Weak Reflection Principle and Static Hedging of Barrier Options

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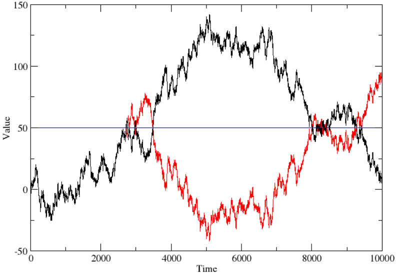
Reflection Principle for Brownian motion

B is a standard Brownian motion on a real line, $M_t = \sup_{u\in[0,t]} B_u$. For any $U > 0$ and $K < U$:

$$\mathbb{P}\left(B_t \leq K, M_t > U\right) = \mathbb{P}\left(B^U_{t-T_U} \leq K, T_U < t\right)$$

$$= \mathbb{P}\left(2U - B^U_{t-T_U} \leq K, T_U < t\right) = \mathbb{P}\left(B_t \geq 2U - K\right)$$



Relevant properties of Brownian motion

- **strong Markov property**: $B_t^U - B_{T_U} = B_{T_U+t} - B_{T_U}$ is a standard Brownian motion, independent of $\mathcal{F}_{T_U}$;
- **continuity**: since the paths of B are continuous, $B_{T_U} = U$ and, in view of the above, B^U is a Brownian motion started from U, and independent of $\mathcal{F}_{T_U}$;
- **symmetry**: the distribution of B_t^U is symmetric with respect to the initial level U, i.e $\text{Law}\left(B_t^U\right) = \text{Law}\left(2U - B_t^U\right)$.

Applications of Refection Principle

- **Computation** of the joint distribution of the process and its running maximum (minimum).
- **Static hedging** of barrier options: given payoff h, upper barrier $U = 0$, and maturity $T > 0$, find G such that

$$\mathbb{E}\left(h(B^x_T)\mathbf{1}_{\{\sup_{u\in[0,T]} B^x_u<0\}} \middle| \mathcal{F}_{t\wedge T_0}\right) = \mathbb{E}\left(G(B^x_T) \middle| \mathcal{F}_{t\wedge T_0}\right), \quad \forall t \in [0, T]$$

  - if $T_0 \geq T$, we obtain: $G(z)\mathbf{1}_{\{z<0\}} = h(z)$, hence $G(z) = h(z) - g(z)$, where $\mathrm{supp}(g) \subset (0, \infty)$;
  - if $T_0 < T$, $\mathbb{E}\left(h\left(B^0_{T-T_0}\right)\mathbf{1}_{\{T_0<T\}}\right) = \mathbb{E}\left(g\left(B^0_{T-T_0}\right)\mathbf{1}_{\{T_0<T\}}\right)$.

    The choice $g(z) = h(-z)$ satisfies the above condition, since

$$\mathbb{E}\left(h\left(B^0_t\right)\right) = \mathbb{E}\left(g\left(B^0_t\right)\right), \quad \text{for all } t > 0,$$

Thus, $G(z) = h(z) - h(-z)$ solves the static hedging problem.

Static hedging of Up-and-Out Put

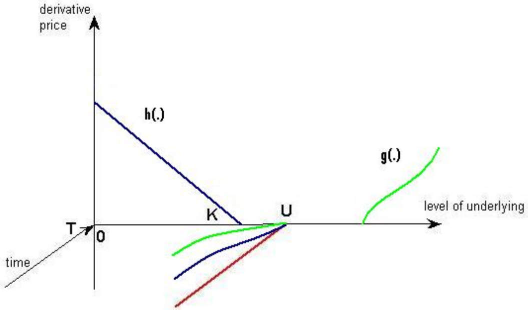


Figure : Price functions of the options with payoffs h (blue) and g (green), along the barrier $S = U$ (red)

Key properties

- **What do we need for the above to work?**
  - We need the symmetry of the conditional distribution of X_{T_U+t}, given $\mathcal{F}_{T_U}$, with respect to the initial level U.
  - **Strong Markov property** and **continuity** allow us to reduce the problem to the actual expectations, as opposed to conditional ones.
- **Keeping the strong Markov property and continuity, how far can we extend the Reflection Principle beyond Brownian motion?**
  - **diffusions** whose **coefficients are symmetric** with respect to U;
  - **monotone continuous functions** of the above.
- **This is not quite satisfactory for existing applications...**

Strong and Weak symmetries

- Stochastic process X, defined on a real line and started from zero, possesses a **strong symmetry** (with respect to zero) if there exists a mapping $\mathbf{S} : \mathbb{R} \to \mathbb{R}$, such that $x\mathbf{S}(x) \leq 0$, for all $x \in \mathbb{R}$, and $\text{Law}(\mathbf{S}(X_t)) = \text{Law}(X_t)$, for all $t > 0$.

- Fix two spaces of Lebesgue measurable test functions, $\mathcal{B}^-$ and $\mathcal{B}^+$, such that all functions in $\mathcal{B}^-$ have support in $(-\infty, 0)$, and all functions in $\mathcal{B}^+$ have support in $(0, \infty)$. We say that X possesses an **upper weak symmetry** (with respect to zero) if there exists a mapping $\mathbf{W}^+ : \mathcal{B}^- \to \mathcal{B}^+$, such that

$$\mathbb{E}\, h(X_t) = \mathbb{E}\left(\mathbf{W}^+(h)(X_t)\right), \quad \text{for all } t > 0,$$

  for any $h \in \mathcal{B}^-$. We will refer to $\mathbf{W}^+(h)$ as the **mirror image** of h.

- If X has a **strong symmetry S** (with respect to zero), then, the it also **possesses a weak symmetry**, with $\mathbf{W}^{\pm}(h) = h \circ \mathbf{S}$

Weak Reflection Principle

- The **Standard (Strong) Reflection Principle** arises as a combination of the **strong Markov property**, **continuity**, and the **strong symmetry**.
- Similarly, the **Weak Reflection Principle** requires the **strong Markov property**, **semi-continuity**, and the **weak symmetry**.
- The **Weak Reflection Principle (WRP)** is **sufficient for the applications we described**!
- The important questions are:
  - **existence** of the weak symmetry transformation **W**;
  - its **uniqueness**;
  - and **numerical approximation**.

Time-homogeneous diffusions

In *Carr-N.* (2011), we develop **WRP** for processes X of the form.

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t,$$

where we assume that

- $\inf_{x\in\mathbb{R}} \sigma(x) > 0$,
- $\mu, \sigma \in C^3(\mathbb{R})$,
- μ, σ and their first three derivatives have finite limits at $-\infty$,
- and, for any $k = 1, 2, 3$, the functions $e^{(3-k)x}|\mu^{(k)}(x)|$ and $e^{(3-k)x}\sigma^{(k)}(x)$ are bounded as $x \to +\infty$.

We **do not make any symmetry assumptions** on μ and σ!

Weak Symmetry

Given h, with support in $(-\infty, U)$, find its mirror image g, s.t. it has support in (U, ∞) and

$$\mathbb{E}\left[h\left(X_\tau\right)\middle|\, X_0 = U\right] = \mathbb{E}\left[g\left(X_\tau\right)\middle|\, X_0 = U\right], \quad \text{for all } \tau > 0$$

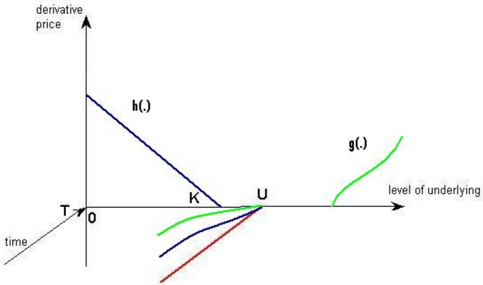


Figure : Expectations of $h(X_\tau)$ (blue) and $g(X_\tau)$ (green), along the barrier $X = U$ (red)

WRP for time-homogeneous diffusions

- Assume $U = 0$.
- Consider arbitrary function h, with support in $(-\infty, 0)$, such that its weak derivative has finite variation and is exponentially bounded.
- Then, its **Mirror Image** $\mathbf{W}(h)$ is given by

$$\mathbf{W}(h)(x) =$$

$$\frac{2}{\pi i}\int_{\varepsilon-\infty i}^{\varepsilon+\infty i} \frac{z\psi_1(x,z)}{\partial_x\psi_1(0,z)-\partial_x\psi_2(0,z)}\int_{-\infty}^{0}\frac{\psi_1(s,z)}{\sigma^2(s)}e^{-2\int_0^s \frac{\mu(y)}{\sigma^2(y)}dy}h(s)ds\,dz,$$

where the functions $\psi^1(x,z)$ and $\psi^2(x,z)$ are the **fundamental solutions** of the associated **Sturm-Liouville** equation:

$$\frac{1}{2}\sigma^2(x)\psi_{xx}(x,z) + \mu(x)\psi_x(x,z) - z^2\psi(x,z) = 0$$

Solution

- Recall that, in order to solve the static hedging problem, we only need to compute the **mirror image of the put payoff**.

$$h(x) = (K - x)^+$$

- Thus, the Static Hedge of an UOP option (with barrier $U = 0$ and strike $K < 0$) is given by

$$G(x) = (K - x)^+ - g(x),$$

where

$$g(x) = \frac{1}{\pi i} \int_{\varepsilon - \infty i}^{\varepsilon + \infty i} \frac{\psi^1(x, z)\, \psi^1(K, z)}{\psi^1_x(0, z) - \psi^2_x(0, z)} \frac{dz}{z},$$

Constant Elasticity of Variance: $\mu = 0$, $\sigma(S) = S^{1+\beta}$

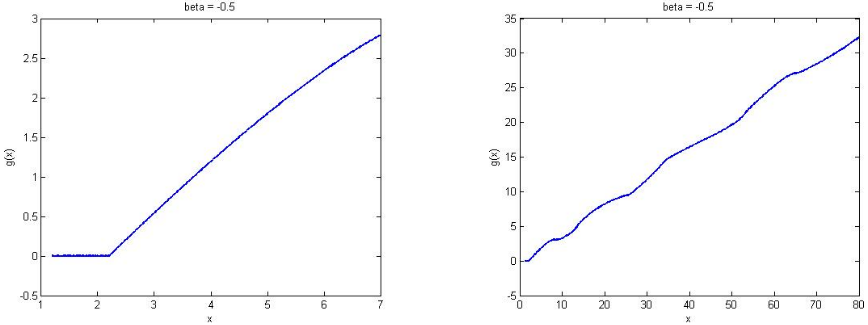


Figure : The "mirror image" g in the zero-drift CEV model with barrier $U = 1.2$ and strike $K = 0.5$: the case of $\beta = -0.5$, for small (left) and large (right) values of the argument.

Other CEV: Bachelier and Black-Scholes

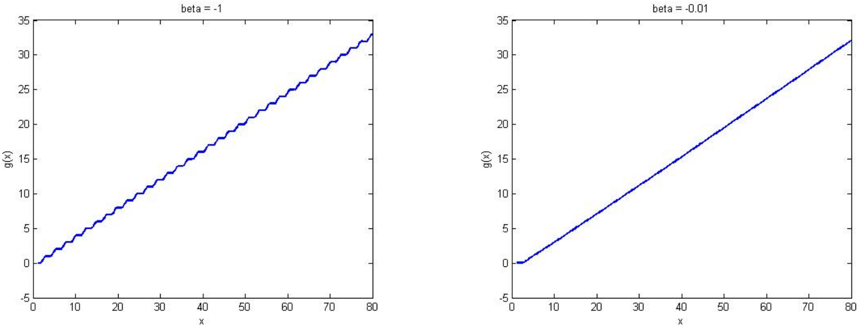


Figure : The "mirror image" g in the zero-drift CEV model with barrier $U = 1.2$ and strike $K = 0.5$: the cases of $\beta = -1$ (left) and $\beta \approx 0$ (right).

WRP for Lévy processes with one-sided jumps

- In our recent work *Bayraktar-N. (2013)*, we develop **WRP for L'evy processes** X with the characteristic exponent

$$\psi(\lambda) = \frac{1}{t} \log \left(\mathbb{E} e^{\lambda X_t} \right) = \mu\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{-\infty}^{0} \left(e^{\lambda x} - 1 - \lambda x \right) \Pi(dx),$$

where Π has support in $(-\infty, 0)$ and satisfies

$$\int_{-\infty}^{-1} |x| e^{-\zeta x} \Pi(dx) < \infty,$$

with some $\zeta \geq 0$.

- Consider arbitrary function h, such that its support is in $(-\infty, 0)$ and there exists $\hat{h} \in \mathbb{L}^1(\mathbb{R})$, such that the function $y \mapsto e^{\zeta y} h(y)$ is a **Fourier transform of** $\hat{h}$. Then the mirror image of h is given by

$$\mathbf{W}(h)(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{zx} \int_{\mathbb{R}} \left(\frac{\psi'(z)}{\psi(z) - \psi(-\zeta - is)} - \frac{1}{z + \zeta + is} \right) \hat{h}(s) ds dz$$

Applications of WRP

- The Weak Reflection Principle allows us to **compute the joint distribution of the process and its running maximum (minimum)**, through the marginal distribution of the process itself:

$$\mathbb{P}(X_T \leq K, \sup_{t\in[0,T]} X_t \leq U) = \mathbb{E}(\mathbf{W}^{+,U}\left(\mathbf{1}_{(-\infty,K)}\right)(X_T)),$$

where $\mathbf{W}^{+,U}\left(\mathbf{1}_{(-\infty,K)}\right)$ is the mirror image of the indicator function $\mathbf{1}_{(-\infty,K)}$, with respect to the barrier U.

- **Static hedge** of a up-and-out option with terminal payoff function h is given by a European-type option with payoff

$$G(X_T) = h(X_T) - \mathbf{W}^{+,U}(h)(X_T)$$

- The **connection to PDE's and PIDE's** yields potential applications in **Physics** and **Biology**.

Summary

- **Weak Reflection Principle** is an extension of the classical reflection principle for Brownian motion that can be applied to processes **without any strong symmetries**.
- The **Weak Reflection Principle** allows us to
  - control the (conditional) expectation of a function of the process,
  - at the first time when the process hits the barrier of a given domain,
  - by changing the function outside of this domain.
- Existing applications are in **Finance, Computational Methods**. Possibly, Physics, Biology, ...
- Further extensions:
  - **Specific applications in Physics and Biology?**
  - **More general domains**?
  - **More general stochastic processes**?

Non-existence result

- *Bardos-Douady-Fursikov* (2004) treat this problem for a general parabolic PDE, and prove the existence of **approximate solutions** g_ε, such that

$$\sup_{t\in[0,T]} \left| u^h(U,t) - u^{g_\varepsilon}(U,t) \right| < \varepsilon$$

- They show that an exact solution doesn't exist in general...
- Their proof is **not constructive** - finding even an approximate solution is left as a separate problem.
- The example of **non-existence** relies heavily on the **time-dependence of the coefficients** in the corresponding PDE!

Naive numerical approximation

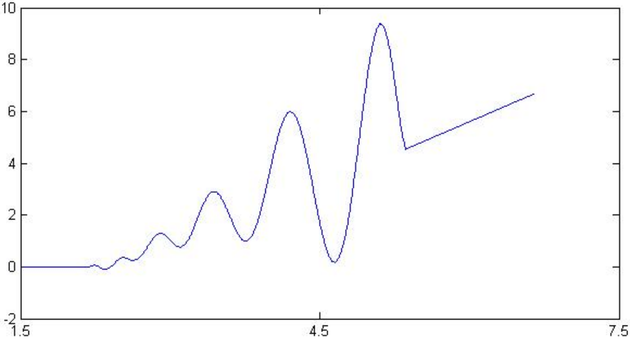


Figure : Payoff function g_ε as a result of the naive least-square optimization approach

Square root process revisited

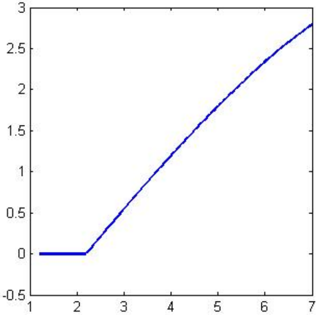


Figure : 2. Function g, for $\beta = -0.5$, $U = 1.2$, $K = 0.5$

Notice that there is a constant $K^* \geq U$, such that the support of g is exactly $[K^*, \infty]$.

Computation and extensions

$$g(S) = \frac{1}{\pi i} \int_{\varepsilon - \infty i}^{\varepsilon + \infty i} \frac{\psi^1(S,z)\,\psi^1(K,z)}{\psi^1_S(U,z) - \psi^2_S(U,z)} \frac{dz}{z},$$

- If $\sigma(S)$ is **piece-wise constant**, the fundamental solutions $\psi_1(S,z)$ and $\psi_2(S,z)$ can be easily computed as **linear combinations of exponentials**, on each sub-interval in S.
- This family of models is **sufficient for all practical purposes**.
- The proposed static hedge also succeeds in all models that arise by running the time-homogeneous diffusion on an **independent continuous stochastic clock**.
- One can obtain a **semi-robust extension of this static hedging strategy**. More precisely, a strategy that succeeds in all models, as long as the market implied volatility stays within given bounds (beliefs about implied volatility are fulfilled).

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Robust static hedge with beliefs on implied volatility

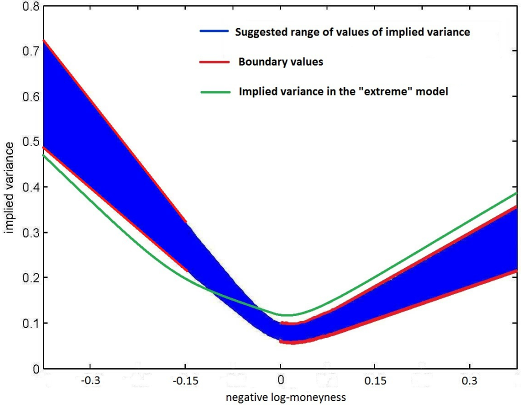

Figure : Range of possible values of (beliefs on) implied volatility (blue), and the extremal implied volatility produced by a diffusion model (green)

Short-Maturity Behavior and Single-Strike Hedge

- **Key observation: when time-to-maturity is small, only the values of g around K^* matter**!
- Thus, for small maturities, the approximation of the payoff function g with a **scaled call payoff** should perform well.
- We have:

$$\int_K^U \frac{dy}{\sigma(y)} = \int_U^{K^*} \frac{dy}{\sigma(y)}, \qquad \eta = \sqrt{\frac{\sigma(K)}{\sigma(K^*)}}.$$

- Using the above, we can construct the **single-strike sub- and superreplicating strategies**: there exists $\delta > 0$, such that, whenever $S_t = U$,

$$[1 - \delta(T-t)]\, P_t(K) - \eta C_t(K^*) \le 0 \le [1 + \delta(T-t)]\, P_t(K) - \eta C_t(K^*)$$

Function g: properties and numerical computation

- There exists a constant $K^* \geq U$, such that the support of g is exactly $[K^*, \infty]$.
- Introduce the "*signed geodesic distance*":

$$Z(x) := \sqrt{2} \int_U^x \frac{dy}{\sigma(y)}$$

- Then K^* is a solution of the equation

$$Z(K^*) + Z(K) = 0$$

- The function g is "*analytic with respect to the geodesic distance Z*" in (K^*, ∞):

$$g(x) = \sum_{k=1}^{\infty} c_k \left(Z(x) - Z(K^*)\right)^k ,$$

and the exists an algorithm for computing c_k's.