Nonlocal Artificial Boundary Conditions for external problems & near-wall turbulence modelling

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1. Introduction to Calderón-Ryaben'kii's Potentials

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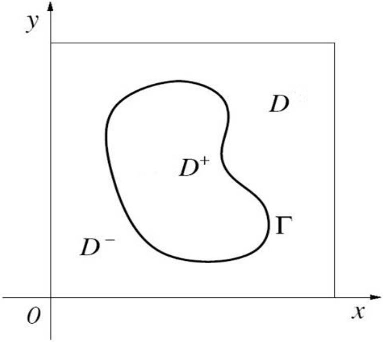
2. Artificial Boundary Conditions

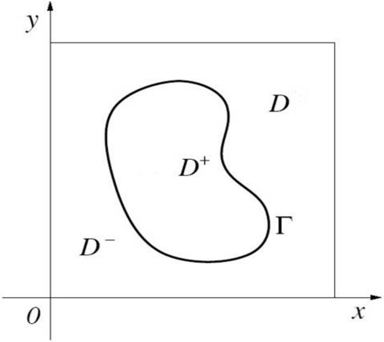


$$D := D^+ \cup \Gamma \cup D^-,$$

where Γ is a Lipschitz boundary

Consider solution of
a well-posed linear BVP:

$$LU = f,$$
$$U \in \Xi_D,$$
$$D \subseteq \mathbb{R}^m, U \in \mathbb{R}^p, f \in \mathbb{R}^p.$$



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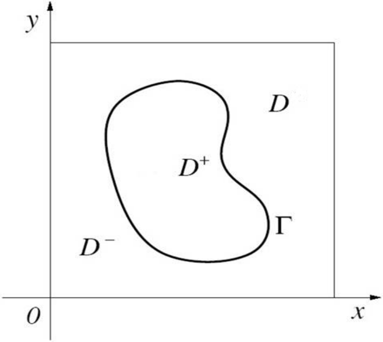
$$\Xi_D \subset H^s(D^+) \cap H^s_0(D^-), \ s > k - 1/2,$$
$$f \in H^{s-k}_{loc}(D^+) \cap H^{s-k}_{loc}(D^-),$$

k is the order of operator L.



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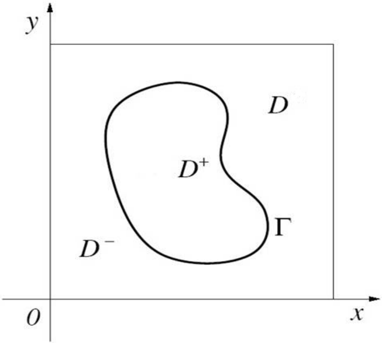
SU, Generalized Calderón-Ryabenkii's Potentials, IMA J.Appl.Math., 2009



$$LU = f,$$
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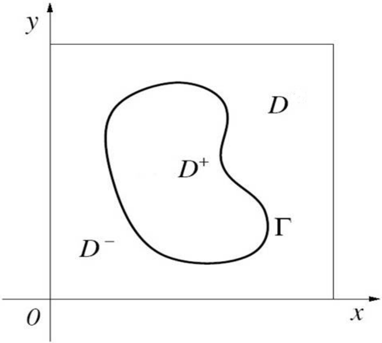
$$P_{D^+} V_{D^+} = V_{D^+} - \int_{D^+} G(\mathbf{x}, \mathbf{y}) LV(\mathbf{y}) d\mathbf{y},$$
$$V \in \Xi_D, \; \mathbf{x} \in D^+.$$



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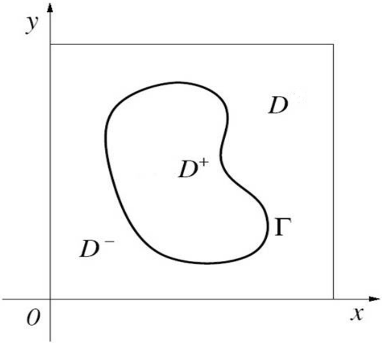
CR potential is a projection:



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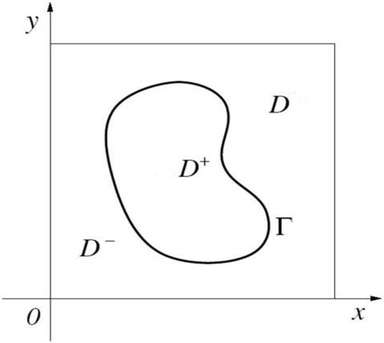
$$V \in \Xi_D, \ \mathbf{x} \in D^+.$$

CR potential is a <u>projection</u>:

If $\operatorname{supp} LV \subset D^+$, then $P_{D^+} V_{D^+} = V_{D^+} - V_{D^+} = 0_{D^+}$;

if $\operatorname{supp} LV \subset D^-$, then $P_{D^+} V_{D^+} = V_{D^+} - 0_{D^+} = V_{D^+}$.



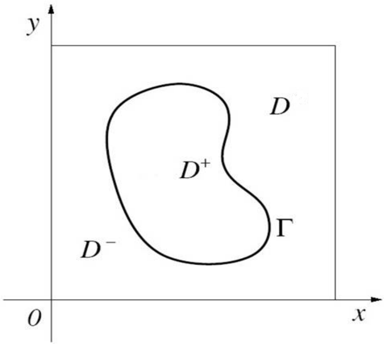
Consider $\mathrm{Tr}_\Gamma : H^s(D^+) \to H^{s-1/2}(\Gamma)$

$$\mathrm{Tr}_\Gamma\, U_{D^+} \overset{def}{=} \lim_{\epsilon \to 0} \mathrm{Tr}_{\Gamma_\epsilon}\, U_{D^+},$$

where

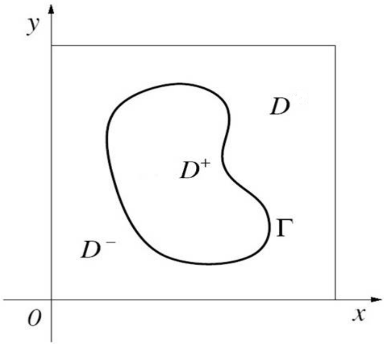
$$\mathrm{Tr}_{\Gamma_\epsilon}\, U_{D^+} \overset{def}{=} U_{D^+}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_\epsilon,$$

$\Gamma_\epsilon \subset D^+$, $\Gamma_\epsilon \to \Gamma$ if $\epsilon \to 0$.



Consider Tr_{Γ}^{+} :

$$\Xi_{D^+} \to \Xi_{\Gamma} \subset \oplus_0^{k-1} H^{s-1/2-j}(\Gamma).$$



Consider Tr_Γ^+ :

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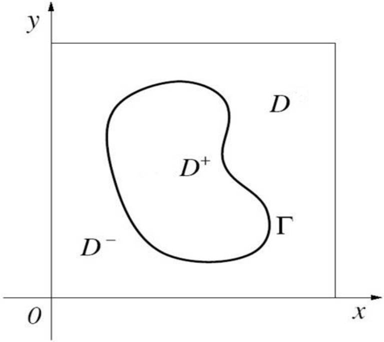
Clear trace: $(\mathrm{Tr}_\Gamma^+, \Xi_\Gamma)$ iff

$$\mathrm{Tr}_\Gamma^+ V_{D^+} = \mathrm{Tr}_\Gamma^+ W_{D^+} \Rightarrow$$
$$P_{D^+} V_{D^+} = P_{D^+} W_{D^+},$$
$$V, W \in \Xi_D.$$



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Then, $P_{D^+} U_{D^+} = P_{D^+\Gamma} \mathrm{Tr}_\Gamma^+ U_{D^+}$

Consider BVP for second-order equation:

$$Lu \equiv \nabla(p\nabla u) + qu = f,$$
$$u \in \Xi_D,$$
where $p \in C^1(D), q \in C(\overline{D}^0), f \in L^1_{loc}(D).$

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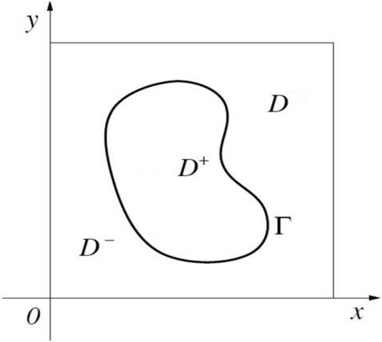
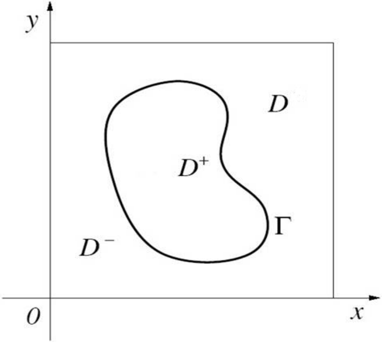
Clear trace

$$\mathrm{Tr}^+_\Gamma v = \begin{bmatrix} v \\ \frac{\partial v}{\partial n} \end{bmatrix}_\Gamma .$$



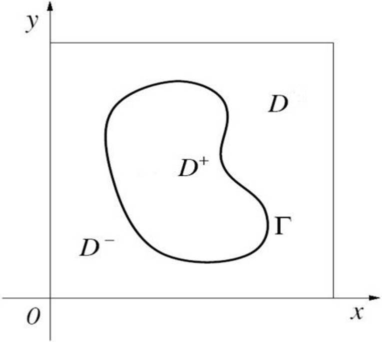
Generalised Green's formula:



Generalised Green's formula:

$$P_{D^+}V_{D^+} = V_{D^+} - \int_{D^+} G(\mathbf{x},\mathbf{y})LV(\mathbf{y})d\mathbf{y},$$

$$V \in \Xi_D, \ \mathbf{x} \in D^+.$$



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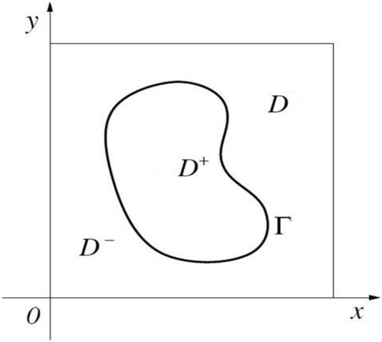
Hence,

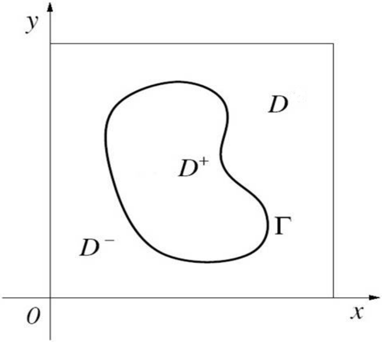
$$V_{D^+} = P_{D^+\Gamma} \operatorname{Tr}_\Gamma^+ V_{D^+} + G_{D^+} f_{D^+}.$$



$$L(u) = f,$$
$$u \in \Xi_D,$$
$$L(u) = 0 \Leftrightarrow u = 0.$$

Nonlinear Potential:

$$P_{D^+}(v_{D^+}) = L_{D^+}^{-1}(L(v) - \theta_{D^+} L(v)),$$
$$v \in \Xi_D.$$



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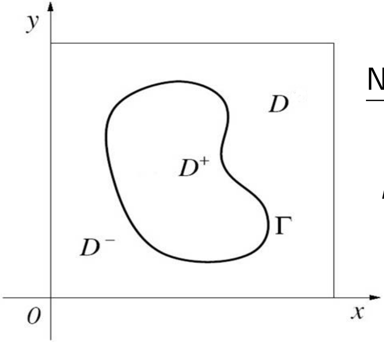
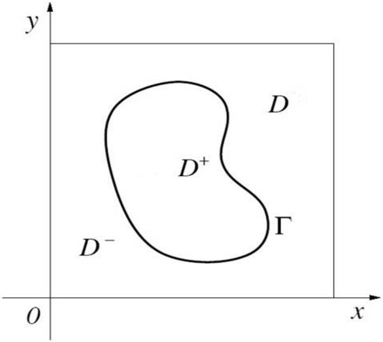
If $\operatorname{supp} L(v) \subset D^+$, then $P_{D^+}(v_{D^+}) = 0_{D^+}$,
if $\operatorname{supp} L(v) \subset D^-$, then $P_{D^+}(v_{D^+}) = v_{D^+}$.

SU, J. Computat. & Appl. Math. (2010)



Nonlinear Green's Identity

$$P_{D^+\Gamma}\,\mathrm{Tr}_\Gamma^+(G_{D^+}(f)) = G_{D^+}(f - f_{D^+}).$$

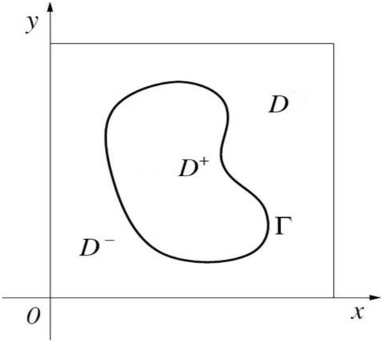


Consider BVP:

$$LU = f,$$
$$U \in \Xi_D.$$

Introduce

$$\xi_\Gamma = \mathrm{Tr}_\Gamma^- V_{D^-},$$
$$P_{D^-\Gamma}\xi_\Gamma = P_{D^-} V_{D^-},$$
$$V \in \Xi_D.$$



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$$V \in \Xi_D.$$

Boundary equation on Γ:

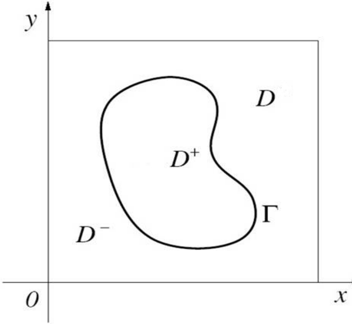
$$\xi_\Gamma = \mathrm{Tr}_\Gamma^- P_{D^-\Gamma}\xi_\Gamma + \mathrm{Tr}_\Gamma^- G_{D^-} f_{D^-}.$$

Original BVP:

$$LU = f \quad (\operatorname{supp} f \subset D^+),$$
$$U \in \Xi_D.$$

Artificial boundary condition:

$$\xi_\Gamma = \mathrm{Tr}_\Gamma^- P_{D^-\Gamma}\xi_\Gamma.$$



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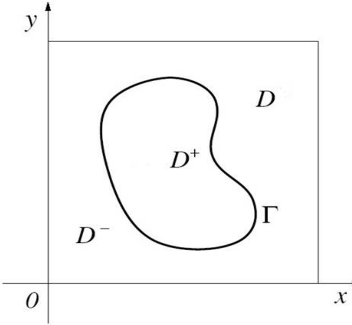

Reduced BVP on D^+:

$$L_{D^+} U_{D^+} = f_{D^+},$$
$$\mathrm{Tr}_\Gamma^+ U_{D^+} = \xi_\Gamma.$$

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$$LU = f \quad (\operatorname{supp} f \subset D^{+}),$$
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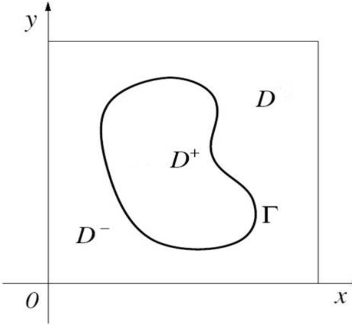
Artificial boundary condition:

$$\xi_\Gamma = \mathrm{Tr}_\Gamma^- P_{D^-\Gamma} \xi_\Gamma.$$

ABC:



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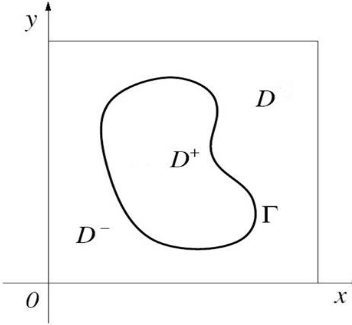

ABC:

- nonlocal

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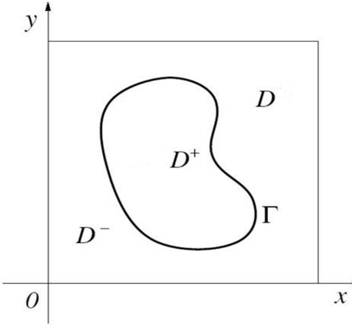


Artificial boundary condition:

$$\xi_\Gamma = \mathrm{Tr}_\Gamma^- P_{D^-\Gamma}\xi_\Gamma.$$

ABC:

- nonlocal
- asymptotically exact

Nonstationary Problem

Consider IBVP in $D \times (0, T)$:

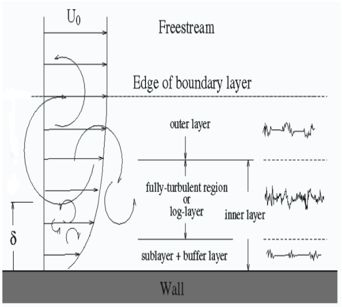
$$U_t = -LU + f,$$
$$U \in \Xi_D,$$
$$U(\mathbf{x}, 0) = 0.$$

Definition

$$P_{D^+} V_{D^+}(\mathbf{x}, t) = V_{D^+} - \int_T \int_{D^+} G(\mathbf{x}|\mathbf{y}, t|\tau) LV(\mathbf{y}, \tau) d\mathbf{y} d\tau.$$

SU, Adv. Appl. Math. (2009)

The turbulent boundary layer



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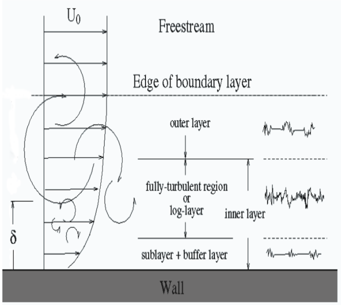
- near-wall sublayer significantly affects mean flow
- resolution of near-wall area requires up to 90% of CPU time
- standard approach: another solution is set at $y = \delta$ (wall functions)
- limited applications

The turbulent boundary layer



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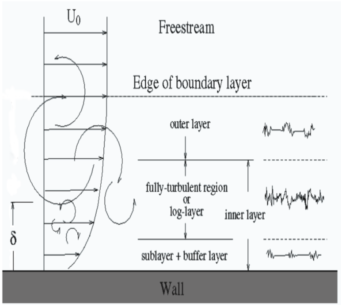
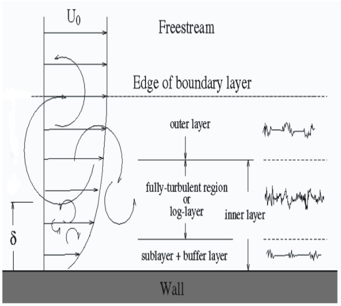


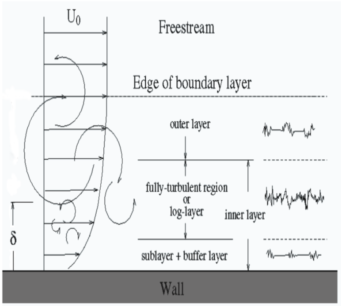
- the b.c can be transferred from the wall to $y = \delta$

The turbulent boundary layer



- the b.c can be transferred from the wall to $y = \delta$
- the interface near-wall b.c. (INBC) is nonlocal

The turbulent boundary layer



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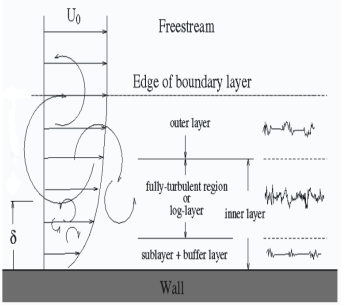
- the b.c can be transferred from the wall to $y=\delta$
- the interface near-wall b.c. (INBC) is nonlocal
- general formulation of b.c.

$$\frac{\partial u}{\partial n}_{|\delta} = u(\delta)S_\delta(1) + f_\delta,$$

S_δ is the Steklov-Poincaré operator

The turbulent boundary layer



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- general formulation of b.c.

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S_δ is the Steklov-Poincaré operator

SU, Computers & Fluids (2009)

Model 1D Equation

Consider equation

$$(\mu u_y)_y = R(y)$$

in $D = [0\ y_e]$ with boundary condition at $y = 0$:

$$u(0) = u_0.$$

INBC is set at δ

$$0 < \delta < y_e,\ \ D^- := [0\ \delta].$$

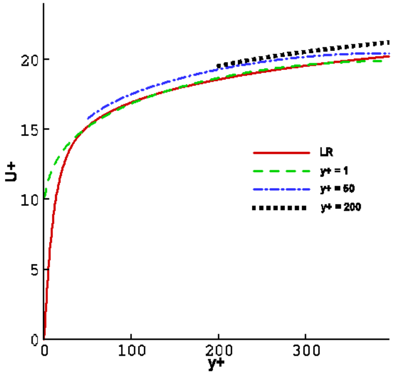
Consider

$$(\mu u_y)_y = R(y),$$
$$u(0) = u_0.$$

INBC at $y=\delta$:

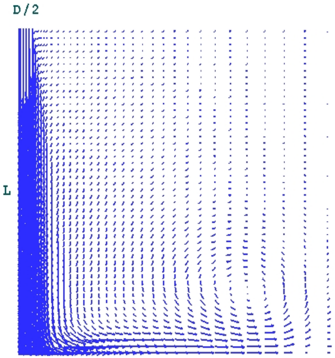
$$u(\delta) = u_0 + u'(\delta)\int_0^\delta \frac{\mu(\delta)}{\mu(y)}dy - \frac{1}{\mu(\delta)\delta}\int_0^\delta (\frac{\mu(\delta)}{\mu(y)}\int_y^\delta R(y')dy')dy.$$

Velocity Profile. $Re = 395$



Solid line is Reichardt's profile;
$y^{+*} = u_\tau \delta / \nu = 1, 50, 200$; u_τ is friction velocity

Input data

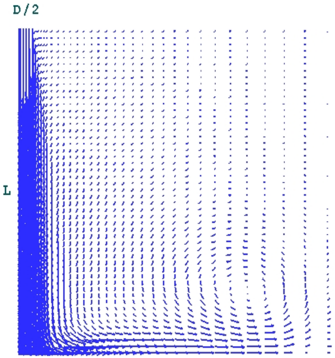


$$L/D = 2,\ 4,\ 6,\ 10,\ 14$$

$$Re = 23000,\ 70000.$$

SU, J. Appl. Numer. Math. (2008)

Input data



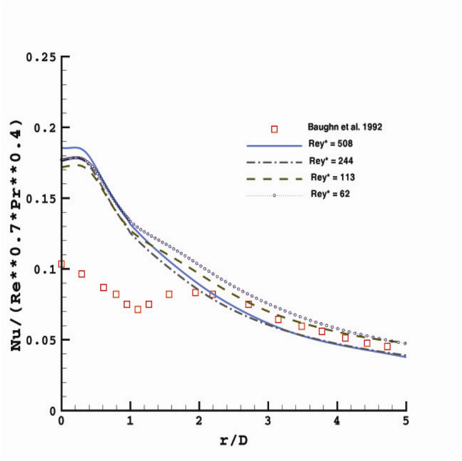
$L/D = 2, 4, 6, 10, 14$

$Re = 23000, 70000.$

SU, J. Appl. Numer. Math. (2008)

- INBC: locally 1D

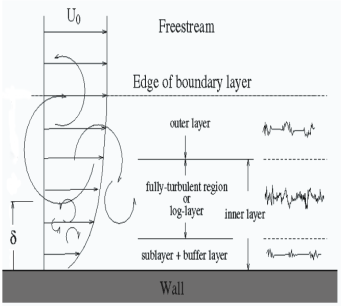
- no free parameters



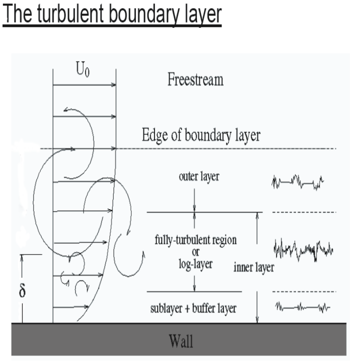
Local scaled Nusselt number ($L/D = 0.4$, $Re = 70000$).
$Re_\delta \equiv \rho\sqrt{k^*}\delta/\mu_l = 62, 113, 244, 508.$

Viscosity profile in the near-wall domain

The turbulent boundary layer



Viscosity profile in the near-wall domain



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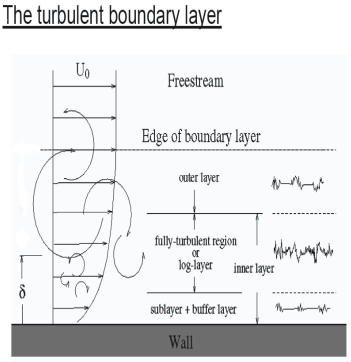
1. Piece-wise linear profile (identical to AWFs)

Viscosity profile in the near-wall domain

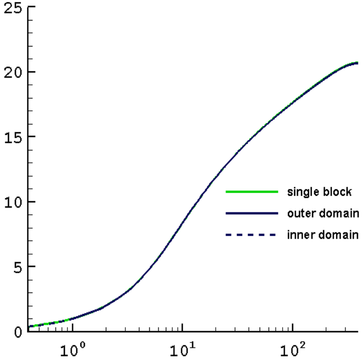


1. Piece-wise linear profile (identical to AWFs)
2. Nonlinear approximation (Cabot-Moin, 1999):

$$\nu_t = \nu \kappa y^+ (1 - \exp(-y^+/A))^2,$$

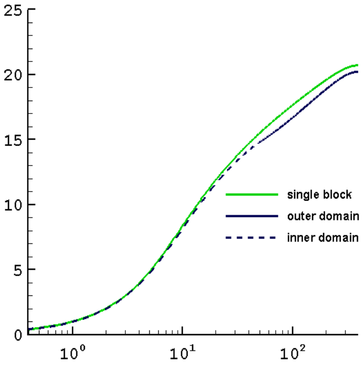
where $\kappa = 0.41$, $A = 19$.

Low-Re Velocity Profile. $Re = 395$



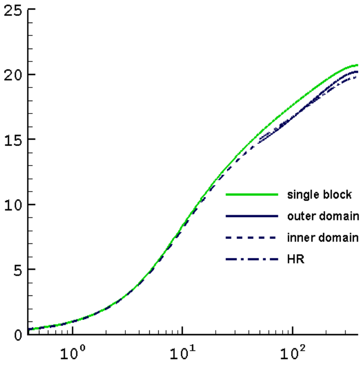
$y^{+*} = u_\tau \delta/\nu = 1$; u_τ is friction velocity

Low-Re Velocity Profile. $Re = 395$



$$y^{+*} = u_\tau \delta / \nu = 50$$

Low-Re Velocity Profile. $Re = 395$



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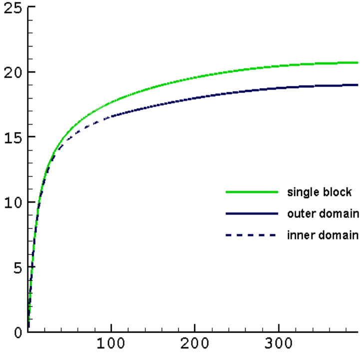
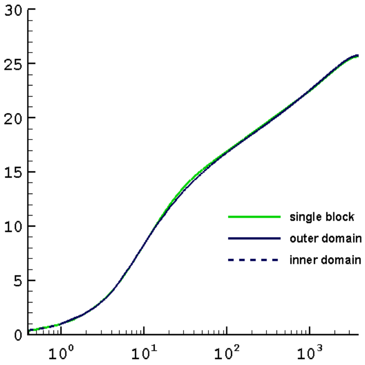


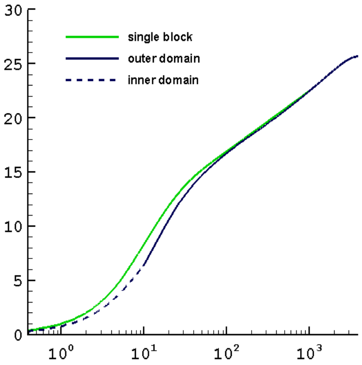
$$y^{+*} = u_\tau \delta/\nu = 100$$

Low-Re Velocity Profile. $Re = 3950$



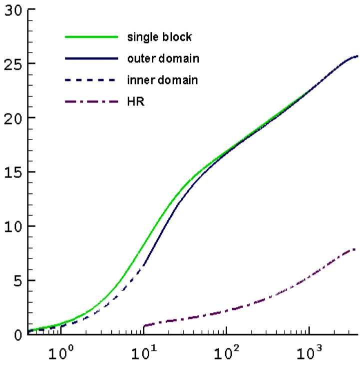
$$y^{+*} = u_\tau \delta / \nu = 1$$

Low-Re Velocity Profile. $Re = 3950$



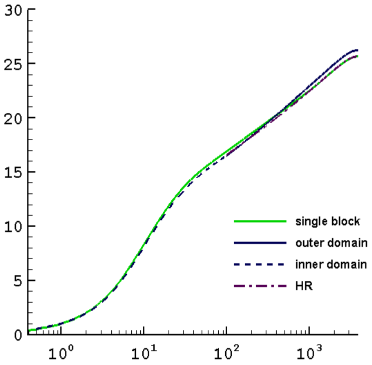
$$y^{+*} = u_\tau \delta / \nu = 10$$

Low-Re Velocity Profile. $Re = 3950$



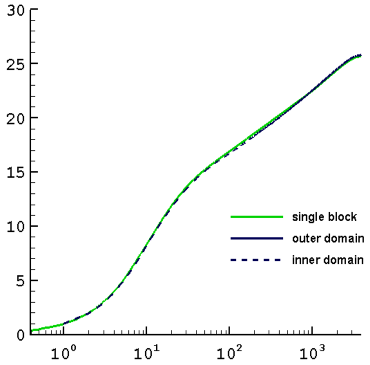
$$y^{+*} = u_\tau \delta / \nu = 10$$

Low-Re Velocity Profile. $Re = 3950$



$$y^{+*} = u_\tau \delta / \nu = 100$$

Low-Re Velocity Profile. $Re = 3950$



$$y^{+*} = u_\tau \delta / \nu = 200$$

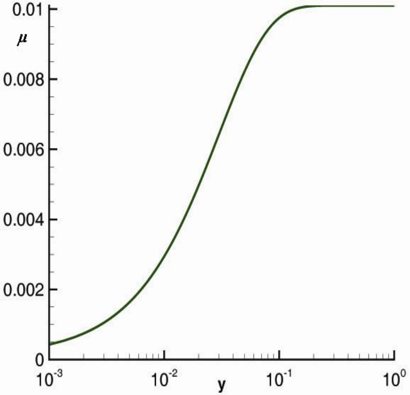
Consider BVP for second-order equation:

$$(\mu(y)u_y)_y + \alpha\mu(y)u_{xx} + \beta(y)u_y + \gamma(y)u = f(x,y),$$

$$l_y u(x,0) = \alpha_w(x),$$
$$u(0,y) = u_0(y),$$
$$u(1,y) = u_1(y),$$
$$u(x,1) = u_s(x),$$

where $\alpha > 0$, $\mu = (1 - \exp(-y/\epsilon) + \delta_0)/Re$, $\epsilon \ll 1, \delta_0 \ll 1, Re \gg 1$, $\beta = Cy^p \; > \; 0$, $p > 0$.

Profile of μ



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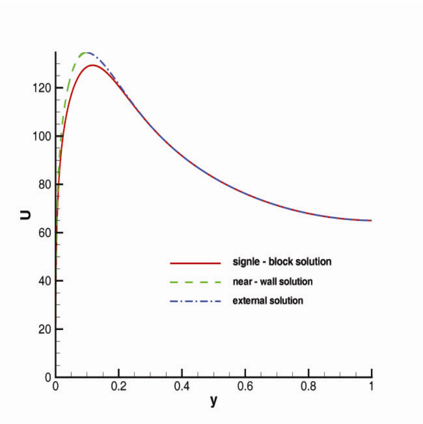
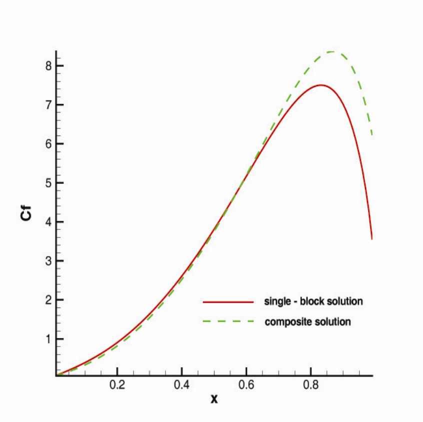
1D INBCs. $\alpha = 1,\ \delta = 0.1,\ Re_\delta = 1.2 * 10^3$



Profile of u at $x = 0.8$



$C_f = \mu \frac{du}{dy}(x,0)$

Consider BVP for second-order equation:

$$(\mu(y)u_y)_y + \alpha\mu(y)u_{xx} + \beta(y)u_y + \gamma(y)u = f(x,y),$$

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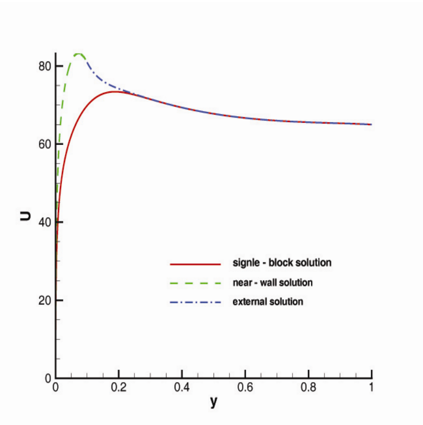
$\beta = Cy^p \ > \ 0$, $p > 0$.

1D INBCs. $\alpha = 10, \ \delta = 0.1, \ Re_\delta = 0.7 * 10^3$



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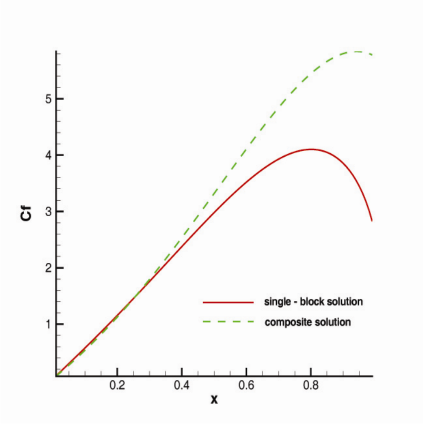
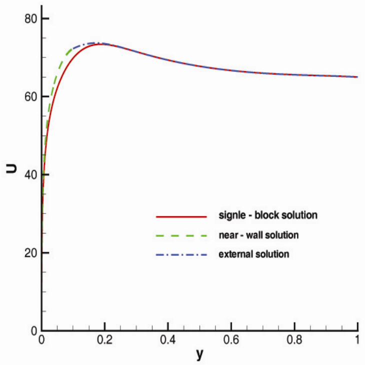


$C_f = \mu \frac{du}{dy}(x, 0)$

Nonlocal Boundary Conditions. $\alpha = 10, \ \delta = 0.1$



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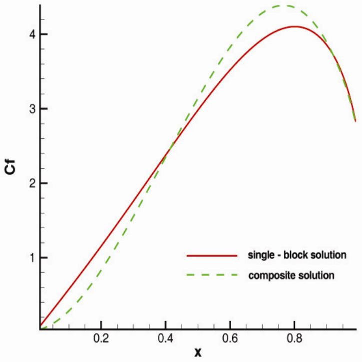

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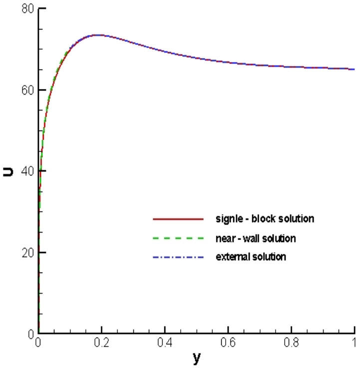
Nonlocal INBC. $\alpha = 10, \ \delta = 0.1$



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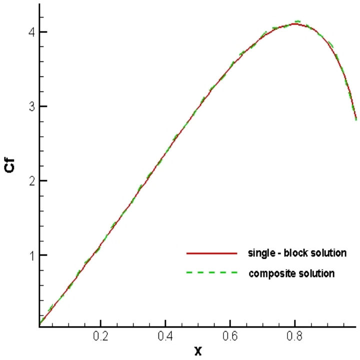
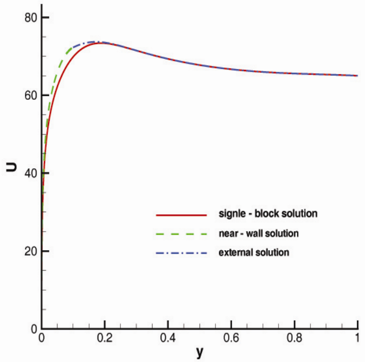


$C_f = \mu \frac{du}{dy}(x, 0)$

INBC: $\frac{\partial u}{\partial n}_{|\delta} = u(\delta)S_\delta(1) + f_\delta$. $\alpha = 10, \;\; \delta = 0.1$



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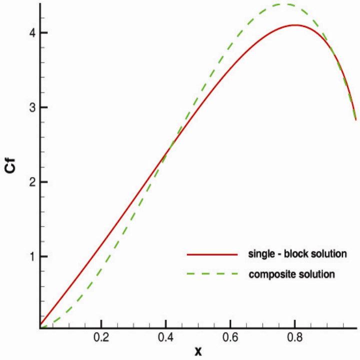

$C_f = \mu \frac{du}{dy}(x, 0)$

Consider BVP for second-order equation:

$$u_t = (\mu(y)u_y)_y + \beta(y)u_y + \gamma(y)u - f(y),$$

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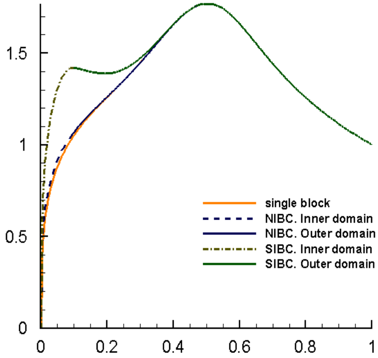
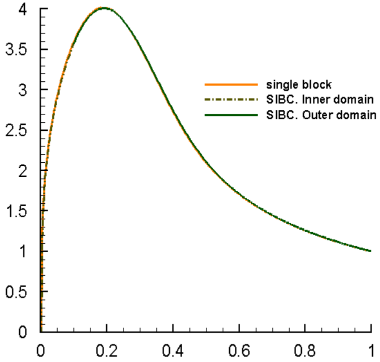
Nonstationary IBCs



Non-stationary vs stationary IBCs. $t = 1$

Nonstationary IBCs



Non-stationary vs stationary IBCs. $t = 10$

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  - can be calculated in advance
- Nonlocal INBC can be important for complex geometries
- Nonstationary INBC have been obtained

Literature

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