z})$$

For example, $\sum_j \beta^{-1,2j} = |N(K)|$.

Problem

To find a criterion for $s(K) = 2$ in terms of bigraded Betti numbers.

Problem

To find an exact formula for $s(K)$ in the case $\dim K = 2$.

References I

A. Ayzenberg,
The problem of Buchstaber number and its combinatorial aspects,
arXiv:1003.0637v1 [math.CO], 2010.

A. Ayzenberg,
Relation between the Buchstaber invariant and generalized chromatic numbers,
Far-Eastern Math. J.,11:2(2011),113-119.(in russian).

A. Ayzenberg,
Relation between the Buchstaber invariant and generalized chromatic numbers,
Far-Eastern Math. J.,11:2(2011),113-119.(in russian).

Victor M. Buchstaber, Taras E. Panov,
Torus Actions and Their Applications in Topology and Combinatorics,
Providense, R.I.: American Mathematical Society,2002.(University Lecture Series; V.24).

M. Davis, T. Januszkiewicz,
Convex polytopes, Coxeter orbifolds and torus actions,
Duke Math. J., 1991. V.62, N2, 417-451.

Y. Fukukawa, M. Masuda,
Buchstaber invariants of skeleta of a simplex,
Osaka J. Math. Volume 48, Number 2 (2011), 549-582; arXiv:0908.3448v2 [math.AT].

Y. Fukukawa, M. Masuda,
Buchstaber invariants of skeleta of a simplex,
Osaka J. Math. Volume 48, Number 2 (2011), 549-582; arXiv:0908.3448v2 [math.AT].

References II

N. Erokhovets,
Buchstaber invariant of simple polytopes,
UMN, 2008, 63:5, 962-964.

N. Erokhovets,
Buchstaber invariant of simple polytopes,
arXiv:0908.3407 [math.AT], 2009.

N. Erokhovets,
Moment-angle manifolds of simple n-polytopes with $n+3$ facets,
UMN, 2011, 66:5, 1006-1008.

M. Joswig,
The group of projectivities and colouring of the facets of a simple polytope,
Russian Mathematical Surveys, 2001, 56:3, 584-585.

I. V. Izmestiev,
Three-dimensional manifolds defined by coloring a simple polytope,
Math. Notes, 2001, 69:3, 340-346.

I. I. Izmestiev,
Free torus action on the manifold $\mathcal{Z}_P$ and the group of projectivities of a polytope P,
Russian Math. Surveys, 2001, 56:6, 582-583.

Thank You for Your Attention!