Optimization of bank credit policy and the governmental guarantees in a model of investing in a risky projects

Vadim Arkin, Alexander Slastnikov
Central Economics and Mathematics Institute,
Russian Academy of Sciences, Moscow

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General framework

There is a project that, in a certain period after its financing (the lag) can bring some income flow. The project is risky, i.e. after beginning of financing it can fail with some probability, even before starting functioning.

To finance the project it is necessary to obtain funds on credit from a bank.

Since a risky project has the probability of a failure to return the credit, the interest rate on a credit can be rather high, but the bank is ready to reduce it provided the credit is partially returned.

To attract investment to risky projects, there is a mechanism of governmental guarantees on credit. This means that if the project fails and the investor does not return the credit, the bank receives from the state reimbursement of a part of the credit to the project. This mechanism allows, in particular, to reduce the interest rate on the loan.

Aims

The aim of the work is to study:

1. optimization of credit policy of a bank and a mechanism of governmental guarantees on loan;
2. risk influence on efficiency of mechanism of governmental loan guarantees for the state, a bank and an investor.

The basic model. I

I is the amount of required investment for the project.

h is the duration of a lag of fixed assets.

The project is risky, i.e. after a financing and a lag of fixed assets it:

- with probability q, $0 < q < 1$ fails and the investor defaults;
- with probability $1 - q$ begins to operate and brings some profit flow.

The project is assumed to be infinitely-lived, and profit flow is described by a random process.

At any moment, investor can either *accept* the project and proceed with the investment or *delay* the decision until he obtains new information regarding its business environment (product and resource prices, product demand, etc.).

The basic model. II

All funds for the required investment are borrowed from a bank.

τ is the moment of financing the project (= moment of borrowing).

- If the project becomes operational, the repayment of loan and accrued interest on it starts after the ending of the lag of fixed assets at time $\tau + h$.
- If the project fails, the loan is not returned, but the bank receives (at the end of the lag $\tau + h$) reimbursement of credit from the state, as a part θ of gross loans.

Financing

A credit policy of the bank (in relation to given investment project) is described by set $\{L,\ \lambda,\ (f_t)\}$, where:

L is the loan period, λ is the interest rate, (f_t) is the flow of repayments (without interest repayments) per unit of credit:

$f_t \geq 0,\ h \leq t \leq L : \displaystyle\int_h^L f_t dt = 1.$

The credit burden on the project, i.e. the total repayments (including interest) for borrowing (per unit of credit), discounted to the investment moment τ, are:

$$\begin{aligned} k &= \lambda h e^{-\rho h} + \int_{\tau+h}^{\tau+L} \left(f_{t-\tau} + \lambda \int_t^{\tau+L} f_{s-\tau} ds \right) e^{-\rho(t-\tau)} dt \\ &= \lambda h e^{-\rho h} + 1 + (\lambda/\rho - 1)(1 - M), \end{aligned}$$

where ρ is the discount rate, $\lambda > \rho$, $M = \displaystyle\int_h^L f_t e^{-\rho t} dt.$

Choosing the investment moment

Let the present value of the implemented project X_t (expected integral future profit, discounted to present time t) be described by diffusion process

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dw_t,\ \ t \geqslant 0; \quad w_t \text{ — standard Wiener process.}$$

An investor's behavior is specified by an investment moment τ (which is supposed to be a stopping time)

Investor's objective function is the net present value from the project, discounted to zero time (NPV):

$$N(\tau) = (1-q)\mathbf{E}[(1-\gamma)X_\tau - kI]e^{-\rho\tau},$$

where γ is a coefficient of the tax burden (a share of tax payments in profits), k is a rate of the credit burden (on the project).

Credit policy of the bank

For determing a credit burden on the investment project the bank follows the criteria of discounted bank profit from granting a loan for the project:

$$C(k,\theta,\tau) = \mathbf{E} e^{-\rho\tau} \left((1-q)kI + q\theta I e^{-\rho h} - I \right),$$

where k is the credit burden (per unit of credit), θ is the part of loan reimbursement.

The natural restriction: $\quad k \geq k_0, \;\; (k_0 > 1)$.

There can be other restrictions, associated for example with competition from other banks.

A choice of guaranteed reimbursement on loan by the state

The criteria to compare different variants of guaranteed reimbursement on loan in financing a risky project is a budgetary effect, i.e. a difference between expected tax payments to the budget from the implemented project and expected cost of loan guarantees for supporting the project:

$$B(\theta, \tau) = \mathbf{E} e^{-\rho\tau} \left((1-q)\gamma X_\tau - q\theta I e^{-\rho h} \right).$$

Restrictions on acceptable parts: $\Theta = \{\underline{\theta} \leqslant \theta \leqslant \overline{\theta}\}$
(may be related to legislation or significance of the project for the state)

Three-stage optimization problem

1. Investor problem. For any credit policy of bank k investor chooses the optimal investment moment $\tau^*(k)$ as a solution to optimal stopping problem

$$N(\tau) \to \max_{\tau}, \quad \text{over all stopping times } \tau.$$

2. Bank problem. Knowing the dependence of optimal investor behavior $\tau^*(k)$ on credit terms, the bank chooses the optimal credit burden on project $k^*(\theta)$ for any part θ of guaranteed reimbursement of credit, solving the bank problem

$$C(k, \theta, \tau^*(k)) \to \max_{k \geq k_0}.$$

3. The state problem. Knowing the optimal credit burden on project $k^*(\theta)$ and the corresponding optimal investor behavior $\tau^*(k^*(\theta))$, the government determines the optimal part of a loan reimbursement θ^*, solving the budget problem

$$B(\theta, \tau^*(k^*(\theta))) \to \max_{\underline{\theta} \leqslant \theta \leqslant \overline{\theta}}.$$

Mathematical assumptions

Let present values of the investment project $X_t,\ t \geq 0$ be modelled by a diffusion process with values in the interval $(l, r)\ \ (-\infty \leq l < r \leq \infty)$, and be described by the stochastic differential equation

$$dX_t = a(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x,$$

The process X_t is assumed to be regular, and the boundary point l of the process values is a natural boundary.

$\mathbb{L}f(x) = \alpha(x)f'(x) + 0.5\sigma^2(x)f''(x)$ be an infinitesimal operator of X_t,

$\psi(x)$ be an increasing solution to ODE $\mathbb{L}f = \rho f$,

$\varphi(x)$ be a decreasing solution.

Assume that $\lim\limits_{x \downarrow l} x/\varphi(x) = 0$.

Define a function $F(p) = p - \tilde{\psi}(p)/\tilde{\psi}'(p)$, for $l < pI < r$, where $\tilde{\psi}(x) = \psi(xI)$.

It is supposed that the following condition holds:

(A) $F(p)$ be an increasing function, $\lim\limits_{pI \to r} F(p) = \infty$

Further, define functions $\xi(k) = \tilde{\psi}\left(F^{-1}(k\delta)\right)$ and $G(k) = k - \xi(k)/\xi'(k)$ for $k \ge k_0$, where $\delta = (1-\gamma)^{-1}$.

Let the following conditions hold:

(B) $G(k)$ be an increasing function, $\lim\limits_{k \to \infty} G(k) = \infty$

(C) $\alpha(pI) \le \rho(p - k\delta)I$ for $p > F^{-1}(k\delta)$

The existence theorem

If conditions **(A)–(C)** hold, then there exists a solution to the three-stage optimization problem, i.e. there are the unique functions $p^*(k)$, $k^*(\theta)$ and the value θ^*, such that:

$$\tau^*(k) = \min\{t \geq 0 : X_t \geq p^*(k)I\} = \underset{\tau}{\operatorname{argmax}}\, N(\tau),$$

$$k^*(\theta) = \underset{k \geqslant k_0}{\operatorname{argmax}}\, C(k, \theta, \tau^*(k)),$$

$$\theta^* = \underset{\underline{\theta} \leqslant \theta \leqslant \overline{\theta}}{\operatorname{argmax}}\, B(\theta, \tau^*(k^*(\theta))).$$

The triple $(\theta^*, k^*(\theta^*), \tau^*(k^*(\theta^*)))$ can be viewed as a Stackelberg equilibrium point in three-person game "State–Bank–Investor".

Characterization of solution

If conditions **(A)–(C)** hold, the solution to the three-stage optimization problem can be represented as follows:

$p^*(k) = F^{-1}\left(\frac{k}{1-\gamma}\right)$ — optimal threshold for investment;

$k^*(\theta) = \max\{k_0, G^{-1}\left(\frac{1-q\theta e^{-\rho h}}{1-q}\right)\}$ — optimal credit burden (on the project);

$\theta^* = \underset{\underline{\theta}\leqslant\theta\leqslant\bar{\theta}}{\operatorname{argmax}}\left[\frac{(1-q)\gamma m(\theta)-qe^{-\rho h}\theta}{\psi(m(\theta)I)}\right]$ — optimal part of a loan reimbursement,

where $\quad m(\theta) = F^{-1}\left(\frac{1}{1-\gamma}G^{-1}\left(\frac{1-q\theta e^{-\rho h}}{1-q}\right)\right)$

For proving the existence theorem we essentially use threshold strategies in optimal investment timing problems.

Let us consider the following investment timing problem (optimal stopping with linear payoff function):

$$\mathbf{E}^x (X_\tau - c) e^{-\rho\tau} \to \sup, \qquad (*)$$

over all stopping times.

Theorem

Threshold stopping time $\tau_{p^*} = \inf\{t \geq 0 : X_t \geq p^*\}$ is optimal for the problem (*) if and only if the following conditions hold:

$$\frac{p-c}{\psi(p)} \leq \frac{p^*-c}{\psi(p^*)} \quad \text{for } p < p^*,$$

$$\psi'(p^*)(p^*-c) = \psi(p^*);$$

$$\alpha(p) \leq \rho(p-c) \quad \text{for } p > p^*,$$

where $\alpha(p)$ is the drift function of the process X_t.

The case of geometric Brownian motion

Let present value of the implemented project be described by geometric Brownian motion (GBM) with parameters (α, σ):

$$dX_t = X_t(\alpha dt + \sigma dw_t), \ \alpha < \rho.$$

For example, if the profit flow from the implemented project π_t is GBM(α, σ), then

$$X_t = \mathbf{E}\left(\int_{t+h}^{\infty} \pi_s e^{-\rho(s-t)} ds \middle| \mathcal{F}_t \right), \quad \text{где } \mathcal{F}_t = \sigma\{X_s, \ 0 \le s \le t\},$$

is also GBM(α, σ).

In this case $\psi(x) = x^{\beta}, \ \beta > 1$, where β is a positive root of the quadratic equation

$$\tfrac{1}{2}\sigma^2\beta(\beta - 1) + \alpha\beta - \rho = 0.$$

The conditions **(A)–(C)** hold with

$$F(y) = G(y) = \frac{\beta - 1}{\beta} y.$$

Optimal investment moment

$$\tau^*(k) = \min\{t \geq 0 : X_t \geq p^*(k)I\},$$

where $p^*(k) = F^{-1}\left(\frac{k}{1-\gamma}\right) = \frac{\beta}{\beta-1} \cdot \frac{k}{1-\gamma}$
is the optimal threshold for investment.

Optimal credit policy of a bank

Optimal credit burden on the project:

$$k^*(\theta) = \max\{k_0, \frac{\beta}{\beta - 1} \cdot \frac{1 - q\theta e^{-\rho h}}{1 - q}\}.$$

Optimal interest rate (as a function of the part of loan reimbursement):

$$\lambda^*(\theta) = \left(k^*(\theta) - M\right)/\mu,$$

where $\mu = \left(\rho h e^{-\rho h} + 1 - M\right)/\rho, \quad M = \int_h^L f_t e^{-\rho t} dt.$

Optimal part of loan reimbursement

Let denote $\widehat{\theta} = \dfrac{\beta^2\tilde{\gamma} - (\beta-1)}{\beta^2\tilde{\gamma} + (\beta-1)^2} \cdot \dfrac{e^{\rho h}}{q}$, where $\tilde{\gamma} = \dfrac{\gamma}{1-\gamma}$.

Then the optimal part of loan reimbursement is as follows:

$$\theta^* = \begin{cases} \underline{\theta}, & \text{if } \ \widehat{\theta} < \underline{\theta} \\ \widehat{\theta}, & \text{if } \ \underline{\theta} \leqslant \widehat{\theta} \leqslant \overline{\theta} \\ \overline{\theta}, & \text{if } \ \widehat{\theta} > \overline{\theta} \end{cases} .$$

If $\theta^* > \underline{\theta}$, then in the domain of "matching interests" $\{\underline{\theta} \leqslant \theta \leqslant \theta^*\}$ all objective functions:

– budgetary effect $B(\theta, \tau^*(k^*(\theta)))$,
– bank profit from crediting the project $C(k^*(\theta), \theta, \tau^*(k^*(\theta)))$,
– NPV of investor $N(\tau^*(k^*(\theta)))$

increase in the part of loan reimbursement θ.

If the parameters of the model are such that the optimal part of loan reimbursement exceeds minimal acceptable value, then increase in the part of loan reimbursement (in certain limits) is profitable to **all participants** (the state, the bank and the investor). It differs from common opinion that benefits for investors lead to losses of budget.

If $\underline{\theta} = 0$, then the domain of matching interests exists for the set of parameters of the model $\{\gamma > \frac{\beta-1}{\beta^2+\beta-1}\}$. In particular, if $\beta \geqslant 2$ and $\gamma \geqslant 0.2$, then the domain of matching interests exists for *any* parameters of the model.

Dependence of the optimal part of loan reimbursement on parameters

- The part of loan reimbursement $\widehat{\theta}$ declines inversely proportional to the risk q.
- θ^* does not decrease with growth of tax burden on the project γ.
- The dependence of θ^* on volatility σ of the project is determined by the value of the tax burden γ. Namely:

  – if $\gamma < \frac{(\beta-1)^2}{\beta^2+(\beta-1)^2}$, then θ^* does not increase in σ;

  – if $\gamma \geqslant \frac{(\beta-1)^2}{\beta^2+(\beta-1)^2}$, then θ^* does not decrease in σ.

Dependence of the optimal credit policy of a bank on parameters

- The optimal credit burden on project $k^*(\theta)$ (and also the optimal interest rate in credit $\lambda^*(\theta)$) linearly declines in the part of loan reimbursement θ.
- The optimal credit burden on project $k^*(\theta)$ increases in risk q.
- The optimal credit burden on project $k^*(\theta)$ increases in volatility of the project σ.

Dependence of the budgetary effect on parameters. I

- The additional expected tax revenues to the budget, induced by the mechanism of the government loan guarantees (with part of loan reimbursement $\theta \le \theta^*$) exceeds the expected government costs for supporting these guarantees $q\mathbf{E}\theta I e^{-\rho\tau^*(k^*(\theta))}$.
- The optimal budgetary effect $B(\theta^*, \tau^*(k^*(\theta^*)))$ decreases in risk q.
- The budgetary "efficiency" of the optimal оптимальных guarantees $\mathcal{E} = \dfrac{B(\theta^*, \tau^*(k^*(\theta^*)))}{B(0, \tau^*(k^*(0)))}$ does not depend on risk q whenever $\theta^* < \overline{\theta}$.

  If $\theta^* = \overline{\theta}$, then $\mathcal{E}$ increases in q for $q < q_0$ and decreases in q for $q \geqslant q_0$,

  where $q_0 = \dfrac{\beta^2\tilde{\gamma} - \beta + 1}{\beta^2\tilde{\gamma} + (\beta-1)^2} e^{\rho h}, \quad \tilde{\gamma} = \dfrac{\gamma}{1-\gamma}$.

Dependence of the budgetary effect on parameters. II

- The dependence of the budgetary effect $B(\theta, \tau^*(k^*(\theta)))$ on tax burden γ has the Laffer type, i.e. increasing for $\gamma < \gamma_0$ and decreasing for $\gamma > \gamma_0$, where $\gamma_0 = \frac{1}{\beta} + \frac{1}{b} \cdot \frac{\tilde{q}\theta}{b^2(1-\tilde{q}\theta) + \tilde{q}\theta}$, $\quad b = \frac{\beta}{\beta - 1}$, $\quad \tilde{q} = qe^{-\rho h}$.

- With the "optimal" part of loan reimbursement $\widehat{\theta} = \widehat{\theta}(\gamma)$ the budgetary effect $B(\widehat{\theta}, \tau^*(k^*(\widehat{\theta})))$ increases in γ.

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Thank you for your attention!