A Refined Riemann's singularity theorem for sigma functions

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I will talk about the theta function from the view point of integrable systems.

The study on Novikov's conjecture by Shiota, Mulase and others implies that any property on the theta function of an algebraic curve, if it is valid for any algebraic curve, can be understood from the properties of the KP-hierarchy.

In this talk I will take the Riemann's singularity theorem as one of such properties of the theta functions of algebraic curves. To consider this example is motivated by the study of multi-variate sigma functions.

The Riemann's singularity theorem is an excellent theorem which connects the geometry of Riemann surfaces to the analysis of the theta functions. So let me begin by reviewing it.

X: a compact Riemann suface of genus g,

$\{\alpha_i, \beta_i\} \to \{dv_i\} \to \Omega = \left(\int_{\beta_j} dv_i\right) \to$

$\to \theta(z, \Omega), \quad J(X) = \mathbb{C}/(\mathbb{Z}^g + \Omega\mathbb{Z}).$

$p_\infty \in X \to \delta$: Riemann's constant, Abel-Jacobi map

$$I\colon X \to J(X),$$
$$I(p) = \int_{p_\infty}^{p} dv.$$

We identify a degree zero divisor with its Abel-Jacobi image:

$$\sum (p_i - q_i) = \sum (I(p_i) - I(q_i)).$$

I particular, for the Riemann divisor Δ

$$\Delta - (g-1)p_\infty = \delta.$$

For $p_1, ..., p_n \in X$ set

- $L(p_1 + \cdots + p_n) := \{$meromorphic function on X whose pole divisor is bounded by $p_1 + \cdots + p_n$ $\}$,
- $d(p_1 + \cdots + p_n) :=$ dim $L(p_1 + \cdots + p_n)$.

For $\alpha = (\alpha_1, ..., \alpha_g)$ set

- $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_g^{\alpha_g}$, $\quad ||\alpha|| = \alpha_1 + \cdots + \alpha_g$.

  We call $||\alpha||$ the degree of ∂^α.

Riemann's vanishing theorem

$$\theta(z) = 0$$

$$\Leftrightarrow$$

$$z = p_1 + \cdots + p_{g-1} - \Delta \text{ for some } p_i \in X.$$

Riemann's singularity theorem

Suppose that $e = p_1 + \cdots + p_{g-1} - \Delta$ and $r = d(p_1 + \cdots + p_{g-1})$. Then the multiplicity of the zeros of $\theta(z)$ at e is r:

(1) $\partial^\alpha \theta(e) = 0$ for $||\alpha|| < r$.

(2) $\partial^\beta \theta(e) \neq 0$ for $||\beta|| = r$.

In the study of the sigma functions, Onishi('05) found such β explicitly in the case of hyperelliptic curves and $e = -\delta$.

Let me briefly recall Onishi's results.

Consider a hyperelliptic curve X defined by

$$y^2 = 4x^{2g+1} + \cdots$$

and take $p_\infty = \infty$.

The fundamental sigma function $\sigma(u)$ is a certain modification of the Riemann's theta function and is a function of

$$u = {}^t(u_1, u_3, \ldots, u_{2g-1}).$$

We remark that $u = 0$ corresponds to $z = -\delta$ in the Riemann's theta function $\theta(z)$.

The sigma function has a nice Taylor expansion at the origin of the form

$$\sigma(u) = s_\lambda(u) + \cdots ,$$

where $s_\lambda(u)$ is the Schur function corresponding to the partition $\lambda = (g, ..., 2, 1)$.

Examples of Schur functions

$$s_{(1)}(u) = u_1,$$

$$s_{(2,1)}(u) = -u_3 + \frac{1}{3}u_1^3,$$

$$s_{(3,2,1)}(u) = u_1 u_5 - u_3^2 - \frac{1}{3}u_1^3 u_3 + \frac{1}{45}u_1^6,$$

$$\begin{aligned} s_{(4,3,2,1)}(u) = u_3 u_7 - u_5^2 - \frac{1}{3}u_1^3 u_7 + u_1^2 u_3 u_5 \\ + \frac{1}{15}u_1^5 u_5 - u_1 u_3^2 - \frac{1}{105}u_1^7 u_3 \\ + \frac{1}{4725}u_1^{10}. \end{aligned}$$

The fact that the first term of the expansion of the sigma function becomes the Scur function determined from X was found by Buchstaber-Enolski-Leykin in the more general context of (n, s) curves in 1999.

Onishi used this expansion to prove the following theorem.

For $0 \leq k \leq g-1$ and $p_i \in X$ such that $p_i \neq p_j^*$ for $i \neq j$, set

$$m_k := d(p_1 + \cdots + p_k + (g-1-k)\infty) \\ = \left[\frac{g-1-k}{2}\right] + 1.$$

Set

$$\partial_{u_{i_1}} \cdots \partial_{u_{i_n}} \sigma(u) = \sigma_{i_1 \dots i_n}(u).$$

Onishi's Theorem

Let $T_k = \{2g - 2k - 1, 2g - 2k - 5, ...\}$ be the subset of $\{1, 3, ..., 2g - 1\}$ with the cardinality m_k. Then they satisfy the following properties.

(1)
$\sigma_{T_k}(I(p_1) + \cdots + I(p_k))$ does not vanish identically as a function of p_1,...,p_k.

(2)
$$\sigma_{T_k}(I(p_1) + \cdots + I(p_k)) = \\ c_k \sigma_{T_{k-1}}(I(p_1) + \cdots + I(p_{k-1})) z_k^{g+1-k} + \cdots,$$
where $z_k = z(p_k)$ and z is a local coordinate at ∞.

In particular, if $k=0$, (1) tells that

$$
\begin{array}{lll}
g=2, & m_0=1, & \sigma_3(0)\neq 0, \\
g=3, & m_0=2, & \sigma_{51}(0)\neq 0, \\
g=4, & m_0=2, & \sigma_{73}(0)\neq 0, \\
g=5, & m_0=3, & \sigma_{951}(0)\neq 0, \\
g=6, & m_0=3, & \sigma_{11,7,3}(0)\neq 0, \\
 & \vdots & \vdots
\end{array}
$$

Comparison with the Schur functions

Those non-vanishing derivatives correspond to the underlined terms of Schur functions:

$$s_{(2,1)}(u) = \underline{-u_3} + \frac{1}{3}u_1^3,$$

$$s_{(3,2,1)}(u) = \underline{u_1 u_5} - u_3^2 - \frac{1}{3}u_1^3 u_3 + \frac{1}{45}u_1^6,$$

$$\begin{aligned} s_{(4,3,2,1)}(u) = \underline{u_3 u_7} - u_5^2 - \frac{1}{3}u_1^3 u_7 + u_1^2 u_3 u_5 \\ + \frac{1}{15}u_1^5 u_5 - u_1 u_3^2 - \frac{1}{105}u_1^7 u_3 \\ + \frac{1}{4725}u_1^{10}. \end{aligned}$$

$$\vdots$$

In this way Onishi's theorem gives explicitly a non-vanishing derivative predicted by the Riemann's singularity theorem.

This kind of results is extended to several cases.

(1) $y^n = f(x)$, Matsutani-Previato (arXiv:1006.1090)

(2) (n,s) curves, Yori-N (arXiv:1205.6897)

(3) telescopic curves, Ayano-N (arXiv:1303.2878)

In (2),(3) we derived everything from the properties of Schur functions combined with the properties of the integrable systems without using explicit formulas which hold for sigma functions etc.

After we wrote (2), V. Enolski asked me whether other properties like the index of speciality can also be studied from properties of Schur functions or not. In other words his question is whether it is possible to study the case of an aribitrary Riemann surface and an arbitrary zero e (not necessarily $-\delta$) of the theta function in a similar way.

In this talk we will give an affirmative answer to his question.

So we consider an arbitrary compact Riemann surface X of genus g and an arbitrary zero e of the theta function as in the beginning of this talk.

Let us write

$$e = q_1 + \cdots + q_{g-1} - \Delta.$$

Strategy of the arguments

$$(X, p_\infty, e, \ldots)$$

$$\Downarrow$$

Schur functions $\{s_\mu(t)\}_{\mu \geq \lambda}$ (combinatorics)

$$\Downarrow$$

Tau functions $\tau(t)$ (integrable systems)

$$\Downarrow$$

$$\theta(z)$$

.

Schur function

Define the polynomial $p_n(t)$, $t = (t_1, t_2, ...)$, by

$$\exp(\sum_{n=1}^{\infty} t_n k^n) = \sum_{n=0}^{\infty} p_n(t) k^n.$$

Example. $p_0 = 1$, $p_1 = t_1$, $p_2 = t_2 + t_1^2/2$,

$$p_3 = t_3 + t_1 t_2 + t_1^3/3!.$$

Schur function

A partition $\lambda = (\lambda_1, ..., \lambda_l)$ is a decreasing sequence of non-negative integers.

For a partition $\lambda = (\lambda_1, ..., \lambda_l)$ define the Schur function $s_\lambda(t)$ by

$$s_\lambda(t) = \det(p_{\lambda_i - i + j}(t))_{1 \le i,j \le l}.$$

Example

$$s_{(4)}(t) = t_4 + t_3 t_1 + \frac{1}{2} t_2^2 + \frac{1}{2} t_2 t_1^2 + \frac{1}{24} t_1^4,$$

$$s_{(3,1)}(t) = -t_4 - \frac{1}{2} t_2^2 + \frac{1}{2} t_1^2 t_2 + \frac{1}{8} t_1^4,$$

$$s_{(2,2)}(t) = -t_3 t_1 + t_2^2 + \frac{1}{12} t_1^4,$$

$$s_{(2,1,1)}(t) = t_4 - \frac{1}{2} t_2^2 - \frac{1}{2} t_2 t_1^2 + \frac{1}{8} t_1^4$$

$$s_{(1,1,1,1)}(t) = -t_4 + t_3 t_1 + \frac{1}{2} t_2^2 - \frac{1}{2} t_2 t_1^2 + \frac{1}{24} t_1^4, \ .$$

We assign the weight i to t_i. Then the Schur function $s_\lambda(t)$ is homogeneous with the weight $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$.

We specify the partition from the data of Riemann surfaces.

Schur function of a Riemann surface

Let L be the flat line bundle on X defined by

$$L = q_1 + \cdots + q_{g-1} - (g-1)p_\infty.$$

Gaps and non-gaps

A non-negative integer b is a gap of L at p_∞ if there does not exist a meromorphic section of L on $X - \{p_\infty\}$ with a pole of order b.

By Riemann-Roch we can easily prove that there are exactly g gaps for any (L, p_∞).

An element of the complement of gaps in the set of non-negative integers is called a non-gap.

Gaps sequences

We consider two kinds of gap sequences simultaneously.

$1 = w_1 < \cdots < w_g$: the gap sequence at p_∞,

$0 = w_1^* < w_2^* < \cdots$: the non-gaps at p_∞,

$0 \leq b_1 < \cdots < b_g$: the gap sequence of L at p_∞,

$0 \leq b_1^* < b_2^* < \cdots$: the non-gaps of L at p_∞.

Schur function of a Riemann surface

We define

$$\lambda = (b_g, ..., b_1) - (g-1, ..., 1, 0).$$

A special property of the Schur function corresponding to λ is the following.

Proposition

$s_\lambda(t)$ depends only on $t_{w_1},\ldots,t_{w_g}$, where $w_1,\ldots,w_g$ is the gap sequence of X at p_∞.

To discuss the relation of this Schur function $s_\lambda(t)$ with the theta function we need to do a linear change of the variables. This linear change is described by a non-normalized period matrix. Let us introduce it.

We further specify the following data:

z: a local coordinate at p_∞,

$\{du_{w_i}\}$: a basis of holomorphic 1-forms such that

$$du_{w_i} = z^{w_i - 1}(1 + O(z))dz \quad \text{at} \quad p_\infty.$$

Then the period matrices are determined:

$$2\omega_1 = \left(\int_{\alpha_j} du_{w_i} \right),\ 2\omega_2 = \left(\int_{\beta_j} du_{w_i} \right),\ \Omega = \omega_1^{-1}\omega_2.$$

We consider the function

$$\theta((2\omega_1)^{-1}u + e),$$

where

$$u = {}^t(u_{w_1}, ..., u_{w_g}).$$

This enumeration of variables is necessary to connect theta functions to solutions of the KP-hierarchy.

Then the relation of the Schur function $s_\lambda(t)$ with the theta function is given by the following theorem.

The first term of the exapansion with respect to weights

Theorem

$$C\theta((2\omega_1)^{-1}u + e) = s_\lambda(u) + \text{higher weights terms.}$$

Recall that $s_\lambda(u)$ is a polynomial of $u_{w_1},...,u_{w_g}$. So this formula is consistent with the enumeration of the variables $u = {}^t(u_{w_1}, ..., u_{w_g})$ in the theta function.

The theorem is proved by using the Sato's theory on the KP-hierarchy.

This theory makes a one to one correspondence between solutions, called tau functions, of the KP-hierarchy and poinst of a certain infinite dimensional Grassmanian called the universal Grassman manifold (UGM).

Strategy of the proof

$$\tau(t) = e^{\frac{1}{2}q(t)}\theta(At + e) \ : \ \text{a solution to KP}$$

$$\Rightarrow U \in UGM^{\lambda}$$

$$\Rightarrow C\tau(t) = s_{\lambda}(t) + \sum_{\lambda<\mu} \xi_{\lambda\mu} s_{\mu}(t)$$

$\Rightarrow$ the expansion of theta.

Here $\mu \geq \lambda$ means

$$\mu_i \geq \lambda_i \text{ for any } i.$$

A family of Schur functions

In the expansion of the tau function the Schur functions $s_\mu(t)$ satisfying the condition $\mu \geq \lambda$ appear.

So we study the common properties of those Schur functions.

a-sequence

Let us define

$$m_0 = d(q_1 + \cdots + q_{g-1}) < g.$$

We define the "a-sequence" by

$$(a_1, ..., a_{m_0})$$

$$= (b_g, b_{g-1}, ..., b_{g-m_0+1}) - (b_1^*, ..., b_{m_0}^*).$$

Example of a-sequence

hyperelliptic curves, $p_\infty = \infty$, $q_i = p_\infty$ for any i:

$g = 2$: $m_0 = 1$, $a_1 = 3$,

$g = 3$: $m_0 = 2$, $(a_1, a_2) = (5, 1)$,

$g = 4$: $m_0 = 2$, $(a_1, a_2) = (7, 3)$,

$g = 5$: $m_0 = 3$, $(a_1, a_2, a_3) = (9, 5, 1)$,

$g = 6$: $m_0 = 3$, $(a_1, a_2, a_3) = (11, 7, 3)$.

Thus the "a-sequence" is a natural generalization of Onishi's sequence.

Properties of the a-sequence

In general the a-sequence has the following properties.

Proposition

(1) (distinct) $a_1 > \cdots > a_{m_0} > 0$.

(2) (gaps) $a_i \in \{w_j\}$.

(3) (weight) $\sum a_i = |\lambda|$.

Vansihing and non-vanishing theorem for Schur functions

Let $\partial_i = \partial_{t_i}$.

Theorem

Let $\mu \geq \lambda$. Then

(1) $\partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots s_\mu(0) = 0 \quad$ *if* $\quad \sum i\alpha_i \neq |\mu|$. *(weight)*

(2) $\partial_{i_1} \cdots \partial_{i_m} s_\mu(0) = 0 \quad$ *if* $\quad m < m_0$. *(degree)*

(3) $\partial_{a_1} \cdots \partial_{a_{m_0}} s_\lambda(0) \neq 0$.

We can lift these properties to tau functions of the KP hierarchy.

Recall that our tau function has the following expansion:

$$\tau(t) = s_\lambda(t) + \sum_{\lambda < \mu} \xi_{\lambda\mu} s_\mu(t).$$

Using this expansion and the vanishing-non-vanishing theorem for Schur functions we can prove the following theorem for tau functions.

Vansihing and Non-vanishing theorem for tau functions

Theorem

(1) $\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\tau(0)=0$ *if* $\sum i\alpha_i<|\lambda|$. *(weight)*

(2) $\partial_{i_1}\cdots\partial_{i_m}\tau(0)=0$ *if* $m<m_0$. *(degree)*

(3) $\partial_{a_1}\cdots\partial_{a_{m_0}}\tau(0)\neq 0$.

Finally we can transplant these properties of the tau functions to Riemann's theta functions.

Recall that our tau function has the expression in terms of the theta function of the form

$$\tau(t) = Ce^{\frac{1}{2}q(t)}\theta((2\omega_1)^{-1}(Bt) + e).$$

Here $B = (b_{ij})$ is the $g \times \infty$ matrix determined by the expansion of du_{w_i} and has the "triangular structure":

$$b_{ij} = \begin{cases} 0, & j < w_i \\ 1, & j = w_i \end{cases}$$

Using this triangular structure of B and the previous properties of the tau function we can prove the following theorem for theta functions.

Vansihing and non-vanishing theorem for theta functions

Theorem

(1) $\partial_{u_{w_1}}^{\alpha_1} \cdots \partial_{u_{w_g}}^{\alpha_g} \theta(e) = 0 \quad$ *if* $\quad \sum w_i \alpha_i < |\lambda|$. *(weight)*

(2) $\partial_{u_{i_1}} \cdots \partial_{u_{i_m}} \theta(e) = 0 \quad$ *if* $\quad m < m_0$. *(degree)*

(3) $\partial_{u_{a_1}} \cdots \partial_{a_{m_0}} \theta(e) \neq 0$.

(2) and (3) give the Riemann's singularity theorem.
Moreover (3) gives non-vanishing derivatives explicitly.

Summary

We have studied the properties of the Schur functions. By combining them with the properties of the tau functions of the KP-hierarchy we have proved the Riemann's singularity theorem with more precise informations on non-vanishing derivetaives.

Comments and questions

1. The full generalization of Onishi's theorem is possible (the expansion of the theta at $I(p_1) + \cdots + I(p_k) + e$ in $z(p_k)$).

2. There are applications to sigma functions (normailzation constant).

3. What is the meaning of the "a-sequence" in terms of the representation theory of symmetric groups ?

4. What is the first term of the expansion with respect to degrees (related with higher weights terms) ?

Thank you.

and

Happy 70th birthday

古希おめでとうございます

Victor Matveevich!