Approximation of nondivergent type parabolic PDEs in finance

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Outline

The problem

In this talk, we present the **time and space FD approximation** to the **Cauchy problem for a linear parabolic PDE**

$$\frac{\partial u}{\partial t} = Lu + f \text{ in } [0, T] \times \mathbb{R}^d, \quad u(0, x) = g(x) \text{ in } \mathbb{R}^d$$

where

- L is the second-order partial differential operator with real coefficients

$$L(t, x) = a^{ij}(t, x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t, x)\frac{\partial}{\partial x^i} + c(t, x);$$

- f and g are real-valued given functions;
- $T \in (0, \infty)$ is a constant.

The coefficients a^{ij} and b^i, and the free data f and g **are allowed to grow in the spatial variables**.

Connection with the BS model

The equation we consider generalizes the **multidimensional version of the Black-Scholes equation**

$$\frac{\partial V}{\partial t} + \frac{1}{2}\rho^{ij}\sigma^i\sigma^j S^i S^j \frac{\partial^2 V}{\partial S^i \partial S^j} + (r - d^i)S^i \frac{\partial V}{\partial S^i} - rV = 0$$

- by dropping the assumption that ρ, σ, r, and d are constant and allowing their dependence of space and time;
- by considering the more general **nonhomogenous case**.

Possible applications to finance

Some very common types of options, with fixed exercise, are modeled by the multidimensional version of the B-S equation.

In general, they comprise the class of **non path-dependent options written on multiple assets** (e.g., basket options).

Stages of the study

The study is developed **under the strong assumption that the PDE does not degenerate**, in the following stages:

(i) Discretisation of a linear evolution equation in abstract spaces
(in which our problem will later be cast into)

(ii) Semi-discretisation in space of the PDE

(iii) Discretisation of the PDE in space and time

Outline

Gelfand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$

with continuous and dense embeddings.

Evolution equation problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} = A(t)u + f(t) \ \text{ in } \ [0, T], \quad u(0) = g, \qquad \text{(Evol.Eq.)}$$

where

- $A(t)$ is a linear operator from V to V^* for every $t \in [0, T]$;
- $A(\cdot)v : [0, T] \to V^*$ is measurable for fixed $v \in V$;
- $f : [0, T] \to V^*$ is measurable and $g \in H$.

Assumption (Abstract setting – AS)

There exist constants $\lambda > 0$, K, M, and N such that:

1. $\langle A(t)v, v\rangle + \lambda\|v\|_V^2 \leq K\|v\|_H^2, \quad \forall v \in V$ *and* $\forall t \in [0, T]$;
2. $\|A(t)v\|_{V^*} \leq M\|v\|_V, \quad \forall v \in V$ *and* $\forall t \in [0, T]$;
3. $\int_0^T \|f(t)\|_{V^*}^2 \mathrm{d}t \leq N$ *and* $\|g\|_H \leq N$.

Definition (Generalised solution)

$u \in C([0, T]; H)$ is a **generalised solution** of (Evol.Eq.) on $[0, T]$ if

1. $u \in L^2([0, T]; V)$;
2. $$(u(t), v) = (g, v) + \int_0^t \langle A(s)u(s), v\rangle \mathrm{d}s + \int_0^t \langle f(s), v\rangle \mathrm{d}s, \quad \forall v \in V, \ \forall t \in [0, T].$$

Theorem (Existence-uniqueness)

If

- *Assumption AS is satisfied,*

then the problem (Evol.Eq.) has a unique generalised solution on $[0, T]$.

Moreover

$$\sup_{t\in[0,T]} \|u(t)\|_H^2 + \int_0^T \|u(t)\|_V^2 \mathrm{d}t \leq N \left(\|g\|_H^2 + \int_0^T \|f(t)\|_{V^*}^2 \mathrm{d}t \right),$$

where N *is a constant.*

Outline

Grid

- $T \in (0, \infty)$;
- n, a non-negative integer such that $k := T/n \in (0, 1]$;
- the n-grid on $[0, T]$:

$$T_n = \{t \in [0, T] : t = jk, \quad j = 0, 1, \ldots, n\};$$

- A_k, f_k, some discrete versions of A and f, respectively.

Discretised version of the problem (Evol.Eq.) – implicit scheme

$$\Delta^- v_{i+1} = A_{k,i+1} v_{i+1} + f_{k,i+1} \quad \text{for} \quad i = 0, 1, \ldots, n-1, \quad v_0 = g, \tag{TimeDiscr.}$$

where Δ^- is the backward finite difference quotient.

Assumption (Implicit discretisation – ID)

1. $\langle A_{k,j+1} v, v\rangle + \lambda \|v\|_V^2 \leq K\|v\|_H^2, \quad \forall v \in V,\ j = 0, 1, \dots, n-1,$

2. $\|A_{k,j+1} v\|_{V^*} \leq M\|v\|_V, \quad \forall v \in V,\ j = 0, 1, \dots, n-1,$

3. $\sum_{j=0}^{n-1} \|f_{k,j+1}\|_{V^*}^2 k \leq N$ *and* $\|g\|_H \leq N,$

where λ, K, M, *and* N *are the constants in Assumption AS.*

Theorem (Existence-uniqueness)

If

- *Assumption ID is satisfied*

then, for small k *and for all* $n \in \mathbb{N}$*, there exists a unique vector* $v_0, v_1, \ldots, v_n$ *in* V *satisfying the problem (TimeDiscr.).*

Theorem (Stability)

If

- *Assumption ID is satisfied;*
- $v_{k,j}$, *with* $j = 0, 1, \ldots, n$, *is the unique solution to (TimeDiscr.);*

then there exists a constant N *independent of* k *such that*

1. $$\max_{0\leq j\leq n} \|v_{k,j}\|_H^2 \leq N \left(\|g\|_H^2 + \sum_{j=1}^{n} \|f_{k,j}\|_{V^*}^2 k \right);$$

2. $$\sum_{j=0}^{n} \|v_{k,j}\|_V^2 k \leq N \left(\|g\|_H^2 + \sum_{j=1}^{n} \|f_{k,j}\|_{V^*}^2 k \right).$$

Assumption (Smoothness – S)

There exist a fixed number $\delta \in (0,1]$ *and a constant* C *such that*

$$\frac{1}{k}\int_{t_i}^{t_{i+1}} \|u(t_{i+1}) - u(s)\|_V \mathrm{d}s \leq Ck^\delta,$$

for all $i = 0, 1, \ldots, n-1$, *where* u *is the solution to (Evol.Eq.).*

Theorem (Convergence)

Suppose that

- *Assumption AS is satisfied;*
- *Assumption ID is satisfied;*
- *Assumption S is satisfied;*
- $u(t)$ *is the unique solution to (Evol.Eq.);*
- $v_{k,j}$, $j = 0, 1, \ldots, n$, *is the unique solution to (TimeDiscr.).*

Then, for k small enough, there exists a constant N independent of k such that

1. $$\max_{0\leq j\leq n}\|v_{k,j}-u(t_j)\|_H^2 \leq N\Bigg(k^{2\delta}+\sum_{j=1}^{n}\frac{1}{k}\Big\|A_{k,j}u(t_j)k-\int_{t_{j-1}}^{t_j}A(s)u(t_j)\mathrm{d}s\Big\|_{V^*}^2 + \sum_{j=1}^{n}\frac{1}{k}\Big\|f_{k,j}k-\int_{t_{j-1}}^{t_j}f(s)\mathrm{d}s\Big\|_{V^*}^2\Bigg);$$

2. $$\sum_{j=0}^{n}\|v_{k,j}-u(t_j)\|_V^2 k \leq N\Bigg(k^{2\delta}+\sum_{j=1}^{n}\frac{1}{k}\Big\|A_{k,j}u(t_j)k-\int_{t_{j-1}}^{t_j}A(s)u(t_j)\mathrm{d}s\Big\|_{V^*}^2 + \sum_{j=1}^{n}\frac{1}{k}\Big\|f_{k,j}k-\int_{t_{j-1}}^{t_j}f(s)\mathrm{d}s\Big\|_{V^*}^2\Bigg).$$

An example

If A_k and f_k are specified by the integral averages

$$\bar{A}_{k,j+1} z := \frac{1}{k} \int_{t_j}^{t_{j+1}} A(s) z \mathrm{d}s \quad \text{and} \quad \bar{f}_{k,t_{j+1}} := \frac{1}{k} \int_{t_j}^{t_{j+1}} f(s) \mathrm{d}s,$$

for all $z \in V$, $j = 0, 1, \ldots, n-1$, then Assumption ID is satisfied. Moreover,

$$\sum_{j=1}^{n} \frac{1}{k} \left\| \bar{A}_{k,j} u(t_j) k - \int_{t_{j-1}}^{t_j} A(s) u(t_j) \mathrm{d}s \right\|_{V^*}^2$$

$$= \sum_{j=1}^{n} \frac{1}{k} \left\| \frac{1}{k} \int_{t_{j-1}}^{t_j} A(s) u(t_j) \mathrm{d}s \, k - \int_{t_{j-1}}^{t_j} A(s) u(t_j) \mathrm{d}s \right\|_{V^*}^2 = 0$$

and

$$\sum_{j=1}^{n} \frac{1}{k} \left\| \bar{f}_{k,j} k - \int_{t_{j-1}}^{t_j} f(s) \mathrm{d}s \right\|_{V^*}^2 = \sum_{j=1}^{n} \frac{1}{k} \left\| \frac{1}{k} \int_{t_{j-1}}^{t_j} f(s) \mathrm{d}s k - \int_{t_{j-1}}^{t_j} f(s) \mathrm{d}s \right\|_{V^*}^2 = 0.$$

Outline

The PDE problem

$$\frac{\partial u}{\partial t} = Lu + f \quad \text{in} \quad Q, \qquad u(0,x) = g(x) \quad \text{in} \quad \mathbb{R}^d, \qquad \text{(PDE)}$$

where

- L is the second-order partial differential operator with real coefficients

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x)\frac{\partial}{\partial x^i} + c(t,x);$$

- $Q = [0, T] \times \mathbb{R}^d$, with $T \in (0, \infty)$ a constant;
- f and g are real-valued given functions.

The coefficients a^{ij} and b^i, and the free data f and g **are allowed to grow in the spatial variables**.

The well-weighted Sobolev space $W^{m,2}(r,\rho)$

- The family of the so-called well-weighted Sobolev spaces was first introduced in [Purtukhia (1984)],
- and further generalised in [Gyöngy & Krylov (1990)].

Let $r > 0$ and $\rho > 0$ smooth functions in U, a domain in $\mathbb{R}^d$, and $m \geq 0$ an integer. The weighted Sobolev space $W^{m,2}(U;r,\rho)$ is the Banach space of locally integrable functions $v : U \to \mathbb{R}$ such that

- $D^\alpha v$ exists in the weak sense, for each multi-index α, with $|\alpha| \leq m$;
- $\|v\|_{W^{m,2}(U;r,\rho)} := \left(\sum_{|\alpha|\leq m} \int_U r^2 \left|\rho^{|\alpha|} D^\alpha v\right|^2 \mathrm{d}x \right)^{1/2} < \infty.$

Endowed with the inner product which generates the above norm $W^{m,2}(U;r,\rho)$ is a Hilbert space.

When $U = \mathbb{R}^d$ we simply write $W^{m,2}(r,\rho) := W^{m,2}(\mathbb{R}^d;r,\rho)$.

Assumption (Weights – W)

Let $m \geq 0$ be an integer, and $r, \rho > 0$ smooth functions on $\mathbb{R}^d$.

There exists a constant K such that

1. $|D^\alpha \rho| \leq K \rho^{1-|\alpha|}$, *for all α such that $|\alpha| \leq m-1$ if $m \geq 2$;*
2. $|D^\alpha r| \leq K \dfrac{r}{\rho^{|\alpha|}}$, *for all α such that $|\alpha| \leq m$.*

Example

$$\rho(x) = (1+|x|^2)^\gamma, \gamma \leq \frac{1}{2} \quad \text{and} \quad r(x) = (1+|x|^2)^\beta, \beta \in \mathbb{R}.$$

Assumption (PDE)

Let $r, \rho > 0$ be smooth functions on $\mathbb{R}^d$, and $m \geq 0$ an integer.

1. *There exists a constant $\lambda > 0$ such that*
$$\sum_{i,j=1}^{d} a^{ij}(t,x)\xi^i\xi^j \geq \lambda\rho^2(x)|\xi|^2,$$
for all $t \geq 0$, $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$;

2. *The coefficients in L and their derivatives in x up to the order m are measurable functions in $[0,T] \times \mathbb{R}^d$ such that*
    - $|D_x^\alpha a^{ij}| \leq K\rho^{2-|\alpha|} \quad \forall|\alpha| \leq m \vee 1,$
    - $|D_x^\alpha b^i| \leq K\rho^{1-|\alpha|} \quad \forall|\alpha| \leq m,$
    - $|D_x^\alpha c| \leq K \quad \forall|\alpha| \leq m,$

    for any $t \in [0,T], x \in \mathbb{R}^d$, with K a constant;

3. *$f \in L^2([0,T]; W^{m-1,2}(r,\rho))$ and $g \in W^{m,2}(r,\rho)$.*

Definition (Generalised solution)

$u \in C([0,T]; W^{0,2}(r,\rho))$ is a generalized solution to the problem (PDE) on $[0,T]$ if

1. $u \in L^2([0,T]; W^{1,2}(r,\rho))$;
2. For every $t \in [0,T]$,

$$
\begin{aligned}
(u(t),\varphi) = (g,\varphi) + \int_0^t \Big\{ & -(a^{ij}(s)D_{x^i}u(s), D_{x^j}\varphi) \\
& + (b^i(s)D_{x^i}u(s) - D_{x^j}a^{ij}(s)D_{x^i}u(s), \varphi) \\
& + (c(s)u(s),\varphi) + \langle f(s),\varphi\rangle \Big\} \mathrm{d}s
\end{aligned}
$$

holds for all $\varphi \in C_0^\infty$.

The following existence-uniqueness result for the solution of the problem (PDE) can be obtained from the corresponding general result for an evolution equation by using the suitable triple of spaces.

Theorem (Existence-uniqueness)

Under

- *Assumption W, with* $m+1$ *in place of* m*;*
- *Assumption PDE;*

the problem (PDE) admits a unique generalised solution u *on* $[0,T]$*. Moreover*

$$u \in C\left([0,T]; W^{m,2}(r,\rho)\right) \cap L^2\left([0,T]; W^{m+1,2}(r,\rho)\right)$$

and, with N *a constant,*

$$\sup_{0\le t\le T} \|u(t)\|^2_{W^{m,2}(r,\rho)} + \int_0^T \|u(t)\|^2_{W^{m+1,2}(r,\rho)} \mathrm{d}t \le N\Big(\|g\|^2_{W^{m,2}(r,\rho)} + \int_0^T \|f(t)\|^2_{W^{m-1,2}(r,\rho)} \mathrm{d}t\Big).$$

Outline

h-grid on $\mathbb{R}^d$

$$Z_h^d = \Big\{ x \in \mathbb{R}^d : x = h \sum_{i=1}^{d} e_i n_i, \quad n_i = 0, \pm 1, \pm 2, \dots \Big\}.$$

with $h \in (0, 1]$.

Discrete operator

$$L_h(t,x) = a^{ij}(t,x)\partial_j^- \partial_i^+ + b^i(t,x)\partial_i^+ + c(t,x)$$

where ∂_i^- and ∂_i^+ are the backward and forward difference quotients in space, respectively.

Discrete problem

$$u_t = L_h u + f_h \quad \text{in} \quad Q(h), \qquad u(0,x) = g_h(x) \quad \text{in} \quad Z_h^d, \quad (\text{SpaceDiscr.})$$

where

- $Q(h) = [0, T] \times Z_h^d$;
- $f_h : Q(h) \to \mathbb{R}$;
- $g_h : Z_h^d \to \mathbb{R}$.

Discrete version of the weighted Sobolev space $W^{0,2}(r, \rho)$

For functions $v : Z_h^d \to \mathbb{R}$, we define the space

$$l^{0,2}(r) = \{v : \|v\|_{l^{0,2}(r)} < \infty\},$$

where the norm $\|v\|_{l^{0,2}(r)}$ is given by

$$\|v\|_{l^{0,2}(r)} = \left(\sum_{x \in Z_h^d} r^2(x)|v(x)|^2 h^d \right)^{1/2}.$$

We define, for any $v, w \in l^{0,2}(r)$, the inner product

$$(v, w)_{l^{0,2}(r)} = \sum_{x \in Z_h^d} r^2(x) v(x) w(x) h^d,$$

which induces the norm.

The inner product space $l^{0,2}(r)$ has a good structure: it can be easily shown that it is complete, therefore a Hilbert space.

Discrete version of the weighted Sobolev space $W^{1,2}(r,\rho)$

For functions $w : Z_h^d \to \mathbb{R}$, we define also the space

$$l^{1,2}(r,\rho) = \{w : \|w\|_{l^{1,2}(r,\rho)} < \infty\},$$

with norm

$$\|w\|^2_{l^{1,2}(r,\rho)} := \|w\|^2_{l^{0,2}(r)} + \sum_{i=1}^{d} \|\rho\partial_i^+ w\|^2_{l^{0,2}(r)}.$$

We endow $l^{1,2}(r,\rho)$ with the inner product, inducing the norm,

$$(w,z)_{l^{1,2}(r,\rho)} = (w,z)_{l^{0,2}(r)} + \sum_{i=1}^{d} (\rho\partial_i^+ w, \rho\partial_i^+ z)_{l^{0,2}(r)},$$

for any functions $w, z \in l^{1,2}(r,\rho)$.

Assumption (Data – D)

1. $f_h \in L^2\left([0, T]; l^{0,2}(r)\right)$;
2. $g_h \in l^{0,2}(r)$;

where $r > 0$ is a smooth function on $\mathbb{R}^d$.

Definition (Generalised solution)

$u \in C([0, T]; l^{0,2}(r)) \cap L^2([0, T]; l^{1,2}(r, \rho))$ is a generalized solution of the discrete problem (SpaceDiscr.) if, for every $t \in [0, T]$,

$$
\begin{aligned}
(u(t), \varphi) = (g_h, \varphi) + \int_0^t \Big\{ &- (a^{ij}(s)\partial_i^+ u(s), \partial_j^+ \varphi) \\
&+ (b^i(s)\partial_i^+ u(s) - \partial_j^+ a^{ij}(s)\partial_i^+ u(s), \varphi) \\
&+ (c(s)u(s), \varphi) + \langle f_h(s), \varphi \rangle \Big\} \mathrm{d}s
\end{aligned}
$$

holds for all $\varphi \in l^{1,2}(r, \rho)$.

The following existence-uniqueness (and stability) result for the solution of the discrete problem (SpaceDiscr.) is obtained as a consequence of the corresponding general result for an evolution equation, by showing that the discrete framework we have set is a particular case of the general framework.

Theorem (Existence-uniqueness)

Under

- *Assumption PDE;*
- *Assumption D;*

the problem (SpaceDiscr.) has a unique generalised solution u in $[0, T]$*. Moreover, for N a constant independent of h,*

$$\sup_{0 \le t \le T} \|u(t)\|^2_{l^{0,2}(r)} + \int_0^T \|u(t)\|^2_{l^{1,2}(r,\rho)} \mathrm{d}t \le N\left(\|g_h(t)\|^2_{l^{0,2}(r)} + \int_0^T \|f_h(t)\|^2_{l^{0,2}(r)} \mathrm{d}t \right).$$

Outline

- We first investigate the **consistency of the discrete scheme**, and prove that the difference quotients approximate the partial derivatives.
- The result is obtained **by using a Sobolev inequality**, and imposing that the weights ρ are bounded from below by a positive constant.
- In practice, the latter restriction amounts to assume that the weights ρ are increasing functions of $|x|$, which is precisely the case we are interested in.

Theorem (Consistency)

Let $r > 0$ and $\rho > 0$ be functions on $\mathbb{R}^d$, and m an integer strictly greater than $d/2$. If

- *Assumption W is satisfied;*
- $\rho(x) \geq C$ *on* $\mathbb{R}^d$*, with* $C > 0$ *a constant;*
- $u(t) \in W^{m+2,2}(r, \rho)$, $v(t) \in W^{m+3,2}(r, \rho)$*, for all* $t \in [0, T]$*;*

then there exists a constant N independent of h such that, for all $t \in [0, T]$,

1. $$\sum_{x \in Z_h^d} r^2(x) \left| u_{x^i}(t, x) - \partial_i^+ u(t, x) \right|^2 \rho^2(x) h^d \leq h^2 N \| u(t) \|^2_{W^{m+2,2}(r,\rho)};$$

2. $$\sum_{x \in Z_h^d} r^2(x) \left| v_{x^i x^j}(t, x) - \partial_j^- \partial_i^+ v(t, x) \right|^2 \rho^4(x) h^d \leq h^2 N \| v(t) \|^2_{W^{m+3,2}(r,\rho)}.$$

- Finally, owing to the **stability and consistency properties** of the discrete scheme, we prove the **convergence** of the discrete problem's solution to the PDE problem's solution, and compute a **rate of convergence**.

- The accuracy obtained is of order 1.

- Note that **the way we set our discrete framework, in strong connection with the framework for problem (PDE), plays a crucial role in obtaining the convergence rate**.

Theorem (Convergence)

Let $r > 0$ and $\rho > 0$ be functions on $\mathbb{R}^d$, and m an integer strictly greater than $d/2$. Suppose that

- *Assumption W is satisfied;*
- *Assumption PDE is satisfied;*
- *Assumption D is satisfied;*
- *$\rho(x) \geq C$, with $C > 0$ a constant;*
- *u is the unique solution to (PDE);*
- *u_h is the unique solution to (SpaceDiscr.);*
- *$u \in L^2([0, T]; W^{m+3,2}(r, \rho))$.*

Then

$$\sup_{0 \leq t \leq T} \|u(t) - u_h(t)\|^2_{l^{0,2}(r)} + \int_0^T \|u(t) - u_h(t)\|^2_{l^{1,2}(r,\rho)} \mathrm{d}t$$

$$\leq h^2 N \int_0^T \|u(t)\|^2_{W^{m+3,2}(r,\rho)} \mathrm{d}t + N\left(\|g - g_h\|^2_{l^{0,2}(r)} + \int_0^T \|f(t) - f_h(t)\|^2_{l^{0,2}(r)} \mathrm{d}t\right),$$

with N a constant independent of h.

Outline

1. Motivation
2. Discretisation of a linear evolution equation in abstract spaces
   - A general framework – classical results
   - Euler implicit discretisation
3. Semi-discretisation in space of the PDE
   - A particular framework – classical results
   - The discrete framework
   - Approximation results
4. Discretisation of the PDE in space and time

The problem

We considered the Cauchy problem for a linear parabolic PDE

$$\frac{\partial u}{\partial t} = Lu + f \quad \text{in} \quad Q \qquad u(0,x) = g(x) \quad \text{in} \quad \mathbb{R}^d \tag{PDE}$$

where

- L is the second-order partial differential operator with real coefficients

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(t,x)\frac{\partial}{\partial x^i} + c(t,x);$$

- $Q = [0, T] \times \mathbb{R}^d$ with $T \in (0, \infty)$ is a constant;
- f and g are real-valued given functions;

and allowed the growth, in the spatial variables, of the coefficients a^{ij} and b^i, and the free data f and g.

Space-discretised problem

We obtained problem (PDE) FD discretisation in space

$$u_t = L_h u + f_h \ \text{ in } \ Q(h), \quad u(0,x) = g_h(x) \ \text{ in } \ Z_h^d, \ (\text{SpaceDiscr.})$$

where

- $Q(h) = [0, T] \times Z_h^d$;
- $L_h(t,x) = a^{ij}(t,x)\partial_j^- \partial_i^+ + b^i(t,x)\partial_i^+ + c(t,x)$, with ∂_i^- and ∂_i^+ respectively the backward and forward difference quotients in space.
- $f_h : Q(h) \to \mathbb{R}$;
- $g_h : Z_h^d \to \mathbb{R}$.

Fully discretised problem

We consider now the time-discretization of the problem (SpaceDiscr.), by using the implicit FD scheme

$$\Delta^- v_{i+1} = L_{hk,i+1} v_{i+1} + f_{hk,i+1} \quad \text{for} \quad i = 0, 1, \dots, n-1, \qquad v_0 = g_h, \tag{FullyDiscr.}$$

where Δ^- and Δ^+ are the backward and the forward difference quotients in time, respectively.

- The existence-uniqueness results for the problems PDE and (SpaceDiscr.) were proved by showing that the frameworks they were set in are particular cases of the general framework for an evolution equation.

- Therefore, under the hypotheses for the existence and uniqueness of the generalised solutions to problems (PDE), (TimeDiscr.), and (SpaceDiscr.), the problem (FullyDiscr.) has a unique generalised solution.

- It remains only to determine the rate of convergence when the discretisation is considered both in space and time.

Theorem (Convergence)

Let $r > 0$ *and* $\rho > 0$ *be functions on* $\mathbb{R}^d$*, and* m *an integer strictly greater than* $d/2$*. Suppose that*

- *Assumption ID is satisfied;*
- *Assumption W is satisfied;*
- *Assumption PDE is satisfied;*
- *Assumption D is satisfied;*
- $\rho(x) \geq C$*, with* $C > 0$ *a constant.*

Let

- u *be the unique solution to (PDE);*
- u_h *be the unique solution to (SpaceDiscr.);*
- $v_{hk,j}$*,* $j = 0, 1, \ldots, n$*, be the unique solution to (FullyDiscr.).*

Assume further that

- u *satisfies Assumption S;*
- $u \in L^2([0, T]; W^{m+3,2}(r, \rho))$*.*

Theorem (cont.)

Then

$$
\begin{aligned}
&\max_{0\le j\le n}\|v_{hk,j}-u(t_j)\|^2_{l^{0,2}(r)}+\sum_{0\le j\le n}\|v_{hk,j}-u(t_j)\|^2_{l^{1,2}(r,\rho)}k\\
&\quad\le N\Big(k^{2\delta}+h^2\int_0^T\|u(t)\|^2_{W^{m+3,2}(r,\rho)}\mathrm{d}t\Big)\\
&\qquad+N\Big(\sum_{1\le j\le n}\frac{1}{k}\Big\|L_{hk,j}u_h(t_j)k-\int_{t_j}^{t_{j+1}}L_h(s)u_h(t_j)\mathrm{d}s\Big\|^2_{l^{0,2}(r)}\\
&\qquad\qquad+\sum_{1\le j\le n}\frac{1}{k}\Big\|f_{hk,j}k-\int_{t_j}^{t_{j+1}}f_h(s)\mathrm{d}s\Big\|^2_{l^{0,2}(r)}\\
&\qquad\qquad+\|g-g_h\|^2_{l^{0,2}(r)}+\int_0^T\|f(t)-f_h(t)\|^2_{l^{0,2}(r)}\mathrm{d}t\Big),
\end{aligned}
$$

with N a constant independent of h and k.

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