Equivariant cohomology and syzygies

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(mostly joint work with Chris Allday and Volker Puppe)

Equivariant cohomology

$T = (S^1)^r$ compact torus, X smooth T-manifold, $\dim H^*(X) < \infty$.

$A = H^*(BT) = \mathbb{R}[t_1, \ldots, t_r]$ with $\deg(t_i) = 2$.

Equivariant cohomology: $H^*_T(X) = H^*(\Omega^*_T(X))$

Cartan model: $\Omega^*_T(X) = \Omega^*(X)^T \otimes A$ with twisted differential,

$$d(\gamma \otimes f) = d\gamma \otimes f + \sum_{i=1}^{r} \iota_{\xi_i}\gamma \otimes t_i f.$$

Here $\xi_1, \ldots, \xi_r$ are generating vector fields.

$H^*_T(X)$ is a f. g. A-module (even an A-algebra),
and $\Omega^*_T(X)$ is a dg-A module (even a dg-A algebra).

Two exact sequences

Chang–Skjelbred sequence (1974): $H^*_T(X)$ free $/A \implies$

$$0 \to H^*_T(X) \to H^*_T(X^T) \to H^{*+1}_T(X_1, X^T)$$

is exact, where $X_1 =$ union of orbits of dimension ≤ 1.

This is an efficient way to compute $H^*_T(X)$, in particular if X^T is finite and X_1 a union of 2-spheres ("GKM method" 1998).

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The CS / GKM method only uses a small part of this sequence!

Equivariant homology

(cf. Jones, Brylinski, Edidin–Graham, ...)

Defined via the A-dual of the Cartan model:

$$H_*^T(X) = H_*(\mathrm{Hom}_A(\Omega_T^*(X), A))$$

This is not the homology of the Borel construction!

Universal coefficient theorem:

$$E_2 = \mathrm{Ext}_A^*(H_*^T(X), A) \;\Rightarrow\; H_T^*(X)$$

Poincaré duality: X compact orientable $\implies$

$$H_T^*(X) \xrightarrow{\cap o_T} H_*^T(X) \quad \text{iso of } A\text{-modules}$$

where o_T is the equivariant lift of the orientation.

The cohomology of the AB sequence

$H^*(AB^*(X))$ = cohomology of the complex of A-modules

$$H^*_T(X_0) \to H^{*+1}_T(X_1, X_0) \to H^{*+2}_T(X_2, X_1) \to \cdots \to H^{*+r-1}_T(X_{r-1}, X_{r-2}) \to H^{*+r}_T(X_r, X_{r-1})$$

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$H^*(AB^*(X))$ = cohomology of the complex of A-modules

$$H_T^*(X_0) \to H_T^{*+1}(X_1, X_0) \to H_T^{*+2}(X_2, X_1) \to \cdots \to H_T^{*+r-1}(X_{r-1}, X_{r-2}) \to H_T^{*+r}(X_r, X_{r-1})$$

Theorem

$$H^i(AB^*(X)) = \mathrm{Ext}^i_A(H_*^T(X), A) \qquad \textit{for any } i \geq 0$$

Syzygies

M f. g. A-module

M ***j*-th syzygy**: $\exists\ F_1, \ldots, F_j$ f. g. free A-modules such that

$$0 \to M \to F_1 \to \cdots \to F_j \quad \text{exact}.$$

Syzygies interpolate between torsion-freeness and freeness:

$$\begin{aligned} \text{first syzygy} &= \text{torsion-free} \\ \text{second syzygy} &= \text{reflexive} \\ &\vdots \\ r\text{-th syzygy} &= \text{free} \\ (r+1)\text{-st syzygy} &= \text{free} \\ &\vdots \end{aligned}$$

Partial exactness

Theorem

Let $j \geq 0$. The AB sequence is exact at all positions $i \leq j - 2$ $\iff$ $H_T^*(X)$ is a j-th syzygy.

This includes Atiyah–Bredon's result and its converse.

Corollary

The CS sequence is exact $\iff$ $H_T^*(X)$ is reflexive

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Example

X_P toric manifold, $x \neq y \in X^T =$ vertices of P, $Y = X_P \setminus \{x, y\}$
Then $H_T^*(Y)$ is a syzygy of order exactly $\dim Q - 1$, where Q is the supporting face of v and v'

Consequences for Poincaré duality spaces

Corollary

The CS sequence is exact $\iff$ the equivariant Poincaré pairing

$$H_T^*(X) \times H_T^*(X) \to A, \quad (\alpha, \beta) \mapsto \langle \alpha \cup \beta, o_T \rangle$$

is perfect.

This answers a point raised by Guillemin–Ginzburg–Karshon.

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Corollary

If $H_T^*(X)$ *is a syzygy of order* $\geq r/2$, *then it is free over* R.

In other words:
$r \leq 2$: $H_T^*(X)$ torsion-free $\Rightarrow$ free (Allday 1985)
$r \leq 4$: $H_T^*(X)$ reflexive $\Rightarrow$ free
etc.

A geometric criterion for syzygies

Assumption

T-action on X effective with connected isotropy groups
(enough: locally effective action on locally orientable T-orbifold)

X_{r-1} = union of X^K for some "characteristic circles" $K \subset T$.

T-action is *regular* if $\operatorname{codim} X^K = 2$ for all such K.

Otherwise one can blow up the X^K to get a regular T-mf $\tilde{X}$.

Note: In general, $\tilde{X}$ is an orbifold, even if X is a manifold.

Proposition

$H_T^*(X)$ j-th syzygy $\iff$ $H_T^*(\tilde{X})$ j-th syzygy

So enough to consider regular T-actions.

A geometric criterion for syzygies: regular actions

T acts regularly $\Rightarrow$ X/T is a manifold with corners

For a face P of X/T, consider the complex

$$B^i(P) = \bigoplus_{\substack{Q \subset P \\ \text{rank } Q = i}} H_*(Q)$$

where rank Q = dimension of T-orbits over interior of Q.
The differential is induced by the inclusions of facets.

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Theorem

$H_T^*(X)$ j-th syzygy $\iff$
$H^i(B^*(P)) = 0$ *for all P and all $i > \max(\text{rank } P - j, 0)$*

This generalizes criteria for torsion-freeness and freeness due to Barthel–Brasselet–Fieseler–Kaup (2002), Masuda–Panov (2006), Masuda (2006), Goertsches–Rollenske (2011).

A special case

Additional assumption

X compact orientable and $\dim X = 2r$

(toric manifolds, quasi-toric manifolds, torus manifolds, ...)

Corollary

$H_T^*(X)$ torsion-free $\iff$ $H_T^*(X)$ free

A special case

Additional assumption

X compact orientable and $\dim X = 2r$

(toric manifolds, quasi-toric manifolds, torus manifolds, ...)

Corollary

$H_T^*(X)$ *torsion-free* $\iff$ $H_T^*(X)$ *free*

This is not true for $\dim X > 2r$.

Intersections of quadrics

Take $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ such that 0 is in their convex hull, but not on any line segment joining two λ_i's.

Define the compact orientable mf $X \subset \mathbb{C}^{r+2}$ by

$$\sum_{i=1}^{r} \lambda_i |z_i|^2 + u^2 + v^2 = 0,$$
$$\sum_{i=1}^{r} |z_i|^2 + |u|^2 + |v|^2 = 1.$$

$\dim X = 2r + 1$, $T = (S^1)^r$ acts by rotating the z_i's.

Proposition

$H_T^*(X)$ is torsion-free, but not reflexive (hence not free).

The proof uses the geometric criterion and recent results of Gómez Gutiérrez and López de Medrano.

References

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  arXiv:1111.0957, to appear in *Trans. AMS*
- M. Franz
  A geometric criterion for syzygies in equivariant cohomology
  arXiv:1205.4462
- C. Allday, M. Franz, V. Puppe
  Equivariant Poincaré–Alexander–Lefschetz duality and the Cohen–Macaulay property
  arXiv:1303.1146

References

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Thanks.