Hedging Barrier Options via a General Self-Duality

Thorsten Rheinländer (with E. Alos, Z. Chen)

Vienna University of Technology

June 21, 2013

Overview

- Let the price process be modelled as a continuous stochastic volatility model with correlation.
- Consider a down-and-in call with strike higher than the barrier level, or its up-and-in put analogue.
- We provide a replicating portfolio by trading in stock, realized volatility and cumulative volatility.
- In contrast to market completion by trading in stock and a vanilla option, this does not require to solve a PDE.
- Our method relies on a general self-duality result, whereby duality is to be understood in the sense of dual market; see Eberlein, Papapantoleon and Shiryaev (2008).

Motivation

- Let S be the price process of some risky asset, modelled as a geometric Brownian motion.
- Consider a down-and-in call option with strike price K, maturity T and barrier level $B < K$. We denote $\tau := \inf\{t : S_t \leq B\}$ and assume $S_0 > B$ and that the interest rate is zero.
- If the barrier has been hit before T, the fair price of this option at the barrier is

$$E_\tau^{\mathbb{P}}\left[(S_T - K)^+\right],$$

where $E_\tau^{\mathbb{P}}$ denotes the conditional expectation with respect to the Brownian filtration $(\mathcal{F}_t)$.

- Carr & Chou (1997): This conditional expectation is equal to

$$E_\tau^{\mathbb{P}}\left[\frac{S_T}{B}\left(\frac{B^2}{S_T} - K\right)^+\right].$$

Self-Duality

Definition. *A non-negative adapted process S is* **self-dual** *if for any non-negative Borel function g and any stopping time $\tau \in [0, T]$,*

$$E_\tau^{\mathbb{P}}\left[g\left(\frac{S_T}{S_\tau}\right)\right] = E_\tau^{\mathbb{P}}\left[\left(\frac{S_T}{S_\tau}\right) g\left(\frac{S_\tau}{S_T}\right)\right].$$

- The semi-static replication of the down-and-in call works more generally for continuous self-dual price processes: Carr & Lee (2009), Molchanov & Schmutz (2010). A typical example is a stochastic volatility model where price process and volatility are uncorrelated.

Correlated stochastic volatility models

- Consider the following stochastic volatility model on a time interval $[0, T]$ under a risk-neutral measure $\mathbb{P}$:

$$dS_t = \sigma(V_t) S_t \, dZ_t, \qquad S_0 = s_0 > 0,$$
$$dV_t = \mu(V_t) \, dt + \gamma(V_t) \, dW_t, \quad V_0 = v_0 > 0.$$

- Here Z, W are two Brownian motions with correlation $\rho \in [-1, 1]$. Let $Z = \rho W + \overline{\rho} W^{\perp}$, where W and $W^{\perp}$ are independent standard Brownian motions and $\overline{\rho} = \sqrt{1 - \rho^2}$.
- We assume that the functions σ, μ, γ are such that there exists a weak solution (S, V), and that $\sigma(V)$ is non-zero on $[0, T]$. The filtration is set to be $\mathbb{F} = \mathbb{F}^{S,V}$, the filtration generated by S and V.
- Moreover, the risk-free interest rate is assumed to be equal to zero. A non-zero interest rate would require an extension of our methods to quasi self-duality.

- Main idea to deal with the *asymmetry risk*: a multiplicative decomposition

$$S = M \times R$$

of the price process S into a self-dual part M and an asymmetric remainder term R.

- We take R_T as Radon-Nikodym derivative to deal with the asymmetry problem via a change of measure:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} |_{\mathcal{F}_t} = R_t, \qquad t \in [0, T].$$

- The **modified price process** D under the measure $\mathbb{Q}$ is defined as

$$D = \frac{S}{R^2} = \frac{M}{R}.$$

General self-duality

- The **general self-duality** holds in our model: for all positive Borel functions g, stopping times $\tau \in [0, T]$,

$$E_\tau^{\mathbb{P}}\left[g\left(\frac{S_T}{S_\tau}\right)\right] = E_\tau^{\mathbb{Q}}\left[\frac{D_T}{D_\tau} g\left(\frac{D_\tau}{D_T}\right)\right],$$

as well as the **dual general self-duality**:

$$E_\tau^{\mathbb{Q}}\left[g\left(\frac{D_T}{D_\tau}\right)\right] = E_\tau^{\mathbb{P}}\left[\frac{S_T}{S_\tau} g\left(\frac{S_\tau}{S_T}\right)\right].$$

- In the classical self-dual case, self-duality and dual self-duality coincide.

- The fair price of the same down-and-in call as before at the barrier is

$$E_\tau^{\mathbb{P}}\left[(S_T - K)^+\right].$$

- This expectation is difficult to evaluate in our context. By the general self-duality, this equals $(\tau < T)$

$$E_\tau^{\mathbb{Q}}\left[\Gamma_\tau^{\mathbb{Q}}\right] = K E_\tau^{\mathbb{Q}}\left[\left(\frac{B}{K} - \frac{D_T}{D_\tau}\right)^+\right].$$

- In contrast to $S_\tau = B$, here D_τ is a random variable. Moreover, D is *a priori* a hypothetical instrument.

- D can be explicitly written as product of S and some functional of the volatility. Moreover, one can perfectly synthesize D by a dynamic explicit trading strategy in the stock S, realized volatility V and cumulative volatility $\int \sigma^2(V)\, dt$. This way D can be seen as the price process of an asset which, at least potentially, could be traded.
- $\mathbb{Q}$ as introduced before is the corresponding martingale measure for D. Consequently, $\mathbb{Q}$ is the pricing measure for the Γ_τ-claim.
- In fact, we can derive a replicating self-financing strategy in S, V, and $\int \sigma^2(V)\, dt$ for the claim Γ_τ.

Replicating hedging strategy

- Recall that

$$\Gamma_\tau^{\mathbb{Q}} = K\left(\frac{B}{K} - \frac{D_T}{D_\tau}\right)^+ .$$

- We write

$$u(x) = K\left(\frac{B}{K} - x\right)^+ .$$

By Ito's formula,

$$\begin{aligned} u\left(\frac{D_T}{D_\tau}\right) &= u(1) + \int_\tau^T \frac{\partial u}{\partial x} \cdot \frac{D_t}{S_t}\, dS_t - 2\rho \int_\tau^T \frac{\partial u}{\partial x} D_t \frac{\sigma(V_t)}{\gamma(V_t)} dV_t \\ &+ \int_\tau^T \left(\frac{\partial u}{\partial x} D_t \left(\rho^2 + 2\rho \frac{\mu(V_t)}{\sigma(V_t)\gamma(V_t)} \right) \right. \\ &\left. + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} D_t^2 \left(\frac{1}{S_t^2} + 4\rho^2 - \frac{4\rho^2}{S_t} \right) \right) \sigma^2(V_t)\, dt. \end{aligned}$$

- Finally, substitute

$$D_t = S_t \exp\left(-2\rho \int_0^t \frac{\sigma(V_s)}{\gamma(V_s)} dV_s - \int_0^t \frac{\sigma(V_s)\mu(V_s)}{\gamma(V_s)} ds\right) \times \exp\left(\rho^2 \int_0^t \sigma^2(V_s)\, ds\right).$$

- This gives a replicating hedge by dynamically trading in stock, realized variance and cumulative variance.
- By using Malliavin calculus, we obtain pricing formulae involving higher greeks.
- Moreover, we give a second order approximation to the price of the barrier option.

References

- Carr, P., Chou, A. (1997) Breaking barriers, Risk **10**, pp. 139–145
- Carr, P., Lee, R. (2009) Put-call symmetry: extensions and applications. *Mathematical Finance* **19**, 523–560
- Eberlein, E., Papapantoleon, A., Shiryaev, A.N. (2008). On the duality principle in option pricing: semimartingale setting. *Finance and Stochastics* **12**, 265-292
- Molchanov, I., Schmutz, M. (2010) Multivariate extension of put-call symmetry. *SIAM Journal of Financial Mathematics* **1** 398–426
- Xiao, Y. (2009) R-minimizing hedging in an incomplete market: Malliavin calculus approach. Available at SSRN