# Existence of an endogenously complete equilibrium driven by a diffusion

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# Outline

Radner equilibrium

Arrow-Debreu equilibrium

Backward Martingale Representation

Technicalities with intermediate utilities

# Bibliography


K. Existence and uniqueness of Arrow-Debreu equilibria with consumptions in $\mathbf{L}^0_+$. arXiv:1304.3284v2, May 2013. URL http://arxiv.org/abs/1304.3284v2.

K. Existence of endogenously complete equilibrium driven by diffusion. arxiv:1304.3516v1, April 2013. URL http://arxiv.org/abs/1304.3516v1.

# Introduction

Three basic topics in Asset Pricing Theory are

1. Arbitrage,
2. Single-agent optimality,
3. Equilibrium;

see, e.g., the books by Karatzas and Shreve (1998), Duffie (2001), and Dana and Jeanblanc (2003).

- For the first two topics a general mathematical theory is available.
- The situation with equilibrium is more involved.
- Equilibrium becomes the main modeling tool as soon as we leave "small" agent's framework, e.g., in market's micro-structure theory.

# Economic agents

The uncertainty is modeled by $(\Omega, \mathcal{F}_1, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$. There are $M$ agents. They choose cumulative consumption processes

$$C_t = \int_0^t \xi_s ds + \Xi 1_{\{t=1\}},\ t \in [0,1].$$

- The expected utility of $m$th agent has the form:

$$\mathbb{U}^m(C) \triangleq \mathbb{E}[\int_0^1 u^m(t,\xi_t)dt + U^m(\Xi)],$$

  where $u^m = u^m(t,c)$ and $U^m = U^m(c)$ are Inada-type utility functions for intermediate and terminal consumptions.

- The income process of $m$th agent is given by

$$I_t^m = \int_0^t \lambda_s^m ds + \Lambda^m 1_{\{t=1\}},\ t \in [0,1],$$

  where

$$\lambda^m \geq 0,\ \Lambda^m \geq 0,\ \text{and } \mathbb{P}[I_1^m > 0] > 0.$$

# Financial market

The financial market consists of a zero-coupon bond and $J$ stocks.

- The bond pays the notional $N = 1$ at maturity $t = 1$.
- The stocks pay the cash dividend

$$D_t = \int_0^t \theta_u du + \Theta 1_{\{t=1\}}.$$

By a $(B, S)$-*market* we call an optional process $B = (B_t) > 0$ of bond's prices and a $J$-dimensional semimartingale $S = (S_t)$ having the terminal values

$$B_1 = 1 \quad \text{and} \quad S_1 = \int_0^1 \frac{\theta_u}{B_u} du + \Theta = \int_0^1 \frac{1}{B_t} dD_t.$$

Here $S$ is the *discounted* value of the buy-and-hold strategy.

# Radner equilibrium

Inputs:

- $M$ agents with utility functions $u_m = u_m(t, c)$ and $U_m = U_m(c)$ and income processes $I^m = (I_t^m)$.
- Zero-coupon bond with terminal value $1$ and $J$ stocks with the dividend process $D = (D_t)$.

Definition

*Radner equilibrium* is a $(B, S)$-market where the agents' optimal consumption processes $\widehat{C}^m = (\widehat{C}_t^m)$, satisfy the clearing condition:

$$\sum_{m=1}^{M} \widehat{C}_t^m = \sum_{m=1}^{M} I_t^m, \; t \in [0, 1]. \tag{1}$$

Question

Does a Radner equilibrium *exist*? Is it unique? Is it stable? etc.

# Existence of Radner equilibrium

Similar to optimal investment there are two approaches:

Direct (PDE): a *coupled* system of nonlinear HJB equations.

Dual (martingale): *static problem* + *martingale representation*.

Second approach is usually more powerful.

- Not clear how to formulate the static problem for general (incomplete) case.

Idea: look for a *complete* Radner equilibrium (that is, with a *complete* $(B, S)$-market). In this case,

Complete static equilibrium = Arrow-Debreu equilibrium.

# Arrow-Debreu equilibrium

## Definition

A pair $(P, (\widehat{C}^m))$, consisting of an optional *consumption price* process $P > 0$ and consumptions $(\widehat{C}^m)$ is an *Arrow-Debreu equilibrium* if the clearing condition (1) holds and

$$|\mathbb{U}^m(\widehat{C}^m)| + \mathbb{E}[\int_0^1 P_t dI_t^m] < \infty,$$

the consumption process $\widehat{C}^m$ satisfies the *budget constraint*:

$$\mathbb{E}[\int_0^1 P_t d\widehat{C}_t^m] = \mathbb{E}[\int_0^1 P_t dI_t^m],$$

and $\mathbb{U}^m(\widehat{C}^m) \geq \mathbb{U}^m(C)$ for every consumption process $C$ satisfying same budget constraint.

# Assumptions

- The utility functions $u_m = u_m(t, c)$ and $U_m = U_m(c)$, w.r.t. $c$, are strictly increasing, strictly concave, and

$$\lim_{c \to 0} u'_m(t, c) = \lim_{c \to 0} U'_m(c) = \infty,$$
$$\lim_{c \to \infty} u'_m(t, c) = \lim_{c \to \infty} U'_m(c) = 0$$

- The individual incomes are non-zero:

$$I_1^m = \int_0^1 \lambda_t^m dt + \Lambda^m \neq 0$$

  and the total incomes are strictly positive:

$$\lambda_t \triangleq \sum_m \lambda_t^m > 0 \quad \text{and} \quad \Lambda \triangleq \sum_m \Lambda^m > 0.$$

# Existence of Arrow-Debreu equilibria

The following theorem is an improvement over Dana (1993).

## Theorem (K. (2013))

*Suppose that Assumption holds. Then an Arrow-Debreu equilibrium exists if and only if there are consumption processes*

$$C_t^m = \int_0^t \xi_s^m ds + \Xi^m 1_{\{t=1\}}.$$

*which satisfy the clearing condition* (1) *and such that*

$$\mathbb{E}[\int_0^1 |u_m(t, \xi_t^m)| dt + |U_m(\Xi^m)|] < \infty$$

*and*

$$\mathbb{E}[\int_0^1 u_m'(t, \xi_t^m) \lambda_t dt + U_m'(\Xi^m) \Lambda] < \infty.$$

# Aggregate utility functions

Denote by $\Sigma^M$ the simplex in $\mathbb{R}^M$:

$$\Sigma^M \triangleq \{w \in [0,\infty)^M : \sum_{m=1}^{M} w^m = 1\}.$$

For a weight $w \in \operatorname{int}\Sigma^M$ define the aggregate utility functions:

$$U(c;w) \triangleq \sup\{\sum_{m=1}^{M} w^m U^m(c^m) : \ c^m \geq 0, \ c^1 + \cdots + c^M = c\},$$
$$u(t,c;w) \triangleq \sup\{\sum_{m=1}^{M} w^m u^m(t,c^m) : \ c^m \geq 0, \ c^1 + \cdots + c^M = c\}.$$

# Pareto optimal consumptions

For a weight $w \in \operatorname{int} \Sigma^M$ define the Pareto optimal consumption processes

$$C_t^m(w) \triangleq \int_0^t \pi_s^m(w) ds + \Pi^m(w) 1_{\{t=1\}},$$

where

$$w^m U_m'(\Pi^m(w)) = U_c(\Lambda; w),$$
$$w^m u_c^m(t, \pi_t^m(w)) = u_c(t, \lambda_t; w).$$

Note that $(C^m(w))$ satisfy the clearing condition (1).

# Structure of Arrow-Debreu equilibria

Theorem

*Suppose that Assumption holds and let $(P, (\widehat{C}^m))$ be an Arrow-Debreu equilibrium. Then there there is $w \in \operatorname{int} \Sigma^M$ such that*

$$
\begin{aligned}
P_t &= \mathrm{const}\left(u_c(t, \lambda_t; w) 1_{\{t<1\}} + U_c(\Lambda; w) 1_{\{t=1\}}\right), \\
\widehat{C}^m &= C^m(w), \quad m = 1, \ldots, M,
\end{aligned}
$$

# Existence of Radner equilibria

It remains to verify the *endogenous* completeness of the $(B, S)$-market implied by the Arrow-Debreu equilibrium:

- The (!) martingale measure $\mathbb{Q}$ for stocks' prices is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const}\, P_1 = \text{const}\, U_c(\Lambda; w).$$

- The bond price process:

$$B_t = Y_t/P_t, \quad \text{where} \quad Y_t \triangleq \mathbb{E}[P_1|\mathcal{F}_t].$$

- The discounted process of buy and hold strategy for stocks:

$$S_t = \mathbb{E}_{\mathbb{Q}}[\psi|\mathcal{F}_t],$$

  where

$$\psi \triangleq \int_0^1 \frac{\theta_u}{B_u} du + \Theta.$$

# Martingale Integral Representation

$(\Omega, \mathcal{F}_1, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$: a complete filtered probability space.

$\mathbb{Q}$: an equivalent probability measure.

$S = (S_t^j)$: $J$-dimensional martingale under $\mathbb{Q}$.

We want to know whether any local martingale $M = (M_t)$ under $\mathbb{Q}$ admits an integral representation with respect to $S$, that is,

$$M_t = M_0 + \int_0^t H_u dS_u, \quad t \in [0,1],$$

for some predictable $S$-integrable process $H = (H_t^j)$.

- Completeness in Mathematical Finance.
- Jacod's Theorem (2nd FTAP): the integral representation holds iff $\mathbb{Q}$ is the only martingale measure for $S$.
- Easy to verify if $S$ is given in terms of *local characteristics* ("forward" description).

# Backward Martingale Representation

Inputs: random variables $\zeta > 0$ and $\psi = (\psi^j)$

- The density of the martingale measure $\mathbb{Q}$ is defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const}\,\zeta.$$

- $\psi$ is the terminal value for $S$:

$$S_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t], \quad t \in [0,1].$$

Problem

*Determine (easily verifiable) conditions on $\zeta$ and $\psi$ so that the martingale representation property holds under $\mathbb{Q}$ and $S$.*

# Diffusion framework

The random variables $\psi = S_1$ and $\zeta = \text{const}\,\frac{d\mathbb{Q}}{d\mathbb{P}}$ are given by

$$\zeta \triangleq G(X_1)e^{\int_0^1 \beta(t,X_t)dt},$$

$$\psi^j \triangleq F^j(X_1)e^{\int_0^1 \alpha^j(t,X_t)dt} + \int_0^1 f^j(t,X_t)e^{\int_0^t \alpha^j(s,X_s)ds}dt$$

$$+ \int_0^1 \frac{g^j(t,X_t)}{Y_t} e^{\int_0^t (\alpha^j(s,X_s)+\beta(s,X_s))ds}dt, \quad j = 1, \dots, J,$$

where $Y_t \triangleq \mathbb{E}[\zeta|\mathcal{F}_t]$ and

- $F^j, G : \mathbb{R}^d \to \mathbb{R}$ and $f^j, g^j, \alpha^j, \beta : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ are deterministic functions;
- $X = (X_t^i)$ is a $d$-dimensional diffusion:

$$X_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s, \quad t \in [0,1],$$

with drift and volatility functions $b^i, \sigma^{ij} : [0,1] \times \mathbb{R}^d \to \mathbb{R}$.

# Assumptions on functions

1. The functions $F^j = F^j(x)$ and $G = G(x)$ are weakly differentiable and have exponential growth:

$$|\nabla F^j| + |\nabla G| \leq Ne^{N|x|}.$$

2. The Jacobian matrix $\left(\frac{\partial F^j}{\partial x^i}\right)$ has rank $d$ almost surely under the Lebesgue measure on $\mathbb{R}^d$.
3. The maps $t \mapsto e^{-N|\cdot|} f^j(t, \cdot) \triangleq \left(e^{-N|x|} f^j(t, x)\right)_{x \in \mathbb{R}^d}$, $t \mapsto e^{-N|\cdot|} g^j(t, \cdot)$ and $t \mapsto \alpha^j(t, \cdot)$, $t \mapsto \beta(t, \cdot)$ of $[0, 1]$ to $\mathbf{L}_\infty$ are analytic on $(0, 1)$ and Hölder continuous on $[0, 1]$.

- Careful with item 3: stronger than pointwise analyticity! For instance, $f(t, x) = \sin(te^x)$ will not work. Sometimes is overlooked in the literature.

# Assumptions on the diffusion $X$

1. The map $t \mapsto b^i(t, \cdot)$ of $[0, 1]$ to $\mathbf{L}_\infty$ is analytic on $(0, 1)$ and Hölder continuous on $[0, 1]$.
2. The map $t \mapsto \sigma^{ij}(t, \cdot)$ of $[0, 1]$ to $\mathbf{C}$ is analytic on $(0, 1)$ and Hölder continuous on $[0, 1]$. Moreover, $\sigma = \sigma(t, x)$ is uniformly continuous with respect to $x$:

$$|\sigma(t, x) - \sigma(t, y)| \leq \omega(|x - y|).$$

   for some strictly increasing function $\omega = (\omega(\epsilon))_{\epsilon > 0}$ such that $\omega(\epsilon) \to 0$ as $\epsilon \downarrow 0$, and has a bounded inverse:

$$|\sigma^{-1}(t, x)| \leq N \quad (\text{uniform ellipticity for } \sigma\sigma^*).$$

- Counter-example on $t$-analyticity condition in $\sigma = \sigma(t, x)$.

# Backward Martingale Representation

Theorem (K. Predoiu (2012))

*Assume that* $\mathbb{F} = \mathbb{F}^X$. *Under the conditions above the martingale representation property holds for the probability measure* $\mathbb{Q}$ *with the density*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \frac{\zeta}{\mathbb{E}[\zeta]},$$

*and the* $\mathbb{Q}$*-martingale*

$$S_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t], \quad t \in [0, 1].$$

# Comparison with the literature

Assumptions on functions: In Anderson and Raimondo(2008), Hugonnier, Malamud, and Trubowitz (2012), and Riedel and Herzberg(2012)

- The Jacobian matrix $\left(\frac{\partial F^j}{\partial x^i}\right)$ needs to have full rank only on some open set (counter-example in our setting).
- The "small letter" functions, that is, the functions of $(t, x)$ should be $(t, x)$-analytic (for maps?).

Assumptions on diffusion $X$:

- In Anderson and Raimondo(2008) $X$ is a Brownian motion.
- In Hugonnier, Malamud, and Trubowitz (2012) the diffusion coefficients $b = b(t, x)$ and $\sigma = \sigma(t, x)$ are either *analytic* with respect to $(t, x)$ or the transitional probability is $\mathbf{C}^7$.

# Elements of the proof

Parabolic PDEs:

- Evolution equations in $\mathbf{L}_p$ spaces (maximal regularity, analyticity theorem by Kato and Tanabe).
- Elliptic equations in Sobolev spaces (sectoriality property).
- Interpolation theory ($\mathbf{W}_p^1$ is the midpoint of $\mathbf{L}_p$ and $\mathbf{W}_p^2$ in complex interpolation).

Stochastic Analysis:

- Krylov's variant of Ito's formula (instead of $\mathbf{C}^2$ we can have $\mathbf{W}_p^2$ with $p \geq d$ under the uniform ellipticity condition).

# Back to existence of Radner equilibria

A delicate point is to show that for every $w \in \operatorname{int} \mathbf{S}^M$ there is a function $f = f(t, x)$ such that

$$f(t, X_t) = u_c(t, \lambda_t; w),$$

and the map $t \mapsto e^{-N|\cdot|} f(t, \cdot) \triangleq \left(e^{-N|x|} f(t, x)\right)_{x \in \mathbb{R}^d}$ of $[0, 1]$ to $\mathbf{L}_\infty$ is analytic on $(0, 1)$ and Hölder continuous on $[0, 1]$. Here,

- $(\lambda_t)$ is the total income rate process,
- $u(t, c; w)$ is the $w$-weighted sup-convolution w.r.t. $c$:

$$u(t, c; w) \triangleq \sup\{\sum_{m=1}^{M} w^m u^m(t, c^m) : \; c^m \geq 0, \; c^1 + \cdots + c^M = c\}.$$

# Assumption on income

The total income process has the form:

$$\lambda_t = e^{h_1(t,X_t)+h_2(X_t)}, \; t \in [0,1],$$

where

- $t \mapsto h_1(t,\cdot)$ is a Hölder continuous map of $[0,1]$ to $\mathbf{L}_\infty(\mathbb{R}^d)$ whose restriction on $(0,1)$ is analytic;
- the function $h_2 = h_2(x)$ has a linear growth: for some $N \geq 0$,

$$|h_2(x)| \leq N(1+|x|), \; x \in \mathbb{R}^d.$$

# Assumption on intermediate utility

The function $u^m(t,\cdot)$ is of Inada type. The derivatives $u^m_{ct}$ and $u^m_{cc}$ exist and $u^m_{cc} < 0$. There are a constant $N > 0$ such that

$$|u^m(t, e^y)| \leq e^{N(1+|y|)},$$

and an open set $V \subset (0,\infty)^2$ containing $(0,1) \times \{1\}$ such that

$$(t,s) \mapsto (g(t, se^y))_{y\in\mathbb{R}},$$

is a bounded analytic map of $V$ to $\mathbf{L}_\infty(\mathbb{R}^{d+1})$, where $g = g(t,c)$ stands for $u^m_{ct}/u^m_c$, $cu^m_{cc}/u^m_c$, and $u^m_c/(cu^m_{cc})$.

Example

$$u^m(t,c) = e^{-\nu^m t}\frac{c^{1-a^m}-1}{1-a^m},$$

where $\nu^m \in \mathbb{R}$ is an impatience and $a^m > 0$ is a risk-aversion.

# Summary

- Necessary and sufficient conditions are given for the existence of Arrow-Debreu equilibria.
- In an economy driven by a diffusion $X$ we gave new conditions for the existence of *endogenously* complete Radner equilibrium (the stocks are pre-fixed by their dividends).
- The construction of complete Radner equilibria $=$ Arrow-Debreu (easy) $+$ Backward Martingale Representation (hard).
- The diffusion coefficients of $X$ have only minimal $x$-regularity. However, they are $t$-*analytic* (a counter-example for $\sigma$).
- Technicalities with intermediate utilities: $t$-analyticity is for *maps* to $\mathbf{L}^{\infty}$.