Exponential functionals of Lévy processes

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(joint work with Denis Belomestny)

Setup

Definition.

For a Lévy process $\xi = (\xi_t)_{t\geq 0}$, *exponential functional* of ξ is defined as

$$A_q(\xi) := \int_0^q e^{-\xi_t}\, dt, \qquad q > 0.$$

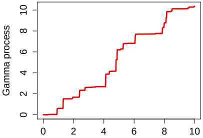
Terminal value:

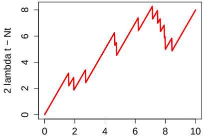
$$A_\infty(\xi) := \lim_{q\to+\infty} A_q(\xi) = \int_{\mathbb{R}_+} e^{-\xi_t} dt.$$

Finiteness condition:

$$A_\infty(\xi) < +\infty \quad \text{a.s.} \quad \Longleftrightarrow \quad \lim_{t\to+\infty} \xi_t = +\infty \quad \text{a.s.}$$

Examples: $\xi_t =$ subordinator (non-decreasing Lévy process);
$\xi_t = 2\lambda t - N_t$, where N_t is a Poisson process with intensity λ.





Motivation

ARCH(1) model (AutoRegresive Conditionally Heteroscedastic) [*Engle, 1982*]

$$Y_n = \sigma_n \varepsilon_n, \qquad \sigma_n^2 = \alpha + \beta Y_{n-1}^2,$$

where ε_n - white noise process (i.i.d., $\mathbb{E}\varepsilon_n = 0, \operatorname{Var} \varepsilon_n = 1$).

GARCH(1,1) model (Generalized ARCH) [*Bollerslev, 1986*]

$$Y_n = \sigma_n \varepsilon_n, \qquad \sigma_n^2 = \alpha + \beta Y_{n-1}^2 + \gamma \sigma_{n-1}^2.$$

COGARCH (COntinious-time GARCH) [*Klüppelberg, Lindner, Maller, 2004*]
Idea: replace ε_n by the increments of a Lévy process X_t.

$$Y_t = \int \sigma_t dX_t, \qquad \begin{aligned} \sigma_t^2 &= e^{-\xi_t}\Big(\alpha \int_{(0,t]} e^{\xi_{s-}} ds + \sigma_0^2\Big), \\ \xi_t &= t \log \gamma + \sum_{0<s\leq t} \log\Big(1 + \frac{\gamma}{\beta} (\Delta X_s)\Big). \end{aligned}$$

$A_\infty(\xi)$ - invariant measure of the Generalized Ornstein-Uhlenbeck process σ_t.

Setup

Model:

$$A_\infty = \int_0^{+\infty} e^{-\xi_t} dt, \quad \text{where} \quad \xi_t = ct + \mathcal{T}_t,$$

where $c > 0$ and $\mathcal{T}_t$ is a pure subordinator with finite Lévy measure, i.e.,

$$\nu(\mathbb{R}_-) = 0, \qquad a := \nu(\mathbb{R}_+) < \infty.$$

Data:

Suppose that n observations $A_{\infty,1}, ..., A_{\infty,n}$ of the integral A_∞ are given.

The aims:

(1) to estimate a, c; (2) to estimate the Lévy measure ν.

Main idea

Recursive formula (*Kuznetzov, A., Pardo, J. and Savov, V.*, 2011):

$$\mathbb{E}\left[A_\infty^{s-1}\right] = \frac{\phi(s)}{s}\mathbb{E}\left[A_\infty^{s}\right],$$

where $\phi(s)$ is a Laplace exponent of the process ξ, i.e.,

$$\phi(s) := -\log \mathbb{E}\left[e^{-s\xi_1}\right],$$

and s is any number from the set

$$\Upsilon := \left\{s \in \mathbb{C} : \ 0 < \mathrm{Re}(s) < \theta\right\} \qquad \text{with} \qquad \theta := \sup\left\{z \geq 0 : \mathbb{E}[e^{-z\xi_1}] \leq 1\right\}.$$

Laplace exponent:

$$\phi(u + \mathrm{i}v) \quad = \quad a + c\,(u + \mathrm{i}v) - \int_0^{+\infty} e^{-\mathrm{i}vx}\bar{\nu}(dx),$$

where $\bar{\nu}(dx) = e^{-ux}\nu(dx)$, and $u + \mathrm{i}v \in \Upsilon$.

Algorithm 1: Estimation procedure for a, c

Consider the set $\left\{u + \mathrm{i}v \in \mathbb{C} : u \in (1,\theta) - \text{fixed}, v \in [\varepsilon V_n, V_n] - \text{varies}\right\} \subset \Upsilon$
Take the values $v_1, ..., v_M$ on the equidistant grid on $[\varepsilon V_n, V_n]$ with a step Δ.

1. $\widehat{\mathbb{E}}_n \left[A_\infty^s\right] := \text{mean}_k \left(A_{\infty,k}^s\right)$ for $s = u + \mathrm{i}v_m$ and $s = (u-1) + \mathrm{i}v_m$.
2. $\hat{\phi}_n(s) := s\, \widehat{\mathbb{E}}_n \left[A_\infty^{s-1}\right] / \widehat{\mathbb{E}}_n \left[A_\infty^s\right]$ for $s = u + \mathrm{i}v_m, \quad m = 1..M$.
3. $\hat{c}_n := \arg\min_c \sum_{m=1}^M w_n(v_m) \left(\operatorname{Im} \hat{\phi}_n(u + \mathrm{i}v_m) - cv_m\right)^2.$

   Motivation: $\operatorname{Im} \phi(u + \mathrm{i}v) = cv - \operatorname{Im} \mathcal{F}_{\bar{\nu}}(-v).$
4. $\hat{a}_n := \arg\min_a \sum_{m=1}^M w_n(v_m) \left(\operatorname{Re} \hat{\phi}_n(u + \mathrm{i}v_m) - \hat{c}_n u - a\right)^2.$

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Algorithm 1: Simulation study

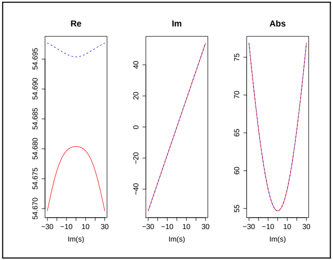
$$A_\infty = \int_0^{+\infty} e^{-\xi_t}\, dt, \qquad \xi_t = ct + \mathcal{T}_t,$$

Let $\mathcal{T}_t$ have a Lévy density $\quad \nu(x) = ab \exp\{-bx\}\, I\{x>0\}, \quad a, b > 0.$

In this case, $\quad A_\infty \stackrel{d}{=} B\Big(b+1, a/c\Big)/c.$

Laplace exponent: $\quad \phi(s) = s\left(c + a\,(b+s)^{-1}\right).$

Estimate: $\quad \hat{\phi}_n(s) = s\, \widehat{\mathbb{E}}_n\left[A_\infty^{s-1}\right] / \widehat{\mathbb{E}}_n\left[A_\infty^{s}\right], \qquad s = u + \mathrm{i}v, \quad u = 30,\ v = 0..30.$



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$$\hat{c}_n = \mathrm{regr}\Big(\mathrm{Im}\,\hat{\phi}_n(u+\mathrm{i}v_m) \sim v_m\Big),$$
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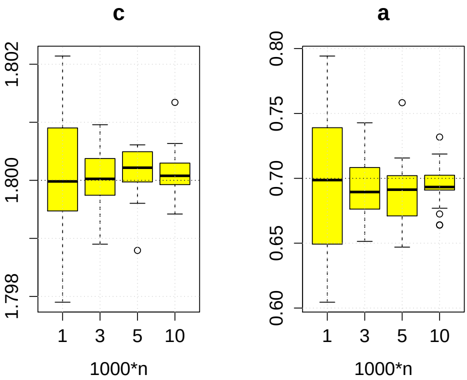

Algorithm 2: Estimation procedure for the Lévy measure ν

Consider the set $\left\{ u + \mathrm{i}v \in \mathbb{C} : u \in (1,\theta) - \text{fixed}, v \in [-V_n, V_n] - \text{varies} \right\} \subset \Upsilon$
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   Motivation: $\quad \phi(u+\mathrm{i}v) = a + c\,(u+\mathrm{i}v) - \mathcal{F}_{\bar{\nu}}(-v)$.
4. $\hat{\nu}(x) = e^{ux} \frac{\Delta}{2\pi} \sum_{m=1}^{M} e^{\mathrm{i}v_m x} \hat{\mathcal{F}}_{\bar{\nu}}(-v_m) \mathcal{K}(v_m h)$.
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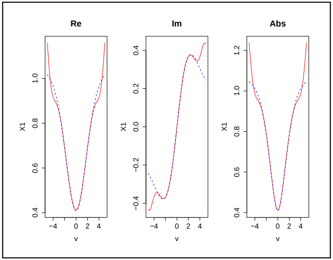
Algorithm 2: Simulation study

$$A_\infty = \int_0^{+\infty} e^{-\xi_t}\, dt, \qquad \xi_t = ct + \mathcal{T}_t,$$

$\mathcal{T}_t$ is a compound Poisson process, $\quad \mathcal{T}_t = -(\log q) \sum_{k=1}^{N_t} \eta_k,$
where $q \in (0,1)$, η_k - i.i.d. $\sim \mathcal{N}_\alpha(0,1)$ truncated on (α, ∞), $\alpha \geq 0$.

Laplace exponent: $\quad \phi(s) = \lambda \left[1 - \frac{1-F(\alpha+(\log q)s)}{1-F(\alpha)} \exp\left\{ -\frac{(\log q)^2 s^2}{2} \right\} \right],$

Estimate: $\quad \hat{\phi}_n(s) = s\, \widehat{\mathbb{E}}_n \left[A_\infty^{s-1} \right] / \widehat{\mathbb{E}}_n \left[A_\infty^{s} \right], \quad s = u + \mathrm{i}v, \quad u = 30, \; v = -5..5.$



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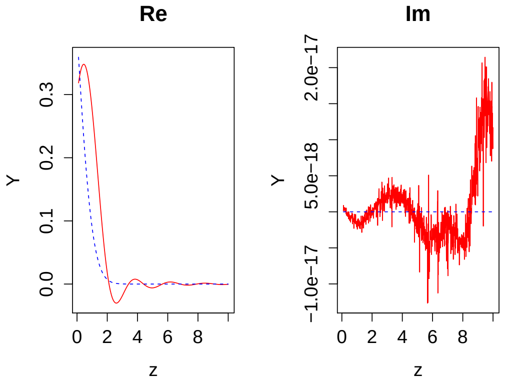

Theoretical study: assumptions

$A_\infty = \int_{\mathbb{R}_+} e^{-\xi_t}\, dt$, where $\xi_t = ct + \mathcal{T}_t$.

(A1) ξ_t is a subordinator with finite Lévy measure, i.e.,

$$c > 0, \quad \nu(\mathbb{R}_-) = 0, \quad a := \nu(\mathbb{R}_+) < \infty.$$

(A2) There exists $u \in \Upsilon$, $c > 0$ such that

$$\left|\mathbb{E}\left[A_\infty^{u+\mathrm{i}v}\right]\right| \asymp \exp\left\{-cv\right\}, \qquad v \to +\infty.$$

(A3) The Fourier transform of the measure $\bar{\nu}(dx) := e^{-ux}\nu(dx)$ has the following asymptotics:

$$|\mathcal{F}_{\bar{\nu}}(-v)| \lesssim v^{-\eta}, \qquad v \to +\infty.$$

Example:

$$\nu(x) = \sum_{j=1}^{M}\sum_{k=1}^{m_j} \alpha_{jk} x^{k-1} e^{-\rho_j x}, \qquad x > 0,$$

with $m_j \in \mathbb{N}$, $\rho_j > 0$, $\alpha_{jk} > 0$.

Quality of $\hat{\phi}(\cdot)$

Theorem

Let the assumptions (A1) - (A3) be fulfilled. Let the sequence V_n tend to ∞ and moreover satisfy the assumption

$$\Lambda_n := V_n^2 \exp\{cV_n\} \sqrt{\log V_n} = o\left(\sqrt{\frac{n}{\log(n)}}\right), \quad n \to \infty.$$

Then there exists a set $\mathcal{W}_n$ such that $\mathbb{P}\{\mathcal{W}_n\} > 1 - \alpha n^{-1-\delta}$ with some positive α and δ and

$$\mathcal{W}_n \subset \left\{ \sup_{v \in [\varepsilon V_n, V_n]} \left| \hat{\phi}_n(u + \mathrm{i}v) - \phi(u + \mathrm{i}v) \right| \le \beta\, \Lambda_n \sqrt{\frac{\log(n)}{n}} \right\},$$

where $\beta > 0$.

Quality of $\hat{a}_n$ and $\hat{c}_n$

Theorem

Consider the setup of the last theorem and take

$$V_n = \kappa \log(n) \quad \textit{with} \quad \kappa < 1/(2c).$$

Let $w_n(v)$ be a weighting function in the form $w_n(v) = w^1(v/V_n)/V_n$ with an integrable non-negative function $w^1(\cdot)$ supported on $[\varepsilon, 1]$. Then

$$\mathcal{W}_n \subset \left\{ |\hat{c}_n - c| \leq \frac{\zeta_1}{\log^{1+\eta}(n)} \right\} \quad \textit{and} \quad \mathcal{W}_n \subset \left\{ |\hat{a}_n - a| \leq \frac{\zeta_2}{\log^{\eta}(n)} \right\},$$

where $\zeta_1, \zeta_2 > 0$.

Law of A_∞

Theorem (*Carmona, Petit, Yor, 1997*)

Let

$$\xi_t = ct + \sigma W_t + \tau_t^+ - \tau_t^-,$$

where τ_t^+, τ_t^- are two subordinators without drift. Then (under some mild assumptions)

1. *the exponential functional $A_\infty = \int_0^{+\infty} e^{-\xi_t} dt$ is absolutely continuous with respect to Lebesgue measure;*
2. *A_∞ has a $\mathbb{C}^\infty$ density $k(x)$ which solves the following equation:*

$$-\frac{\sigma^2}{2}\frac{d}{dx}\Big(x^2 k(x)\Big) + \Big((\sigma^2/2 - c)x + 1\Big)\, k(x) = \int_x^\infty \bar{\nu}^+\Big(\log(u/x)\Big) k(u) du - \int_0^x \bar{\nu}^-\Big(\log(u/x)\Big) k(u) du$$

where $\bar{\nu}^+(\cdot) = \nu^+(\cdot, +\infty)$ and $\bar{\nu}^-(\cdot) = \nu^-(\cdot, +\infty)$ are tail integrals for the Lévy measures of τ_t^+, τ_t^- resp.

Why do we need complex numbers in this task at all?

Inverse Laplace transform (Bromwich integral):

$$\nu(x) = \frac{1}{2\pi \mathrm{i}} \int_{\Omega} e^{sx} \mathcal{L}(s) ds, \qquad \Omega := \{\mathrm{Re}(s) = \gamma\},$$

where γ is greater than the real part of all singularities of $\mathcal{L}(s)$.

The shortest path between two truths in the real domain passes through the complex domain (Jacques Hadamard).

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Thank you for your attention.