On a connection between superhedging prices and the dual problem in utility maximization

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Outline

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General incomplete semimartingale model

Stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathrm{P})$
S is a semimartingale satisfying the (NFLVR)

$$\mathscr{X} = \{X = 1 + H \cdot S \ge 0\colon H \text{ is predictable and } S\text{-integrable}\}$$

is the set of all nonegative wealth processes of an investor with initial wealth 1
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Utility function

In this paper we always assume that a utility function U is as follows:

$U\colon \mathbb{R} \to [-\infty, +\infty)$ is a concave function, $U(x) \equiv -\infty$ on $(-\infty, 0)$ and $U(x) \in \mathbb{R}$ on $(0, \infty)$, and U is strictly increasing on $(0, \infty)$. No other assumptions on U are imposed. As usual, introduce the conjugate V of U defined by

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Utility maximization problem

The problem of maximizing the expected utility of terminal wealth is to maximize $\mathsf{E}U(X_T)$ over $X \in \mathscr{X}(x)$. The value function is defined by

$$u(x) = \sup_{X \in \mathscr{X}(x)} \mathsf{E}U(X_T).$$

Dual value function

As a concave increasing function, u can be represented as

$$u(x) = \min_{y \geqslant 0}[v(y) + xy], \quad x > 0,$$

where $v(y)$, $y \geq 0$, is a convex decreasing function,

$$v(y) = \sup_{x > 0}[u(x) - xy], \quad y \geqslant 0.$$

Characterization of the dual value function

Kramkov and Schachermayer (1999) proved that v is a solution of the following dual optimization problem:

$$v(y) = \inf_{Y \in \mathscr{Y}} \mathsf{E} V\big(yY_T\big), \quad y \geq 0.$$

where $\mathscr{Y}$ is the class of supermartingale deflators, i.e. nonnegative supermartingales Y with $Y_0 = 1$ such that XY is a supermartingale for every $X \in \mathscr{X}$.

The first step of the proof

Put

$$\mathscr{A} = \{X_T \colon X \in \mathscr{X}\},$$

$$\mathscr{D} = \{\eta \in L^0_+ \colon \mathsf{E}\eta\xi \leq 1 \ \forall \xi \in \mathscr{A}\}.$$

Evidently,

$$\mathscr{D} \supseteq \{Y_T \colon Y \in \mathscr{Y}\}.$$

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An analysis of the proof shows that this statement is based on the following representation for the superhedging price proved in Delbaen and Schachermayer (1998): for any nonnegative random variable B (interpreted as a contingent claim),

$$\pi(B) = \sup_{Q \in \mathscr{M}^e_\sigma} \mathrm{E}_Q B,$$

where $\mathscr{M}^e_\sigma$ is the set of all equivalent (to P) probability measures such that S is a σ-martingale with respect to Q, and $\pi(B)$ is the superhedging price of B:

$$\begin{aligned}\pi(B) &= \inf\{x > 0\colon \exists X \in \mathscr{X}(x) \text{ such that } B \le X_T\} \\ &= \inf\{x > 0\colon \exists \xi \in \mathscr{A} \text{ such that } B \le x\xi\}\end{aligned}$$

(here and below $\inf \varnothing = +\infty$).

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Market model

Let $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability space. Denote by L^0 the space of all (equivalence classes of) real-valued random variables. L^0 is equipped with the convergence in probability, and bar means the closure with respect to this convergence. L^0_+ is the cone in L^0 consisting of nonnegative random variables.

We consider an abstract market model described as a quadruple $(\Omega, \mathscr{F}, \mathsf{P}, \mathscr{A})$. where $\mathscr{A}$ is a convex subset of L^0_+. It is assumed also that $\mathscr{A}$ contains a random variable ξ such that $\mathsf{P}(\xi \geq \varkappa) = 1$ for some $\varkappa > 0$. $\mathscr{A}$ is interpreted as the set of terminal wealths of an investor corresponding to all her strategies with initial wealth 1. If the initial wealth is $x > 0$, then the corresponding set of terminal wealths is $x\mathscr{A}$.

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Superhedging prices

Put

$$\mathscr{A}_0 = (\mathscr{A} - L^0_+) \cap L^0_+, \qquad \mathscr{C} = \overline{\mathscr{A}_0}.$$

Let $B \in L^0_+$. A possible definition of the superhedging price of B is

$$\begin{aligned}\pi(B) &= \inf\{x > 0\colon \text{there is a } \xi \in \mathscr{A} \text{ such that } B \le x\xi\} \\ &= \inf\{x > 0\colon B \in x\mathscr{A}_0\}.\end{aligned}$$

Since we do not assume any kind of closedness of $\mathscr{A}$ here, an alternative (and more natural) definition of the superhedging price is

$$\pi_*(B) = \inf\{x > 0\colon B \in x\mathscr{C}\}.$$

Obviously, $\pi_*(B) \le \pi(B)$ for all $B \in L^0_+$.

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When two definitions coincide

It is easy to check that

$$\pi(B) = \pi_*(B) \text{ for all } B \in L^0_+ \qquad \Longleftrightarrow \qquad \mathscr{C} \subseteq \bigcap_{\lambda>1} (\lambda \mathscr{A}_0).$$

Polar description

Put

$$\mathscr{D} = \{\eta \in L^0_+ \colon \mathrm{E}\eta\xi \leq 1 \text{ for all } \xi \in \mathscr{A}\}.$$

Since $\mathscr{D}$ is bounded in P-probability and closed in L^0, every element in $\mathscr{D}$ is majorized by a maximal element of this set.

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Utility naximization problem

Let U be a utility function as above.
For a probability measure $\mathrm{Q} \ll \mathrm{P}$ define the value function $u_{\mathrm{Q}}(x)$, $x > 0$, in the utility maximization problem relative to Q:

$$u_{\mathrm{Q}}(x) = \sup_{\xi \in x\mathscr{A}} \mathrm{E}_{\mathrm{Q}} U(\xi).$$

Dual problem

The dual minimization problem is defined by

$$v_{\mathsf{Q}}(y) = \inf_{\eta \in \mathscr{D}} \mathsf{E}_{\mathsf{Q}} V\left(\frac{y\eta}{d\mathsf{Q}/d\mathsf{P}}\right).$$

Dual relations

As is shown in Kramkov and Schachermayer (1999), see also Gushchin (2011), the following dual relations hold:

$$u_Q(x) = \min_{y \geq 0} [v_Q(y) + xy], \quad x > 0,$$

$$v_Q(y) = \sup_{x > 0} [u_Q(x) - xy], \quad y \geq 0.$$

Superhedging prices and maximal elements

Let $\mathscr{W}$ be a nonempty convex subset of $\mathscr{D}$. Then

$$\pi_*(B) = \sup_{\eta \in \mathscr{W}} \mathsf{E}\eta B$$

for all $B \in L^0_+$ if and only if $\overline{\mathscr{W}}$ contains all maximal elements from $\mathscr{D}$.

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Theorem

Let $\mathscr{W}$ be a nonempty convex subset of $\mathscr{D}$.
(i) *Assume that for a given utility function U, for all $\mathsf{Q} \ll \mathsf{P}$ and $y \geq 0$,*

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Then

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(ii) *Let (1) be satisfied. Then*

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Thank you!