Lower and upper bounds for Asian-type options: a unified approach

Alexander Novikov*, Nino Kordzakhia†
Tim Ling‡, Scott Alexander§

*University of Technology, Sydney
†Macquarie University, Sydney
‡University of Technology, Sydney
§University of Technology, Sydney


UTS

Outline

1. *Introduction. Notations. Methods for pricing.*

2. *Monotonicity of prices as functions of some parameters.*

3. *Lower and upper bounds.*

4. *Numerical example for Gaussian models.*

5. *The case of VWAP.*

6. *The case of Levy processes.*

7. *Rate of convergence of prices of DMO to CMO.*

8. *Future research: the Square-root, Heston models, Hedging issues.*

1. Introduction. Notations and Methods for pricing.

We aim to obtain accurate bounds for option prices

$$C_T = Ee^{-R_T} F_T(S),$$

where $R_t = \int_0^t r_s ds$, r_s is an interest rate, $r_s \geq 0$, $F_T(S)$ is an Asian-type payoff of the option written on the stock price $S = (S_t, 0 \leq t \leq T)$, T is the maturity time. The typical payoff for Asian-type options is

$$F_T(S) = (\int_0^T S_u d\mu(u) - \xi_T)^+, \tag{1}$$

where $x^+ = \max[x, 0] = (-x)^-$ for any x, $\xi_T = K$ is a fixed strike or $\xi_T = S_T$ is a floating strike, $\mu(u)$ is a distribution function on the interval $[0, T]$.

(We assume that all random processes are defined on the same *filtered probability space* $(\Omega, \{F_t\}_{t\geq 0}, P)$.)

The Asian option was “invented by Phelim P. Boyle and David Emanuel in 1979, but The Journal of Finance rejected their paper since the asset was not traded at that time” (Foufasa & Larson (2008)).

Asian options were introduced partly to avoid a problem common for European options, where the speculators could drive up the gains from the option by manipulating the price of the underlying asset near to the maturity date (Wall Street Journal, Jan. 21, 1982, p. 4).

From Carr et al (2009).

“According to the Handbook of Exotic Options (Nelken, 1996) the name Asian option was coined by employees of Bankers Trust, which sold this type of options to Japanese firms that wanted to hedge their foreign currency exposure. These firms used these options because their annual reports were also based on average exchange rates over the year. Average type options are particularly suited to hedge risk at foreign exchange markets and by reason of the averaging effect, significantly cheaper than plain vanilla options.”

“Effectively such options are traded since the mid 1980's and first appeared in the form of commodity linked bonds. Specific examples are the Mexican Petro Bond and the Delaware Gold Index Bond. Asian options are OTC traded, however market and trading volume appear to grow very fast. A recent study of Canadian Imperial Bank of Commerce world markets revealed that Asian style options are **the most commonly traded exotic** options.”

Using the notation

$$\overline{h} = \int_0^T h_u \mu(du), h \in H,$$

where H is the class of adapted random processes $h = (h_s, 0 \leq s \leq T)$ such that $\int_0^T |h_u|\mu(du) = \overline{|h|} < \infty \quad a.s.$, we can rewrite (**??**) as follows

$$F_T(S) = \overline{(S-K)}^+ = (\overline{S} - K)^+. \tag{2}$$

In relation to discretely monitored options (**DMO**) or continuously monitored options (**CMO**) the distribution function μ can be discrete or continuous respectively.

This setup also includes the case of call options on the volume-weighted average price (VWAP), that is

$$V_T := \frac{\sum\limits_{t_j \le T} S_{t_j} U_{t_j}}{\sum\limits_{t_j \le T} U_{t_j}}, \ F_T(S) = (V_T - K)^+,$$

where U_{t_j} is a traded volume at the moment t_j. By setting

$$\mu(u) = \frac{\sum\limits_{t_j \le u} U_{t_j}}{\sum\limits_{t_j \le T} U_{t_j}} \ , 0 \le u \le T,$$

we obtain the representations (**??**) and (**??**) for options on VWAP.

Our motivations to consider such options: requests from the electric generator company and NAB (National Australia Bank), rules of Australian Taxation Law.

The presentation of classical Asian payoffs in the form (**??**) was mentioned by Rogers and Shi (1995) and Večeř (2005) where they used the PDE approach for finding C_T for CMO under the geometric Brownian motion (**gBm**) model and constant interest rates.

The paper Rogers and Shi (1995) generated a flow of related results about lower and upper bounds under different settings. We would like to mention here the pioneering paper by Curran (1994) and the unpublished paper by Thompson (2000); in fact, the latter contains some ideas which we are developing further here.

Note that all above cited papers assume that the interest rate process is constant.

Other methods of pricing:

Matching moments: Milevsky at al (1998), for VWAP: Stace (2007), Novikov at al (2013);

Recurrent integration: Fusai et al (2008), March (2008);

Laplace transform, series expansions: Geman&Yor (1993), Dufresne D. (2000), Dassios &Nagaradjasarma (2006), Cai&Kou (2012);

PDE or PIDE, finite-differences or finite elements: (Fouque & Han (2003), Foufasa & Larson (2008));

Lower/Upper bounds: (Rogers and Shi (1995), Lord (2006), Chen and Lyuu (2007), Lemmens et al (2010) and others);

Monte-Carlo.

The case of floating strikes, that is options with the payoff $F_T(S) = (\overline{S} - S_T)^+$, can be reduced to the case (**??**).

2. Monotonicity of prices as functions of some parameters.

Consider the setup for models with log-returns in the form

$$X_u = R_u + \sigma W_u - u\sigma^2/2, \; R_u = \int_0^u r_s ds, \; u \le T$$

where W_u is a Brownian motion with variance u.

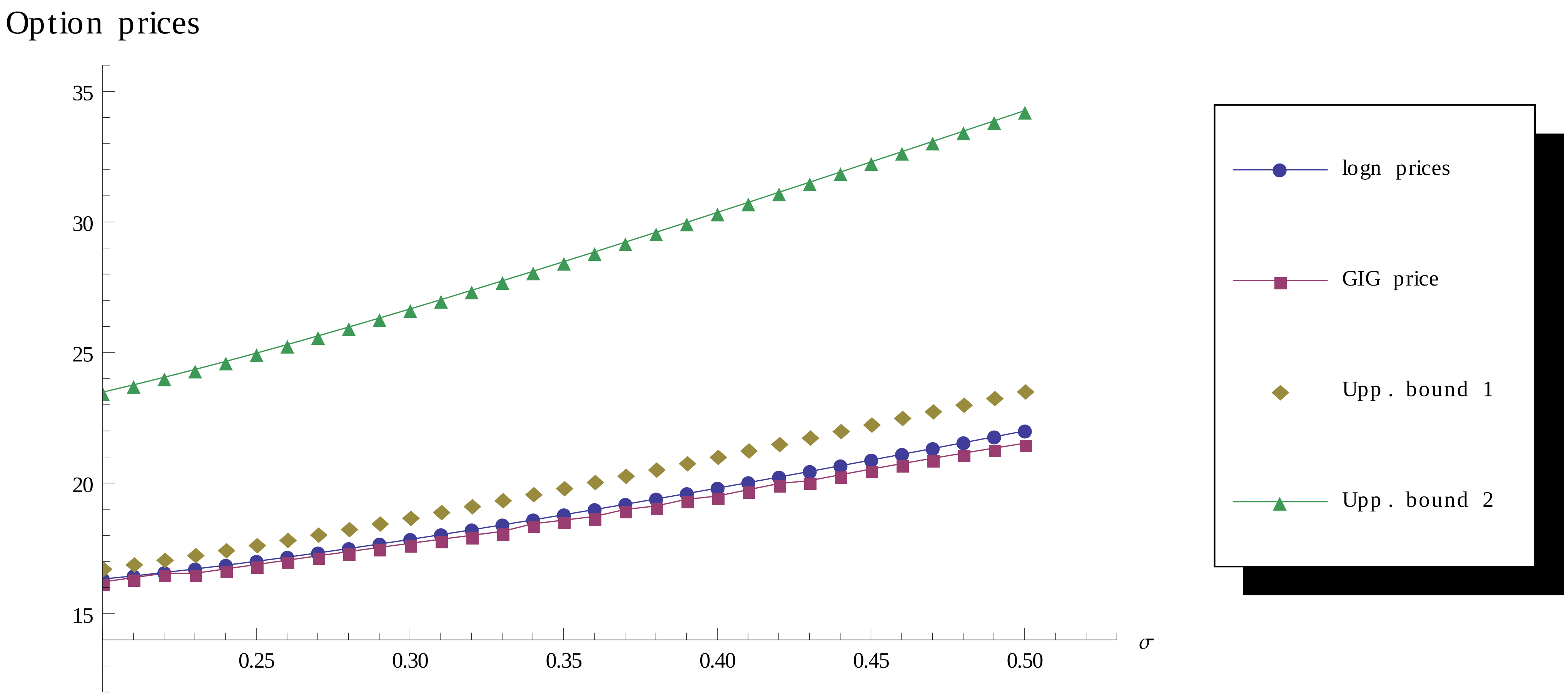
Recall that under these assumptions the discounted stock price

$$e^{-R_t} S_t := S_0 e^{\sigma W_t - \sigma^2/2 \; t + Q_t}$$

is a martingale with respect to the natural filtration, as required by the non-arbitrage theory.

Simple Upper Bounds for the case of fixed strike $\xi_T = K$:

$$C_T(\sigma) := E e^{-R_T}(\overline{S} - K)^+ \le E e^{-R_T}\overline{(S-K)^+} \le E e^{-R_T}(S_T - K)^+.$$


Option prices
35
30
25
20
15
0.25
0.30
0.35
0.40
0.45
0.50
σ
logn prices
GIG price
Upp . bound 1
Upp . bound 2

Proposition 1. *The price $C_T(\sigma)$ is an increasing function of σ.*

Remark. The first result of this type was proved by Carr et al (2008). They showed that under the Black Scholes assumption the price of an arithmetic average Asian call option with fixed strike increases with the level of volatility. They noted that this statement is not trivial to prove and for other models in general wrong (e.g. Binomial two-period). Next it was Baker and Yor (2009) who found a simple proof which we follow here.

Generalisations:

1. instead of W_u one can substitute any continuous martingale Y_u i.e.

$$X_u = R_u + \sigma Y_u - \sigma^2/2 \ < Y >_u, \ u \le T;$$

2. instead of the payoff function $(\overline{S} - K)^+$ one can substitute any convex function $g(\overline{S})$

(Baker and Yor calls this: "increasing in the convex order".

3. if $r_t = r =$*const* then the monotonicity with respect T holds too;

4. the result holds for Bermudian-type options as well etc.

Proof of Proposition 1. It is very simple! Note that

$$\{\sigma W_u, u \le T\} \overset{d}{=} \{B_{u,\sigma^2}, u \le T\}$$

(in distribution), where $\{B_{u,v}, u \ge 0, v \ge 0\}$ is a two-parameter Gaussian random field with

$$E(B_{u,v}) = 0, Cov(B_{u_1,v_1}, B_{u_2,v_2}) = \min(u_1, u_2) \min(v_1, v_2).$$

Obviously, the process $B_{u,} := (B_{u,v}, v \ge 0)$ is a martingale with respect to the filtration $G_{,v} = \sigma\{B_{u,s}, u \le T, s \le v\}$ which can be assumed to be independent on the original filtration $\{F_t\}_{t \ge 0}$.

This implies that

$$E(B_{u,\sigma_2^2} - B_{u,\sigma_1^2})^2 = u|\sigma_2^2 - \sigma_1^2|$$

and hence that the process

$$Y_\sigma = \int_0^T (S_u - K)d\mu(u), \ \sigma \geq 0$$

is also a martingale with respect to the filtration $G_{,\sigma^2}$ ("Peacock" by the terminology of Yor & others).

Indeed, for $\sigma_2 > \sigma_1$

$$\begin{aligned} E(Y_{\sigma_2}|G_{,\sigma_1}) &= \int_0^T (E(S_u|G_{,\sigma_1}) - K)d\mu(u) \\ &= \int_0^T (E(S_0 e^{B_{u,\sigma_1^2} + B_{u,\sigma_2^2} - B_{u,\sigma_1^2} - \sigma_2^2/2\ u + Q_u} |G_{,\sigma_1}) - K)d\mu(u) \\ &= \int_0^T (S_0 e^{B_{u,\sigma_1^2} + \sigma_2^2/2\ u - \sigma_1^2/2\ u - \sigma_2^2/2\ u + Q_u} - K)d\mu(u) = Y_{\sigma_1}. \end{aligned}$$

Therefore by Jensen's inequality we have for any convex function $g(x)$

$$E(g(Y_{\sigma_2})|G_{,\sigma_1}) \geq g(Y_{\sigma_1})$$

and hence Theorem 1 does hold because

$$\begin{aligned} C_T(\sigma_2) &\geq Ee^{-R_T} E(\overline{S}(\sigma_2)|G_{,\sigma_1}) - K)^+) \\ &= Ee^{-R_T} (\overline{S}(\sigma_1) - K)^+ = C_T(\sigma_1). \end{aligned}$$

3. Lower and upper bounds.

Technical tool.

Proposition 2. *Let z be a real number. Then*

$$C_T = \sup_{z, h \in H} E e^{-R_T} (\overline{S} - K) I\{\overline{h} > z\} \tag{3}$$

$$= \inf_{h \in H} E e^{-R_T} \overline{(S - K(1 + h - \overline{h}))^+}, \tag{4}$$

where both supremum and infimum are attained by taking

$$h_u = S_u / K \tag{5}$$

and $z = 1$.

Proof. For any $h \in H$ and z

$$(\overline{S} - K)^{+} I\{\overline{h} > z\} = (\overline{S} - K) I\{\overline{h} > z\} + (\overline{S} - K)^{-} I\{\overline{h} > z\}$$

$$\geq (\overline{S} - K) I\{\overline{h} > z\},$$

thus we obtain

$$C_T = Ee^{-R_T}(\overline{S} - K)^{+} \geq \sup_{z,h \in H} Ee^{-R_T}(\overline{S} - K) I\{\overline{h} > z\}. \qquad (6)$$

Since $(\overline{S} - K)^{+} = (\overline{S} - K) I\{\overline{S}/K > 1\}$, the equalities in (**??**) and correspondingly in (**??**) *are attained* when $z = 1$ and $h_u = S_u/K$ since for the latter case $\overline{h} = \overline{S}/K$.

To prove (**??**) we note that for any $h \in H$

$$C_T = Ee^{-R_T}\overline{(S-K)}^{+}$$

$$= Ee^{-R_T}(\overline{S-K(1+h-\overline{h})})^{+} \le Ee^{-R_T}\overline{(S-K(1+h-\overline{h}))^{+}}, \qquad (7)$$

where the last inequality is due to convexity of x^+. This implies that the C_T is not greater than infimum of the RHS of (**??**) over $h \in H$. The equality in (**??**) *is attained* when $h_u = S_u/K$ since for the latter case

$$\begin{aligned} \overline{(S-K(1+h-\overline{h}))^{+}} &= \overline{(S-K(1+S/K-\overline{S}/K))^{+}} \\ &= \overline{(\overline{S}-K)}^{+} = (\overline{S}-K)^{+}. \end{aligned}$$

Further we use the notation

$$X = (X_t := \log(S_t/S_0), t \le T)$$

and assume that the discounted process $e^{-R_t}S_t = S_0 e^{X_t - R_t}$ is a martingale with respect to the filtration $\{F_t\}_{t\ge 0}$, as required by the non-arbitrage theory.
Of course, if the distribution of $\bar{S}$ (or its Laplace/Fourier transform) is known there is no need to use Theorem 2. But besides the cases when $r = const$ and S_t is a gBm (Geman&Yor (1993)) or square-root process (Dassios & Nagaradjasarma, J. (2006) or jump-diffiusion with exponential jumps (Cai& Kou(2012)) there are no other cases when the distribution of $\bar{S}$ is known. Contrarily, the joint distribution of $(X, \bar{X})$ (or its Laplace/Fourier transform) can be found easily for many cases. Theorem 2 implies that, for all $h \in H$, the following lower and upper bounds hold

$$C_T \ge LB := S_0 \sup_z Ee^{-R_T}(\overline{e^X} - \frac{K}{S_0})I\{\overline{h} > z\}, \tag{8}$$

$$C_T \le UB := S_0 Ee^{-R_T}\overline{(e^X - \frac{K}{S_0}(1 + h - \overline{h}))^+}. \tag{9}$$

How to find a proper h ?

To find a process h producing accurate bounds we need to take into account the complexity of calculations of the joint distribution of $(X, h, \bar{h})$. Obviously, the problem can be made computationally affordable when h_u is a linear function of X_u, that is under the choice

$$h_u = a(u)X_u + b(u)$$

with some nonrandom functions $a(u)$ and $b(u)$. Since both inequalities (**??**) and (**??**) are, in fact, equalities when (**??**) holds, one may try to match the first moments of h_u and S_u/K, that is to set

$$Eh_u = E(S_u/K),\ Var(h_u) = Var(S_u/K).$$

Here we apply a simpler choice with $a(u) = a = const$ and $b(u) = 0$, i.e.

$$h_u = aX_u \ , \tag{10}$$

where the constant a needs to be chosen in the upper bound. For the latter case we have

$$C_T \geq LB1 := S_0 \sup_z Ee^{-R_T}(\overline{e^X} - \frac{K}{S_0}) I\{\overline{X} > z\}, \tag{11}$$

$$C_T \leq UB1 := S_0 \inf_a Ee^{-R_T}\overline{(e^X - \frac{K}{S_0}(1 + aX - a\overline{X}))^+}. \tag{12}$$

Note that the calculation of the lower bound (**??**) does not depend on a choice of the constant a.

Remark 1. Assuming that $R = (R_t, 0 \leq t \leq T)$ and $X = (X_t, 0 \leq t \leq T)$ are independent processes, one may use another lower bound

$$C_T \geq LB2 := S_0 E e^{-R_T} \overline{(E(e^X|\overline{X}) - \frac{K}{S_0})^+} \qquad (13)$$

which holds due to the equality $C_T = S_0 E e^{-R_T} \{E(\overline{e^X - \frac{K}{S_0})}^{\,+} |\overline{X})\}$ and convexity of x^+. (This bound was used by Curran (1994) and Rogers&Shi (1995) and many others.)

Note that under the additional assumption

$$g(x) := \overline{E(e^X|\overline{X} = x)} \text{ is an increasing function of } x \tag{14}$$

we have

$$LB1 \geq LB2.$$

Indeed, one can see that

$$\begin{aligned} LB2 &= S_0 E e^{-R_T} \overline{(E(e^X|\overline{X}) - \frac{K}{S_0})} I\{E\overline{(e^X|\overline{X})} > \frac{K}{S_0}\} \\ &= S_0 E e^{-R_T} \overline{(E(e^X|\overline{X}) - \frac{K}{S_0})} I\{\overline{X} > g^{-1}(\frac{K}{S_0})\}, \end{aligned}$$

where g^{-1} is the inverse function.

Now it is clear that $LB2$ does not exceed $LB1$ since one can use the obvious representation

$$LB1 = S_0 \sup_z Ee^{-R_T} \overline{(E(e^X|\overline{X}) - \frac{K}{S_0})} I\{\overline{X} > z\}.$$

It is easy to check that the condition (**??**) holds in the classical model where X is a Brownian motion and r is a nonrandom function.

4. Numerical illustration for Gaussian models

Here we suppose that the process $X = (X_u, 0 \leq u \leq T)$ is a continuous Gaussian process. **To simplify the exposition further we suppose that the process r_t is a nonrandom bounded function.** The case of stochastic interest rates which are independent of S_t, can be treated in a similar way.

The pair $(X_u, \overline{X})$, obviously, has a Gaussian distribution with

$$\begin{aligned} Cov(X_u, \overline{X}) &= \int_0^T Cov(X_u, X_s) d\mu(s), \qquad (15) \\ Var(\overline{X}) &= \int_0^T \int_0^T Cov(X_u, X_s) d\mu(u) d\mu(s). \end{aligned}$$

Below we consider a numerical example which corresponds to the gBm model with

$$X_u = R_u + \sigma W_u - \sigma^2/2 \ u,$$

where W_u is a standard Bm. For the case of DMO we assume that $\mu(u)$ is an uniform discrete distribution on (0,T] with jumps at points

$$u_i = \frac{i}{N}T, \ i = 1, ..., N,$$

where N is the number of time units (e.g. trading days). In terms of this notation

$$C_T = Ee^{-R_T}F_T(S) = E\frac{e^{-R_T}}{N}\sum_{t_i \le T}(S_{t_i} - K)^+.$$

From (**??**) we obtain

$$\kappa(u_i) := cov(W_{u_i}, \overline{W}) = \sum_{j=1}^{N} \min(u_i, s_j)T/N = u_i(T - \frac{u_i}{2} + \frac{T}{2N}),$$

$$V_N := Var(\overline{W}) = \frac{T}{3}(1 + \frac{3}{2N} + \frac{1}{2N^2}).$$

Note that letting $N \to \infty$ one can obtain the characteristics needed for the pricing of CMO as well.

For numerical illustrations and comparisons we consider the set of parameters $S_0 = K = 100$, $\sigma = 0.3$, the interest rate

$$r_s = 0.09(1 + c/2 \sin(2\pi s)), \tag{16}$$

where the parameter $c = 0$ or $c = 1$.

One can speed up calculations of the bounds using the function $erfc(x)$. For example, using the Girsanov transformation we have obtained the following expression for the lower bound

$$LB1 =$$

$$\frac{e^{-R_T} S_0}{2T\ N} \max_z [\sum_i e^{R_{u_i}}\ erfc\{\sqrt{V_N/2}(z - \sigma\kappa(u_i))\} - \frac{K}{S_0} erfc\{\sqrt{V_N/2} z\}].$$

It takes less than a quarter of a second with Mathematica for any σ to find this lower bound. Computing the upper bound UB2 is also relatively fast (up to 7 seconds using Mathematica for fixed a) but essentially slower with use of the command *FindMinimum* in Mathematica. The optimal value of a for the upper bound (**??**) is usually found in the interval $(0.7, 1)$. In fact, we found that UB1 with the choice $a = 0.9$ produces a reasonable accuracy for approximated prices (the errors are less than 0.5% for wide ranges of T, σ and K).

In Table 1 the numerical results for LB1 and UB1 obtained with Mathematica are reported with three decimal digits. We provide the calculated bounds for two cases $c = 0$ and $c = 1$ in (**??**); the results for $c = 1$ are formatted in bold and placed in brackets. As an estimate for the price we consider a midpoint of the interval $(LB1, UB1)$:

$$\widehat{C}_T = \frac{LB1 + UB1}{2}.$$

The following bound is valid for the relative error of $\widehat{C}_T$:

$$|\widehat{C}_T / C_T - 1| 100\% \leq (UB1/LB1 - 1) 50\%.$$

Table 1. Lower/Upper bounds, gBm model

================================

T	N	$LB1$	$UB1$	$error$ % for $\widehat{C}_T$
	10	12.162 **(12.135)**	12.259 (**12.239**)	0.4 (**0.42**)
1	50	11.782 (**11.785)**	11.829 (**11.807**)	0.1 **(0.11)**
	∞	11.718 (**11.741**)	11.731 **(11.769)**	0.03 **(0.11)**
	10	56.344 **(60.769)**	57.233 **(61.568)**	0.78 **(0.68)**
9	50	56.073 (**60.066)**	56.419 (**60.506)**	0.3 **(0.37)**
	∞	56.012 (**60.014)**	56.146 (**60.197)**	0.1**7** **(0.15)**

================================

As one might anticipate, the prices for options with longer maturities (here $T = 9$) depend essentially on the term structure of interest rates.

5. The case of VWAP

We applied the method of matching moments for finding approximations for VWAP option prices in Novikov et al (2010) under the assumptions that S_t is a gBm, the volume process U_t is a squared Ornstein-Uhlenbeck process, that S_t and U_t are *independent* and $r_t = r = const$. The key point in the approach used in Novikov et al (2010) was the development of technique for finding the function

$$g = (g_t := E\frac{U_t}{\overline{\overline{U}}}, 0 \le t \le T).$$

With the choice of $h_t = aX_t,$ Theorem 2 implies the following bounds

$$C_T \geq S_0 e^{-rT} \sup_z E \overline{\;(e^X - \frac{K}{S_0})gI\{\overline{X} > z\}},$$

$$C_T \leq S_0 e^{-rT} \inf_a E \overline{\;(e^X - \frac{K}{S_0}(1 + aX - a\overline{X}))^+ g},$$

where the averaging is supposed to be with respect to a uniform discrete or continuous distribution on $(0, T]$ for DMO or CMO cases respectively.

The method for calculation of the function g :

$$g_t = \int_0^\infty \frac{\partial}{\partial z} \mathbb{E} \left(e^{zU_t - qV_T} \right) \Big|_{z=0} dq,$$

Numerical example.

We consider the case of CMO with the following parameters (to match the result from Stace (2007)):

$$dS_t = 0.1\ S_t dt + \sigma S_t dW_t,\ \ S_0 = 110, T = 1,\ \ K = 100$$

$$U_t = X_t^2, dX_t = 2(22 - X_t)dt + 5dW_t, X_0 = 22$$

The function g is known in an analytical form (Novikov et al (2013)).

Table 2. Lower/Upper bounds and MC results, gBm model

σ	$LB1$	$MC\ (error)$	$UB1$
0.1	14.1985	14.1991 (0.0019)	14.2042
0.5	19.6127	19.6406 (0.0083)	19.6503
0.8	25.5914	25.6424 (0.0142)	25.7845

For Monte Carlo we used 10 million trajectories and 500 discretisation points.

6. The case of Levy processes

Set

$$X_u = R_u + L_u, \quad L_u = \sigma W_u - \sigma^2/2\ u + J_u,\ R_u = \int_0^u r_s ds,$$

where W_u is a standard Brownian motion (**Bm**), J_u is the jump part of the Levy process L_u,

$$E \exp\{L_u\} = 1.$$

Under these assumptions the discounted stock price

$$M_t = e^{-R_t} S_t = S_0 e^{L_t}$$

is a martingale with respect to the natural filtration, as required by the non-arbitrage theory.

The following Lemma 1 allows us to find the joint characteristic function of the pair $(L_u, \int_0^T L_u b(u)d\mu(u))$ and $(X_u, \int_0^T X_u b(u)d\mu(u))$ as well when the cumulant of the characteristic function of L_t i.e. the function

$$\phi(\lambda) = \frac{1}{t}\log(Ee^{i\lambda L_t}),\ \ \lambda \in (-\infty, \infty),$$

is known explicitly.

Set

$$G(t) = \int_0^t b(u) d\mu(u).$$

Proposition 3. *Assume $g(u)$ is a non-random function of finite variation. Then for $\lambda \in (-\infty, \infty)$ and $z \in (-\infty, \infty)$*

$$E \exp\{i\lambda L_t + iz \int_0^T L_u b(u) d\mu(u)\} \tag{17}$$
$$= \exp\{\int_0^T \phi(\lambda I\{t > u\} + z(G(T) - G(u_-)))du\}.$$

Proof. Integrating by parts we obtain

$$\int_0^T L_u b(u) d\mu(u) = \int_0^T L_u dG(u) =$$
$$L_T G(T) - \int_0^T G(u_-) dL_u = \int_0^T (G(T) - G(u_-)) dL_u.$$

Similarly,

$$\lambda L_t + z \int_0^T L_u b(u) d\mu(u) = \int_0^T [\lambda I\{t > u\} + z(G(T) - G(u_-))] dL_u.$$

The characteristic function of r.v. $\int_0^T f(u)dL_u$ for continuous nonrandom $f(u)$ was found by Lukacs (1969):

$$E\,\text{exp}\{iz\int_0^T f(u)dL_u\} = \text{exp}\{\int_0^T \phi(zf(u))ds\}. \tag{18}$$

The fact that this result does hold under the assumption that $f(u)$ is a function of bounded variation can be shown with the help of a proper limit procedure or directly with use of tools of stochastic calculus, e.g. semimartingale representation of jump processes, see Liptser & Shiryayev (1989).

Since the function $f(u) = \lambda I\{t > u\} + z(G(T) - G(u_-))$ has a bounded variation, we obtain (**??**).

7. Rate of convergence of prices of DMO to CMO

As one may note, the price of options in Table1 for $n = 50$ and $n = \infty$ are not very different. The following result can be used as an explanation of this phenomenon and also, it can be used for a justification of an interpolation formula (*Richardson extrapolation*).

Consider the setup for models with log-returns in the form

$$dS_t = r_t S_t dt + S_t dY_t$$

Y_t is a continuous square-integrable martingale (e.g. a Heston model) such that $d < Y >_t = \sigma_t^2 dt$,

$$\sup_t ES_t^4 < \infty, \sup_t Er_t^2 < \infty, \sup_t E\sigma_t^4 < \infty.$$

Assume that the discounted stock price $e^{-R_t}S_t$ is a martingale with respect to the natural filtration, as required by the non-arbitrage theory.

Set

$$C_T := Ee^{-R_T}(\frac{1}{T}\int_0^T S_u du - K)^+, C_{T,N} := Ee^{-R_T}(\frac{1}{N}\sum_{t_i \le T} S_{t_i} - K)^+.$$

Proposition 4. *Let* $t_i = \frac{iT}{N}, i = 1, ..., N$. *Then:*

1) for any $N \ge 1$

$$|C_T - C_{T,N}| \le \frac{Const}{N};$$

2) as $N \to \infty$

$$N(\frac{1}{T}\int_0^T S_u du - \frac{1}{N}\sum_{t_i \le T} S_{t_i}) \to \frac{S_0 - S_T}{2} \quad (\textit{in probability});$$

3)

$$C_{T,N} = Ee^{-R_T}(\frac{1}{T}\int_0^T S_u du - K + \frac{S_T - S_0}{2NT})^+ + o(\frac{1}{N}), \; N \to \infty.$$

Richardson extrapolation formula:

$$C_{T,N} \approxeq C_T + \frac{A}{N}$$

where A can be estimated when we know C_T and $C_{T,50}$ (e.g.).

Proof of Proposition 4, 1). Due to the elementary inequality

$$|x^+ - y^+| \leq |x - y|$$

we have

$$|C_T - C_{T,N}| \leq E|\frac{1}{T}\int_0^T S_u du - \frac{1}{N}\sum_{t_i \leq T} S_{t_i}| = \frac{1}{T}E|\int_0^T (t^{(N)} - t) dS_t|,$$

where to obtain the last equality we have applied "integration by parts" and used the notation $t^{(N)}$ for a piece-wise constant function such that $t^{(N)} = \frac{iT}{N}$ for $t \in (t_{i-1}, t_i)$.

We have:

$$E|\int_0^T (t^{(N)} - t) dS_t| \leq \int_0^T |t^{(N)} - t| E r_t S_t dt + (\int_0^T (t^{(N)} - t)^2 E(S_t^2 \sigma_t^2) dt)^{1/2}$$

where the last inequality is due to the isometry of stochastic integrals.

Since $\max_t |t^{(N)} - t| = \frac{T}{N}$ and all other functions are bounded, one can see that Theorem 3 does hold.

Remark. Part 2 of Proposition 4 is in fact a stochastic version of "Trapezoid rule" for numerical approximation of integrals:

$$\frac{1}{T}\int_0^T S_u du \approx \frac{1}{N}\sum_{i=1}^{N} S_{t_i} + \frac{S_0 - S_T}{2N} = \frac{1}{N}\sum_{i=1}^{N-1} S_{t_i} + \frac{S_0 + S_T}{2}$$

8. Future research: the Square-root, Heston models, Hedging issues.

Consider a stock price modelled by a square root (SR) process of the form

$$dS_t = rS_t dt + \sigma\sqrt{S_t}dW_t.$$

Dassios and Nagaradjasarma (2006) developed a series expansion for the price of an arithmetic fixed-strike Asian CMO under a SR process. They compared their method to the gBm case but provided no direct numerical comparison for the SR case.

Table 3a demonstrates that their results are in line with those from Monte Carlo methods. Table 3b shows a Richardson extrapolation for a DMO with 50 averaging points based on the Monte Carlo prices of a CMO and a DMO with 10 averaging points. For all examples $T = 1$ and an order-2 scheme with reflection about $S_t = 0$ has been used.

Table 3a.

===========================

S	r	σ	K	$MC\ (error)$	$D\&N$
1.9	0.05	0.69	2.0	0.1909 (1.9e-04)	0.1902
2.0	0.02	0.14	2.0	0.0556 (3.5e-05)	*0.0197*
2.1	0.05	0.72	2.0	0.3091 (2.1e-04)	0.3098
2.0	0.05	0.71	1.0	1.0020 (8.2e-05)	1.0017
2.0	0.05	0.71	3.0	0.0218 (7.8e-05)	0.0210

===========================

1 million trajectories each with 1000 discretisation points have been used. Note the discrepancy in the second D&N price when compared to the MC output (in italics).

Table 3b.

S	r	σ	K	$MC10$ $(error)$	$MC50$ $(error)$	RE
1.9	0.05	0.69	2.0	0.2090 (2.e-04)	0.1945 (2.e-04)	0.1945
2.0	0.02	0.14	2.0	0.0600 (4.e-05)	0.0564 (4.e-05)	0.0564
2.1	0.05	0.72	2.0	0.3287 (2.e-04)	0.3129 (2.e-04)	0.3130
2.0	0.05	0.71	1.0	1.0085 (1.e-04)	1.0032 (8.e-05)	1.0033
2.0	0.05	0.71	3.0	0.0293 (9.e-05)	0.0232 (8.e-05)	0.0233

1 million trajectories each with 1000 discretisation points have been used. "MC10" and "MC50" are the MC prices of DMOs with 10 averaging points and 50 averaging points respectively and "RE" is the Richardson extrapolation for DMOs with 50 averaging points.

References

Albrecher, H.; Mayer, P. A.; Schoutens, W. (2008). General lower bounds for arithmetic Asian option prices. Appl. Math. Finance 15, no. 1-2, 123–149

Baker, D. and Yor,, M. (2009). A Brownian sheet martingale with the same marginals as the arithmetic average of geometric Brownian motion. Electronic Journal of Probability, paper No. 52, pages 1532–1540.

Cai, N. and Kou, S. (2012). Pricing Asian Options Under a Hyper-Exponential Jump Diffusion Model. Operation Research, vol. 60, no. 1, pp. 64–77.

Carr, P., Ewald C. and Xiao Y. (2008). On the qualitative effect of volatility and duration on prices of Asian options. Finance Research Letters, Sept., v.5, 162–171.

Chen K., Lyuu Y. (2007). Accurate pricing formulas for Asian options Applied Mathematics and Computation 188, 1711–1724.

Curran M. (1994). Valuing Asian and portfolio options by conditioning on the geometric mean price Management Science 40, 1705–1711.

Dassios, A. and Nagaradjasarma, J. (2006). The square-root process and Asian options, Quantitative Finance, 6:4, 337-347

Dufresne D. (2000). Laguerre series for Asian and other options. Math. Finance 10 (2000), no. 4, 407–428.

Forsyth, Peter A.; Vetzal, Kenneth R. (2012). Numerical methods for nonlinear PDEs in finance. Handbook of computational finance, 503–528, Springer Handb. Comput. Stat., Springer, Heidelberg.

Foufasa G. , Larson M. (2008). Valuing Asian options using the finite element method and duality techniques. Journal of Computational and Applied Mathematics, 222, 144–158.

Fouque, Jean-Pierre and Han, Chuan-Hsiang. (2003). Pricing Asian options with stochastic volatility. Quant. Finance 3, no. 5, 353–362.

Fusai G., Meucci A. (2008). Pricing discretely monitored Asian options under Lévy processes, J. Bank Finance 32 (10), 2076–2088.

Geman, H. and Yor, M. (1993) Bessel processes, Asian options and perpertuities. Math. Finance, 3, 349–375.

Karatzas I., Shreve St. (1998). Methods of mathematical finance. Applications of Mathematics (New York), 39. Springer-Verlag, New York.

Lemmens D., Liang L.Z.J., Tempere J., De Scheppe A. (2010). Pricing bounds for discrete arithmetic Asian options under Lévy models, Physica A, 389, 5193–5207.

Liptser R. Sh., Shiryayev A. N. (1989). Theory of martingales. Mathematics and its Applications (Soviet Series), 49. Kluwer Academic Publishers Group, Dordrecht.

Lukacs, E. (1969). A characterization of stable processes. J. Appl. Probability, v. 6, 409–418.

March C. (2008). Methods for Pricing Average Rate Options in Exponential Lévy Models. Talk presented at 5th World Congress of the Bachelie Finance Society.

Milevsky M. and Posner S. (1998). Asian Options, the Sum of Lognormals, and the Reciprocal Gamma Distribution. Journal of Financial and Quantitative Analysis, Volume 33, Issue 03, September 1998, 409-422, Nelken I. (1996). The handbook of exotic options : instruments, analysis, and applications.Chicago : Irwin Professional Pub.

Novikov A., Ling T., Kordzakhia N. (2010). Pricing of volume-weighted average options: analytical approximations and numerical results. Festsc M. Musiela. Springer (in print).

Rogers L.C.G., Shi Z. (1995). The Value of an Asian Option. J. Appl. Prob. 32, 1077-1088.

Rogers L.C.G. (1992). Monte Carlo valuation of American options. Mathematical Finance, Vol. 12, No. 3 (July 2002), 271–286.

Stace A.W. (2007). A moment matching approach to the valuation of a volume weighted average price option. International Journal of Theoretical and Applied Finance 10(1), 95-110.

Thompson G. W. P. (2000). Fast narrow bounds on the value of Asian options. Working Paper, Centre for Financial Research, Judge Institute of Management Science, University of Cambridge.

Večeř J. (2002). Unified Asian pricing. Risk 15, 113–116.

QMF 2013

17-20 December

The Quantitative Methods in Finance 2013 Conference

will bring together leading experts in Quantitative Finance Industry and Academia in Sydney Australia

FOCUS

Stochastic Volatility, Interest Rate Term Structure, Credit Risk, Pensions, Insurance, Portfolio Optimisation, Commodity and Emissions Trading and other areas of Quantitative Finance

PLENARY SPEAKERS INCLUDE

Jesper Andreasen, Alexandre Antonov, Marco Avellaneda, Mark Davis, Freddy Delbaen, Robert Elliott, Rüdiger Frey, Martino Grasselli, John Van der Hoek, Constantinos Kardaras, Rüdiger Kiesel, Fima Klebaner, Steve Kou, Ross Maller, Thomas Mikosch, Marek Musiela, Ashkan Nikeghbali, Bernt Øksendal, Marek Rutkowski, Michael Sherris, Stefan Tappe, Harry Zheng

ORGANISERS

Professor Eckhard Platen, ProfessorErik Schlögl and Professor Carl Chiarella and the Quantitative Finance Research Centre, University of Technology, Sydney

ABSTRACT SUBMISSION EXTENDED TO 13 MAY 2013 click here

REGISTRATION click here

Registration Category	Early Bird Fees *Book on or before 30th August 2013 AEST*	Registration Fees *Book on or after 31st August 2013 AEST*
Conference Registration (Industry)	**$2,997.75**	**$3,444.00**
Conference Registration (Academic & Student)	**$787.50**	**$895.00**
Conference Dinner	**$115.00**	**$115.00**

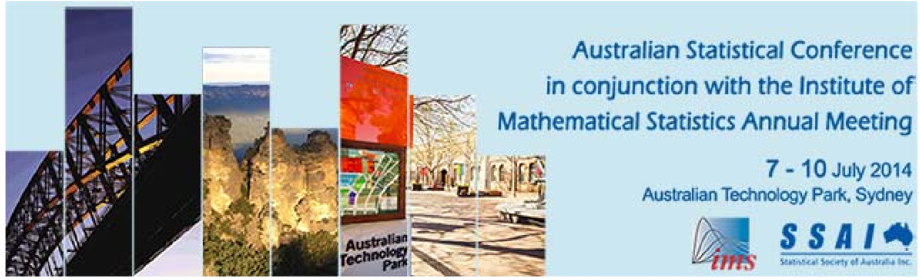
All registration fees are in Australian Dollars and include GST.

Credit card fees will apply (1.96% for Visa/MasterCard and 3.96% for Amex/Diners)

First Announsment

The 2014 IMS-FPS workshop
will be held
at University of Technology,
Sydney
from July 4 to 6, 2014

Web Version | Update preferences | Unsubscribe   Like Tweet Forward



Registration and call for abstracts now open!

We are pleased to announce that registration is now open for ASC-IMS 2014. Save A$125 by registering before 28 February 2014!

Delegates from all areas of statistics will join with world class Australian and International statisticians and mathematicians to develop, network and share their knowledge and expertise. In 2014 the Statistical Society of Australia will hold its biennial Australian Statistical Conference in conjunction with the Institute of Mathematical Statistics Annual Meeting. The Conference will provide opportunities for presentations on a wide range of topics and recognises the role that statistics plays in all aspects of modern life.

For more information on registration inclusions, please visit the Conference website today:

www.asc-ims2014.com

Register Now

Invitation to submit your abstract

You are invited to submit an abstract for consideration for a contributed oral or poster presentation.

As befitting an event of this size, with approximately 15 Keynote presentations and 8 parallel streams, a large portion of the program will be by invitation. However, a substantial part of the program will be set aside for contributed presentations, both oral and poster. While there is no restriction on the topic or number of contributed presentations, the number of oral presentations is by nature limited.