Stochastic Anisotropy-based Robust Control Theory:

Past, Present, and Future

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AGENDA

- Introduction

1. Anisotropy-based theory location in Control Theory
2. Class of control systems anisotropy theory was done for
3. Some Pioneers
4. Pre-conditions of Anisotropy-based theory.
   LQG and H_∞ optimization. Difference and commonality

- Past (1993-2005)
  1. Fundamentals of the theory: anisotropy of the signal, mean anisotropy of the sequence, physical interpretation, how to calculate.
  2. Anisotropic norm: properties, how to calculate. Asymptotic of anisotropic norm
  3. Anisotropy - based optimal control problem. The problem solution. Equations for optimal control design.
  4. Computation tool for control design. Gomotopy method for solving cross-coupled equations.
  5. Anisotropy-based small gain theorem. Criteria of robust stability.

6. Anisotropic-based controller design for flight control in landing approach.
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- Present (2005 - 2012)

1. Anisotropy-based theory for descriptor systems. Analysis problem. Synthesis problem.
2. How to calculate anisotropic norm for descriptor systems.
3. Model reduction in Anisotropy - based theory.
4. Suboptimal anisotropy- based problem. KYP lemma for suboptimal problem. LMI methods in Anisotropic theory. Semidefinite programming in Anisotropic theory.

- Future (2012 - ????)
  1. Suboptimal problem for descriptor system.
  2. How to find generating filter for concrete mean anisotropy level. Signal processing problem.
  3. How to extend anisotropic theory to some non linear systems. Absolute stability.
  4. Adaptive anisotropy-based control.
  5. Anisotropy based theory for the systems with non-zero mean of input disturbance.

VERY DIFFICULT UNSOLVED PROBLEM

Introduction

Anisotropy-based theory location in Control Theory

Андриевский Б.Р., Матвеев А.С., Фрадков А.Л. Управление и оценивание при информационных ограничениях: к единой теории управления, вычислений и связи // АиТ. 2010. N 4. С. 34-99.

Two branches of information theory in Control.

$$\textbf{Control}(C) \times \textbf{Communication}(C) \times \textbf{Computing}(C) = C^3$$

Problems are from control,

performance cost is from information theory

Class of control systems anisotropy theory was done for. Mathematical models for investigation

$$\begin{cases} x_{k+1} = Ax_k + B_1 w_k + B_2 u_k \\ z_k = C_1 x_k + D_{11} w_k + D_{12} u_k \\ y_k = C_2 x_k + D_{21} w_k \end{cases}, \quad -\infty < k < +\infty, \qquad (1)$$

where A, C_i, B_j и D_{ij} are appropriative dimension constant matrixes. System $F(z)$, and its subsystems $F(z)_{ij}$ have following state space realizations:

$$F \sim \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right], \qquad (2)$$

$$F_{ij} \sim \begin{bmatrix} A & B_j \\ C_i & D_{ij} \end{bmatrix}, \quad 1 \le i, j \le 2 \qquad (3)$$

Some pioneers.

- Saridis (1988), IEEE Tr. AC
- Semenov, Vladimirov, Kurdyukov (1994), CDC-33
- Karny (1996), Automatica
- Petersen, James, Dupuis (2000), IEEE Tr. AC

Pre-conditions of Anisotropy-based theory. LQG and H_∞ optimization. Difference and commonality. Common paradigm for LQG/H_2 and H_∞ control problems

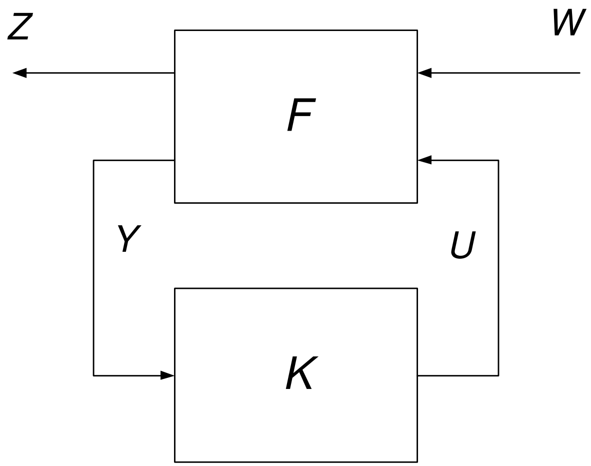


Рис. 1: Common paradigm for H_2/LQG and H_∞ control problems

F is the plant, K is a controller, W и Z are input and output appropriately, Y and U are observing output and control. T_{ZW} is close loop transfer function (transfer function matrix) from W to Z. In both problems: Find control, which minimizes appropriative performance functional.

Standard H_2 optimization problem

Standard H_2 optimization problem: Find the controller K, which

- stabilizes close loop system
- minimizes H_2 norm of close loop transfer function matrix T_{ZW} from W to Z:

$$\|T_{ZW}\|_2 \to \min \tag{4}$$

Definition:

$$\|H\|_2 = \left(Tr \int_{-\pi}^{\pi} \widehat{H}(\omega)(\widehat{H}(\omega))^* d\omega \right)^{1/2}, \tag{5}$$

where

$$\widehat{H}(\omega) \equiv \lim_{r \to 1-0} H\left(r\, e^{i\omega}\right), \quad \omega \in \Omega \equiv [-\pi; \pi],$$

is the angular boundary value of the generating filter H.

Standard H_∞ optimization problem

Standard H_∞ optimization problem: Find the controller K,

- stabilizes close loop system
- minimizes H_∞ norm of close loop transfer function matrix T_{ZW} from W to Z:

$$\|T_{ZW}\|_\infty \to \min \tag{6}$$

Suboptimal H_∞ control problem:

$$\|T_{ZW}\|_\infty \leq \gamma, \tag{7}$$

где $\gamma \geq \gamma_{opt}$, $\gamma_{opt} \geq \|T_{ZW}\|_\infty$.

For transfer function matrix $H(z)$ the define

$$\|H\|_\infty \equiv \sup_{|z|<1} \overline{\sigma}(H(z)) = \operatorname*{ess\,sup}_{\omega\in\Omega} \overline{\sigma}\left(\widehat{H}(\omega)\right), \tag{8}$$

where $\overline{\sigma}(\cdot)$ is maximal singular value of matrix.

Similarity and difference H_∞ and H_2 control problems

Similarity. The solving of both problems are based on solutions of Riccati equations, in H_∞ suboptimal control problem Riccati equation has some parameter γ. If $\gamma \to \infty$ the Riccati equations for H_∞ suboptimal control problem tend to Riccati equations for LQG control problem.

Difference. **Frequency interpretation for H_∞ and H_2 optimal problem for SISO systems** : H_∞ controllers are designed to minimize maximum of amplitude-frequency characteristic of closed-loop system, H_2 control minimizes the average amplitude over all frequencies.

Input signal assumptions: Input disturbance W is to be gaussian white noise in LQG problem. Input disturbance W is quadratic integrable in H_∞ problem.

Singularity of H_∞ and H_2 controllers functioning if input signal assumptions are not true.

The close loop system does not work good with H_2 controller in disturbance attenuation problem if the input signal is «far from» white noise.

The close loop system with H_∞ controller is very conservative (the great amount of control needed) if the input signal is « closed enough» to gaussian white noise.

Convergence (trade-off)between H_∞ and H_2 theories Capability of common (joint) theory construction

- Optimal (suboptimal) H_∞ controllers are not unique. It means we can propose once more performance criterion .
- Natural choice for the new performance criterion is H_2 norm of close loop transfer function matrix.

1. Minimization of close loop system H_2 norm with constraints on H_∞ norm.
Bernstein D.A., Haddad W.M. *LQG* Control with an H_∞ Performance Bound: A Riccati Equation Approach. //IEEE Transactions on Automatic Control, AC-34, N 3, 1989.

2. Minimization of close loop system H_∞ norm with upper bound H_2 norm minimization.
Mustafa D., Glover K. Minimum Entropy H_∞-Control. Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin etc., 1991.

H_∞ optimization problem with minimization of H_∞ entropy

On the set of H_∞ suboptimal controllers to find controller which minimizes H_∞ entropy functional

$$J(\gamma, F) = -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln|\det\left(I_m - \gamma^{-2}(F(j\omega))^* F(j\omega)\right)| d\omega,$$

γ is the number that bounds close loop transfer function H_∞ norm for stable close loop system $F(s)$.

The minimization of H_∞ entropy of the system $F(s)$ is equivalent of the minimization of upper bound of H_2 norm of $F(s)$.

- Designed controller is unique .
- H_∞ control problem with H_∞ entropy minimization is equivalent to risk sensitivity problem.

End of Introduction

Past of the theory. Vladimirov's ideas

- Semyonov, A.V., I.G.Vladimirov, and A.P.Kurdjukov (1994) "Stochastic approach to $\mathcal{H}_\infty$-optimization". *Proceedings of the 33rd Conference on Decision and Control*, Florida, USA, December 14–16, Vol. 3, pp. 2249–2250.

- Vladimirov, I.G., A.P.Kurdjukov, and A.V.Semyonov (1995) "Anisotropy of signals and entropy of linear time invariant systems". *Doklady Akademii Nauk*, Vol. 342, no. 3, pp. 583–585. (in Russian).

- Vladimirov, I.G., A.P. Kurdjukov, and A.V. Semyonov (1995). "Strochastic problem of $\mathcal{H}_\infty$-optimization."*Doklady Akademii Nauk*, Vol.343, No.5, 607–609. (in Russian)

- Vladimirov, I.G., A.P.Kurdjukov, and A.V.Semyonov (1996) "On computing the anisotropic norm of linear discrete-time-invariant systems". *Proceedings of the 13th IFAC World Congress*, San-Francisco, California, USA, June 30–July 5, Vol. G, pp. 179–184, Paper IFAC-2d-01.6.

- Vladimirov I.G., Kurdjukov A.P., Semyonov A.V. State-space solution to anisotropy-based stochastic H-infinity optimization problem. *Proceedings of the 13th IFAC World Congress*, San-Francisco, California, USA, June 30-July 5, 1996, v. H, Paper IFAC-3d-01.6, 1996.

- Diamond, P., I.G.Vladimirov, A.P.Kurdyukov, and A.V.Semyonov (2001) "Anisotropy-based performance analysis of linear discrete time invariant control systems". *Int. J. Control*, Vol. 74, no. 1, pp. 28–42.

- Kurdyukov A.P., Maximov E.A. Robust stability of linear discrete time-invariant systems with anisotropic norm bounded uncertainty. *Automation and Remote Control*, No.12, 2004, pp.129-144 (in Russian)

- Vladimirov, I.G., A.P.Kurdyukov, E.A.Maksimov and V.N.Timin (2005) "Anisotropy-based control theory — a new approach to stochastic robust control". *Plenary addresses of IV conference "System Identification and Control Problems"*, Moscow, Russia, January 25-28, pp. 9–32.

Fundamentals of the theory: anisotropy of the signal, mean anisotropy of the sequence, physical interpretation
How to calculate

Definition 1 *The relative entropy (Kullback-Leibler distance)* $D(f \parallel g)$ *between two densities* $f(x)$ *and* $g(x)$ *is defined by*

$$D(f \parallel g) = \int f(x) \log \frac{f(x)}{g(x)} dx. \tag{9}$$

$D(f \parallel g)$ *is finite, if support set of* $f(x)$ *is contained in the the support set of* $g(x)$ *. It is true that* $0 \log \frac{0}{0} = 0$.

$$D(f \parallel g) \geq 0$$

with equality iff $f = g$ almost everywhere.

Definition 2 *Let* X *and* Y *are two random variable with joint distribution function of probability density* $f(x, y)$ *and probability density functions* $f(x)$ $f(y)$ *appropriately . The mutual information* $I(X;Y)$ *is defined as*

$$I(X;Y) = \int \log f(x, y) \frac{f(x, y)}{f(x)f(y)} dxdy. \tag{10}$$

Definition of anisotropy of the random vector

Denote by $\mathbb{L}_2^m$ the class $\mathbb{R}^m$-dimension absolutely continuously distributed random vectors W with values in R^m satisfying $\mathbf{E}\,|W|^2 < \infty$.

For any $\lambda > 0$ denote as $p_{m,\lambda}$ the probability density function (pdf) on $\mathbb{R}^m$ of gaussian signal with zero mean and scalar covariance matrix λI_m

$$p_{m,\lambda}(w) = (2\pi\lambda)^{-m/2} \exp\left(-\frac{|w|^2}{2\lambda}\right), \qquad w \in \mathbb{R}^m. \tag{11}$$

For any $W \in \mathbb{L}_2^m$ with pdf $f : \mathbb{R}^m \to \mathbb{R}_+$ the relative entropy of $W \in \mathbb{L}_2^m$ according to (11) has the following view

$$D\left(f \| p_{m,\lambda}\right) = \mathbf{E} \ln \frac{f(W)}{p_{m,\lambda}(W)} = -h(W) + \frac{m}{2} \ln(2\pi\lambda) + \frac{\mathbf{E}|W|^2}{2\lambda}, \tag{12}$$

where

$$h(W) = -\mathbf{E} \ln f(W) = -\int_{\mathbb{R}^m} f(w) \ln f(w) \mathrm{d}w \tag{13}$$

is differential entropy of random vector W

Definition of anisotropy of the random vector (continuation)

Definition 3 *The anisotropy* $\mathbf{A}(W)$ *of random vector* $W \in \mathbb{L}_2^m$ *is defined as minimal relative entropy of its pdf from gaussian distribution* $\mathbb{R}^m$ *with zero mean and scalar covariance matrix*

$$\mathbf{A}(W) = \min_{\lambda>0} D\left(f \| p_{m,\lambda}\right). \tag{14}$$

Direct calculation shows, that minimum in (12) over $\lambda > 0$ is obtained if $\lambda = \mathbf{E}|W|^2/m$, so

$$\mathbf{A}(W) = \min_{\lambda>0} D\left(f \| p_{m,\lambda}\right) = \frac{m}{2} \ln\left(\frac{2\pi \mathrm{e}}{m} \mathbf{E}|W|^2\right) - h(W). \tag{15}$$

Properties of random vector anisotropy

Denote by $\mathbb{G}^m(\Sigma)$ the class of $\mathbb{R}^m$-valued gaussian disturbances random vectors W with $\mathbf{E}W = 0$ and nonsingular covariance matrix $\mathbf{cov}(W) = \Sigma$, so that the corresponding pdf is

$$p(w) = (2\pi)^{-m/2}(\det\Sigma)^{-1/2}\exp\left(-\frac{1}{2}\|w\|^2_{\Sigma^{-1}}\right),$$

$\|x\|_Q = \sqrt{x^\top Q x}$ denotes the norm of a vector x, induced by a positive definite symmetric matrix $Q > 0$.

Lemma 1

(a) *For any positive definite matrix* $\Sigma \in \mathbb{R}^{m\times m}$,

$$\min_W \left\{\mathbf{A}(W) : \ W \in \mathbb{L}_2^m, \ \mathbf{E}(WW^\top) = \Sigma\right\} = -\frac{1}{2}\ln\det\frac{m\Sigma}{\operatorname{Trace}\Sigma}, \qquad (16)$$

and the minimum is attained only for $W \in \mathbb{G}^m(\Sigma)$;

(b) *For any* $W \in \mathbb{L}_2^m$, $\mathbf{A}(W) \geq 0$. *Moreover* $\mathbf{A}(W) = 0$ *iff* $W \in \mathbb{G}^m(\lambda I_m)$

Mean anisotropy of random sequences

Let $W \in \mathbb{L}_2^m$ be partitioned into subvectors $w_1, \ldots, w_r$ of dimentions $m_1, \ldots, m_r$, e.g. $m_1 + \ldots + m_r = m$

$$W = \begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix}. \tag{17}$$

For any $1 \leq s \leq t \leq r$, denote by $W_{s:t} = (w_k)_{s \leq k \leq t}$ the $(m_s + \ldots + m_t)$-dimensional subvector of W (17), obtained by "stacking"$w_s, \ldots, w_t$.

Definition 4 *The mean anisotropy of sequence W is defined as:*

$$\overline{\mathbf{A}}(W) = \lim_{N \to +\infty} \frac{\mathbf{A}(W_{0:N-1})}{N}. \tag{18}$$

Mean anisotropy of gaussian random sequences

Let $V \equiv (v_k)_{-\infty<k<+\infty} \in \mathbb{G}^m(I)$, $\quad W \equiv (w_k)_{-\infty<k<+\infty} \equiv GV$,

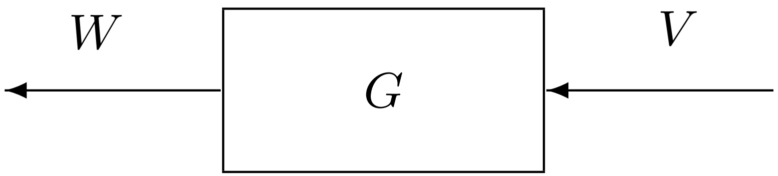


Рис. 2:

The generating filter $G \in H_2^{m\times m}$ is identified with its transfer function $G(z) \equiv \sum\limits_{k=0}^{+\infty} g_k\, z^k$, where $g_k \in \mathbb{R}^{m\times m}$, $\quad k \geq 0$ is input-impulse response.

Theorem 1 *The mean anisotropy (18) can be representable as*

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi} \int\limits_{-\pi}^{\pi} \ln\det\left(\frac{m}{\|G\|_2^2}\, \widehat{G}(\omega) \left(\widehat{G}(\omega)\right)^*\right) \mathrm{d}\omega\,. \tag{19}$$

Properties of gaussian sequence mean anisotropy

- $\overline{\mathbf{A}}(W) > 0$ if rank $\widehat{\mathrm{G}}(\omega) = \mathrm{m}$ for almost all $\omega \in [-\pi, \pi)$,
- $\overline{\mathbf{A}}(W) = +\infty$ if $\widehat{G}$ - not maximum rank,
- $\overline{\mathbf{A}}(W) = 0$ if there is such number $\alpha > 0$ что $\widehat{G}(\omega)\widehat{G}^*(\omega) = \alpha I_m, \quad -\pi \leq \omega < \pi$.

Calculation of mean anisotropy in state space

Let state space representation of generating filter $G \in H_2^{m\times m}$ be

$$\left\{ \begin{array}{ll} x_{k+1} & = Ax_k + Bv_k \\ w_k & = Cx_k + Dv_k \end{array} \right. , \quad -\infty < k < +\infty, \tag{20}$$

where A, B, C, D are matrices of appropriative dimension. The matrix $\rho(A) < 1$ is assumed to be asymptotically stable (with spectral radius $\rho(a) < 1$) and D nonsingular.

Associate with the filter G the Riccati equation in the matrix $R \in \mathbb{R}^{n\times n}$

$$R = ARA^\top + BB^\top - \Lambda\Theta\Lambda^\top, \tag{21}$$

$$\Lambda \doteq (ARC^\top + BD^\top)\,\Theta^{-1}, \tag{22}$$

$$\Theta \doteq CRC^\top + DD^\top. \tag{23}$$

A solution R of equation (21)–(23) is said to be *admissible* if R is symmetric and positive semidefinite and matrix $A - \Lambda C$ is asymptotically stable.

Calculation of mean anisotropy in state space

The equation (21)–(23) can be written in a form

$$ARA^\top - R - (ARC^\top + BD^\top) \times (CRC^\top + DD^\top)^{-1}(CRA^\top + DB^\top) + BB^\top = 0. \quad (24)$$

Theorem 2 *Let a generating filter $G \in H_2^{m\times m}$ have state-space realization (20) with A asymptotically stable and D nonsingular. Then the mean anisotropy (19) of the sequence $W = GV$ is*

$$\overline{\mathbf{A}}(G) = -\frac{1}{2} \ln\det\left(\frac{m\,\Theta}{\mathrm{Trace}\,(CPC^\top + DD^\top)}\right), \quad (25)$$

where $\Theta = CRC^\top + DD^\top$, R is admissible Riccati equation (21)–(23), and P is controllability gramian of the filter satisfying Lyapunov equation

$$P = APA^\top + BB^\top. \quad (26)$$

Algorithm for mean anisotropy calculation

- The Riccati equation (21)–(23) or (24):

$$ARA^\top - R - (ARC^\top + BD^\top) \times (CRC^\top + DD^\top)^{-1}(CRA^\top + DB^\top) + BB^\top = 0.$$

  is solved , and R and $\Theta = CRC^\top + DD^\top$ is found

- Lyapunov equation

$$P = APA^\top + BB^\top.$$

  is solved.

- The mean anisotropy is calculated by formula

$$\overline{\mathbf{A}}(G) = -\frac{1}{2} \ln\det\left(\frac{m\,\Theta}{\mathrm{Trace}\,(CPC^\top + DD^\top)}\right).$$

Anisotropic norm: properties, how to calculate.
Asymptotic of anisotropic norm.

Anisotropic norm of linear time invariant systems

Let $F(z) \in H_\infty^{p\times m}$ be linear time invariant system and $Z = FW$, e.g. $F(z)$ is analitic in open unit ball and has finite H_∞ norm $\|F\|_\infty = \sup\limits_{|z|<1} \overline{\sigma}(F(z)) = \operatorname*{ess\,sup}\limits_{-\pi\le\omega\le\pi} \overline{\sigma}(\widehat{F}(\omega))$, where $\overline{\sigma}(\,\cdot\,)$ is maximum singular value of $F(z)$.

Definition 5 *For given $a \ge 0$, a-anisotropic norm of the system F is defined as*

$$\|F\|_a = \sup_G \{\|FG\|_2/\|G\|_2 : \ G \in \mathbf{G}_a\}, \tag{27}$$

$$\mathbf{G}_a = \left\{G \in H_2^{m\times m} : \ \overline{\mathbf{A}}(G) \le a\right\} \tag{28}$$

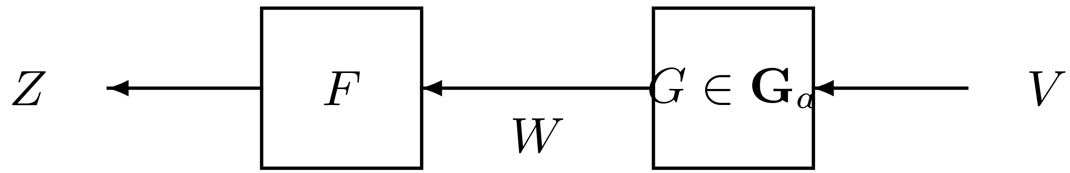


Рис. 3:

Properties of anisotropic norm for linear system

For any fixed system $F \in H_\infty^{p\times m}$, its a-anisotropic norm (58) is nondecreasing continuous function of $a \geq 0$ satisfying

$$\frac{1}{\sqrt{m}}\|F\|_2 = |||F|||_0 \leq |||F|||_a \leq \lim_{a\to+\infty} |||F|||_a = \|F\|_\infty. \tag{29}$$

By (29), computing the norm $|||F|||_a$ is only of interest if $a > 0$ and

$$\|F\|_2 < \sqrt{m}\,\|F\|_\infty \tag{30}$$

(there is a particular interest if $\|F\|_\infty \gg \|F\|_2/\sqrt{m}$). This equality is not true iff, F is an inner (inner system) up to a nonzero constant multiplier $\lambda > 0$ such that $(\widehat{F}(\omega))^*\widehat{F}(\omega) = \lambda I_m$ for almost all $\omega \in [-\pi, \pi)$. For nonzero system $F \in H_\infty^{p\times m}$, the inequality $p < m$ implies (30).

Anisotropic norm of linear system

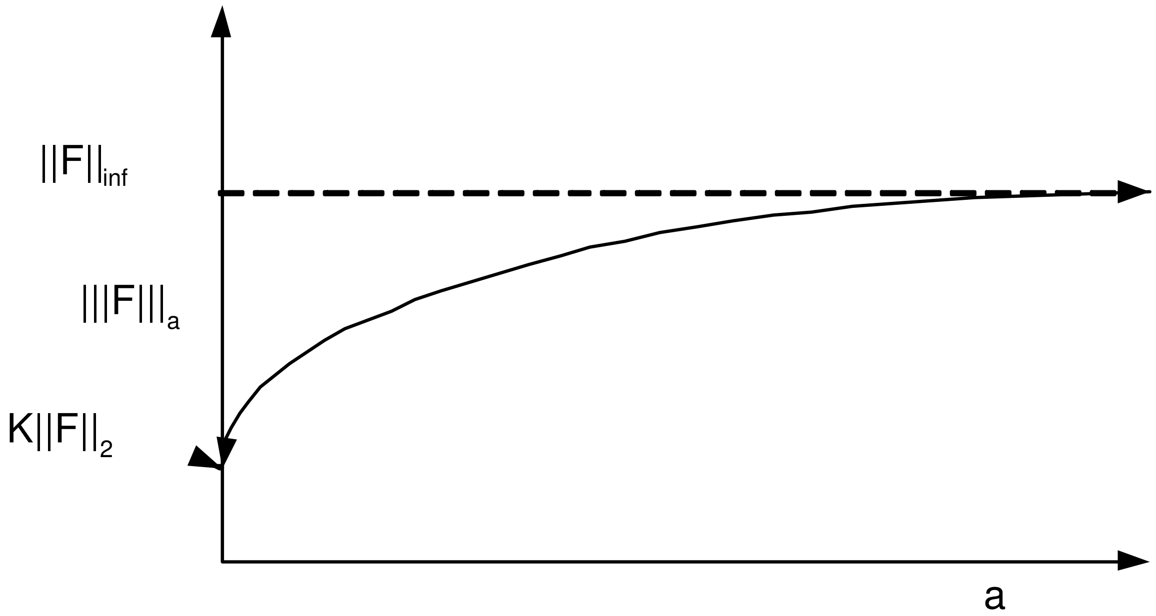


Рис. 4: Changes of anisotropic norm

$$K = \frac{1}{\sqrt{m}}$$

Asymptotic behavior of a - anisotropic norm

$$\left|\!\left|\!\left| F \right|\!\right|\!\right|_a - \frac{\|F\|_2}{\sqrt{m}} \quad \sim \quad \frac{\sqrt{\|F\|_4^4/m - (\|F\|_2^2/m)^2}}{\|F\|_2}\sqrt{a} \quad \text{if } a \to 0+, \tag{31}$$

$$\|F\|_\infty - \left|\!\left|\!\left| F \right|\!\right|\!\right|_a \quad \sim \quad \frac{1}{2}\,\|F\|_\infty \exp\left(-\frac{2}{m}\left(J(\|F\|_\infty) + a\right)\right) \text{ if } a \to +\infty, \tag{32}$$

For any positive integer k, the norm of the system $F \in H_\infty^{p\times m}$ in Hardy space $H_{2k}^{p\times m}$ is defined as

$$\|F\|_{2k} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \text{Trace}\left((\widehat{F}(\omega))^*\widehat{F}(\omega)\right)^k d\omega\right)^{1/(2k)}$$

(particularly, for $k = 1$, it gives H_2-norm).

$$J(\gamma, F) = -\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty} \ln|\det\left(I_m - \gamma^{-2}\left(F(j\omega)\right)^* F(j\omega)\right)|d\omega,$$

Calculation of $\|F\|_4$ norm in state space
Vladimirov's result

Let system F has the following state-space presentation

$$F = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Lemma 2 *H_4-norm of asymptotical stable system F is given as*

$$\|F\|_4^4 = \mathrm{Trace}\,(CPC^\top + DD^\top)^2 + 2\,\mathrm{Trace}\,\Big((CPA^\top + DB^\top)Q(APC^\top + BD^\top)\Big), \tag{33}$$

where P and Q are gramian of controllability and observability of system F. P and Q can be found as solutions of Lyapunov's equations.

$$P = APA^\top + BB^\top, \quad Q = A^\top QA + C^\top C.$$

Pseudo multiplicative property of anisotropic norm

The ring property of H_∞-norm , (sub multiplicative property)

$$\|FG\|_\infty \le \|F\|_\infty \|G\|_\infty$$

is not true for anisotropic norm $|\!|\!| \cdot |\!|\!|_a$.

But there is the analog of ring property.

Theorem 3 *For any* $a \ge 0$ *and any systems* $F \in H_\infty^{p\times m}$ *u* $G \in H_\infty^{m\times m}$,

$$|\!|\!| FG |\!|\!|_a \le |\!|\!| F |\!|\!|_b \, |\!|\!| G |\!|\!|_a \tag{34}$$

где

$$b = a + \overline{\mathbf{A}}(G) + m \, \ln\left(\sqrt{m}\, |\!|\!| G |\!|\!|_a / \|G\|_2\right). \tag{35}$$

Corollary 1 *ANISOTROPIC-BASED SMALL GAIN THEOREM*

Robust stability in anisotropic theory

Let P be the object with follow description

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{36}$$

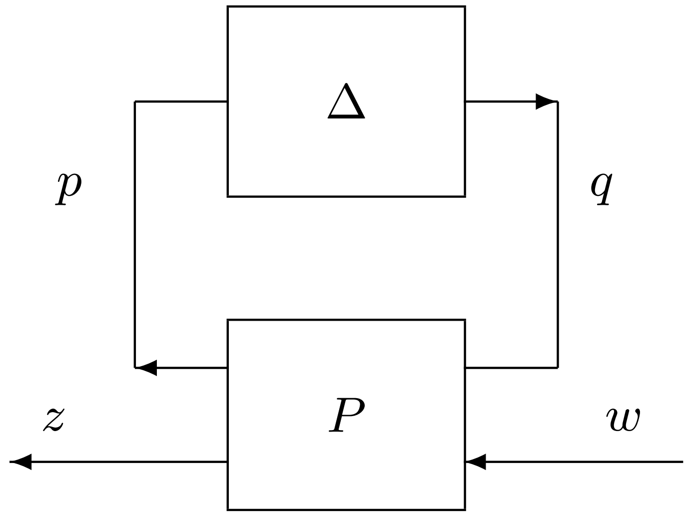


Рис. 5: P–Δ конфигурация.

Theorem 4 *Consider $\mathcal{F}_u(P,\Delta)$, where $\Delta : l_2 \to l_2$ and $P : l_2 \to l_2$ are causal linear systems.*

- *Let P be stable and*

$$\left\| P_{11} \right\|_c < \epsilon^{-1}, \text{ where} \quad c = a + m \ln \frac{\epsilon}{\operatorname*{ess\,inf}_{-\pi \le \omega \le \pi} \underline{\sigma}(\Delta(j\omega))}, \tag{37}$$

$\underline{\sigma}(\Delta) = \sqrt{\lambda_{min}(\Delta^* \Delta)}$ – *minimum singular value of Δ, $\epsilon > 0$.*

- *Let*

$$a = -\frac{1}{2} \ln \det \frac{m\Sigma}{\operatorname{tr} \Sigma} - m \ln \frac{\epsilon}{\operatorname{ess\,sup} \underline{\sigma}(\Delta(j\omega))},$$

where $\Sigma = (I_m - qP_{11}^* P_{11})^{-1}$, and parameter $q \in [0, \|P_{11}\|_\infty^{-2})$ satisfies inequality

$$\operatorname{tr} \left[\left(I_m - \epsilon^2 P_{11}^* P_{11} \right) \left(I_m - q P_{11}^* P_{11} \right)^{-1} \right] \le 0. \tag{38}$$

Then for all $\Delta \in D_a(\epsilon)$ close-loop system $\mathcal{F}_u(P,\Delta)$ is internal stable.

How to calculate the anisotropic norm in state space

Let system F has the following state space representation

$$F = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

It is well known fact for calculation of $\|F\|_2$ norm of the system F it is necessary to solve Lyapunov equation.

It is well known fact for calculation of $\|F\|_\infty$ norm of the system F it is necessary to solve Riccati equation (Bounded real lemma).

As far as anisotropic norm $|\!|\!|F|\!|\!|_a$ of the system lies "between"normalized $\|F\|_2$ and $\|F\|_\infty$ norms, it natiural to propose that we have to use Lyapunov and Riccati equation for anisotropic norm calculation. It is really true, but for the calculation algorithm we have to add some special time equation.

How to calculate the anisotropic norm in state space II

Anisotropic norm is calculated by the formula

$$\|F\|_a = \left(\frac{1}{q}\left(1 - \frac{m}{\mathrm{Trace}\,(LPL^\top + \Sigma)}\right)\right)^{1/2}.$$

q, P, L, Σ are unknown parameters. The can be calculated by solving coupled equations: (39) ia Riccati equation , (40) is Lyapunov equation, (41) is a special time equation

$$\begin{aligned} R &= A^\top RA + qC^\top C + L^\top \Sigma^{-1} L, \\ L &= \Sigma(B^\top RA + qD^\top C), \\ \Sigma &= (I_m - B^\top RB - qD^\top D)^{-1}. \end{aligned} \tag{39}$$

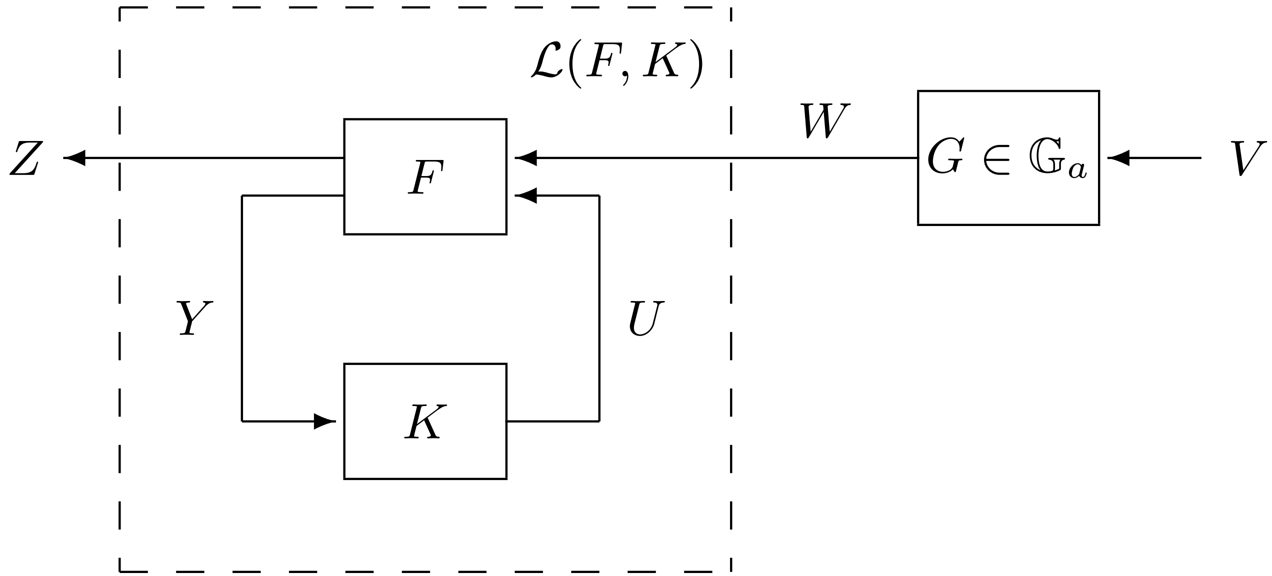
$$P = (A + BL)P(A + BL)^\top + B\Sigma B^\top, \tag{40}$$

$$a = -\frac{1}{2}\ln\det\left(\frac{m\,\Sigma}{\mathrm{Trace}\,(LPL^\top + \Sigma)}\right). \tag{41}$$

Anisotropy-based control design problem

Let W be generated from m_1-dimensional gaussian white noise V c with zero expectation and unit covariance matrix by unknown generating filter G from

$$\mathbb{G}_a \equiv \left\{ G \in H_2^{m_1 \times m_1} \; : \; \overline{A}(G) \leq a \right\} \; . \tag{42}$$



Anisotropic-based optimization problem:

Problem 1 *For given system F and mean anisotropy level $a \geq 0$ of input disturbance W find the controller $K \in \mathcal{K}$, that minimizes the a-anisotropic norm of closed loop system $\mathcal{F}_l(F, K)$:*

$$\left\|\mathcal{F}_l(F, K)\right\|_a \equiv \sup\left\{\frac{\|\mathcal{F}_l(F, K)G\|_2}{\|G\|_2} : \; G \in \mathbb{G}_a\right\} \to \inf\,, \quad K \in \mathcal{K}\,. \qquad (43)$$

Let us note if $a = 0$, the above problem 4 is coincided with standard H_2 optimization problem (Kolmogorov - Wiener-Hopf-Kalman optimization problem).

Solution of anisotropic-based design problem

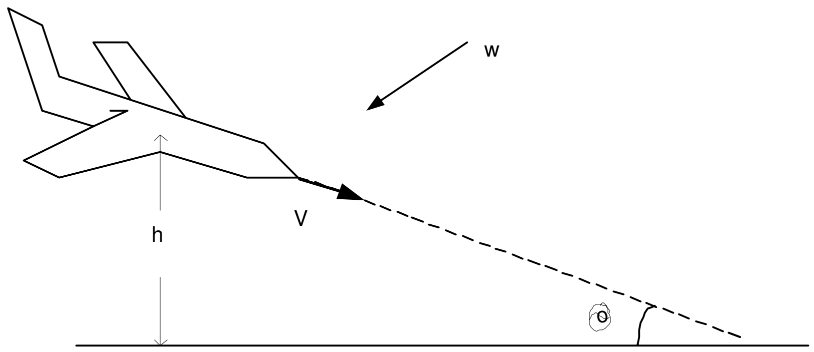
The solution of the problem is reduced to the solving of three algebraic matrix Riccati equations, Lyapynov equation and one algebraic equation of special type. If $a = 0$ the four matrix equations turn into well known two Riccati equations from Kalman theory and the equation of special type cancels.

How to find the solution by computer?
Vladimirov's Package

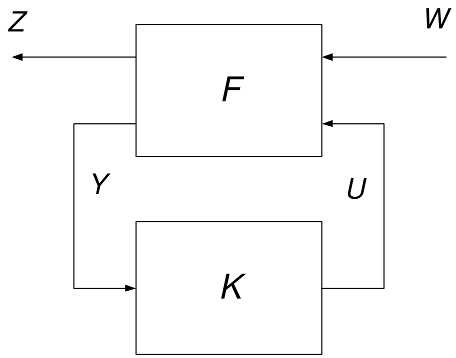
Crossed- coupled three matrix algebraic Riccati equations, Lyapunov equation and special type equation have been solving by homotopy method. We reduced the solution of the algebraic system to differential equation system. The anisotropy level was the independent variable in those differential systems. The initial conditions were the solutions of the problem if $\alpha = 0$, the LQG problem.

I.G. Vladimirov create the application package (software kit) for MathLab and programmed it.

Anisotropic controllers in landing approach



h is vertical coordinate of aircraft mass center V is an aircraft speed relative to wind speed frame
Θ is relative flight-path angle
W is a wind disturbance
$Z = [h, V]^T$
$U = [\Delta\vartheta_{cy}, \delta_t]^T$ are elevator and power lever .



Problem. Design LQG, H_∞ and anisotropy controllers, that solve disturbance attenuation problem .

Model of wind shear

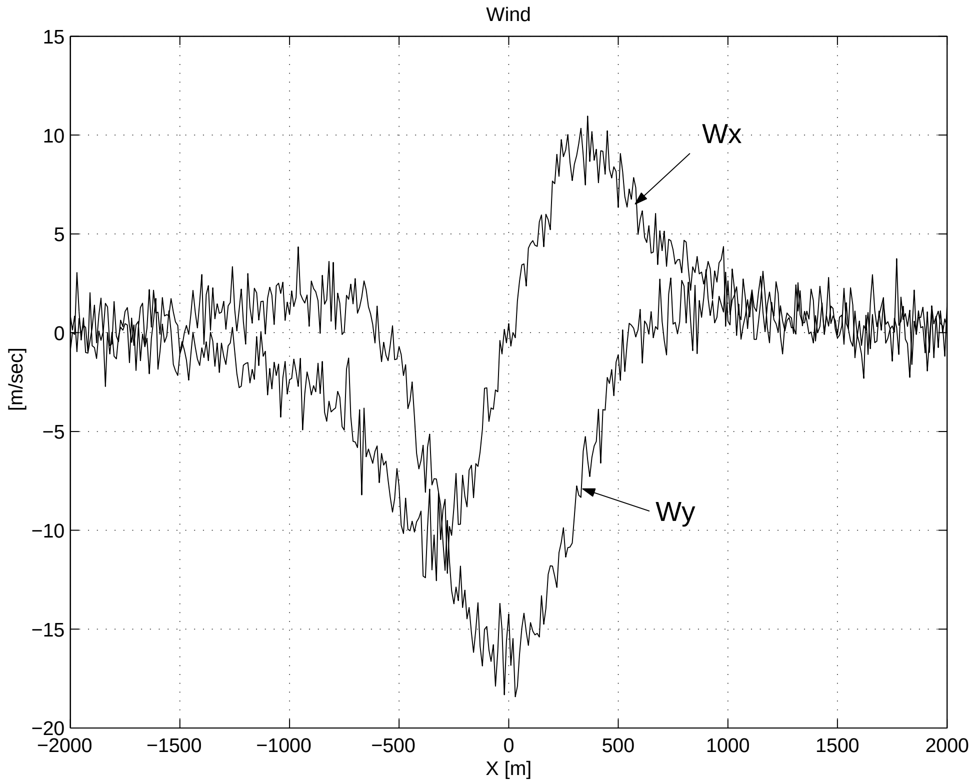


Рис. 6: Plots of vertical and horizontal ingredients of wind share profile

Observation coordinates with different types of controllers

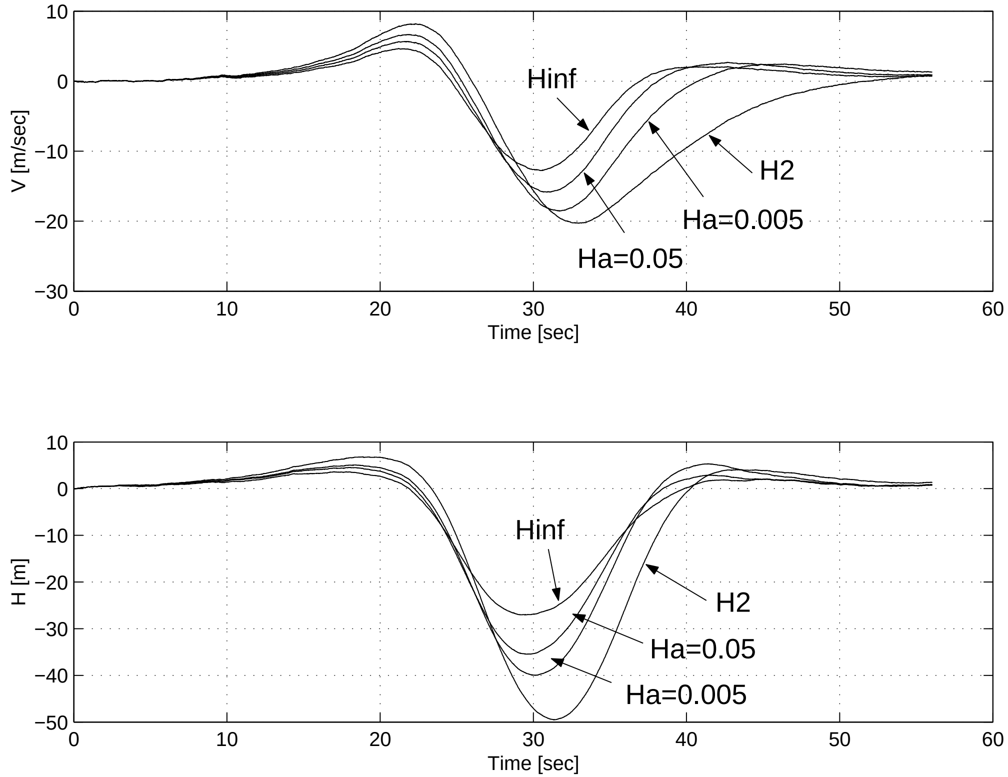


Рис. 7: Plots of V и h for different types of controllers

Control for different types of controllers

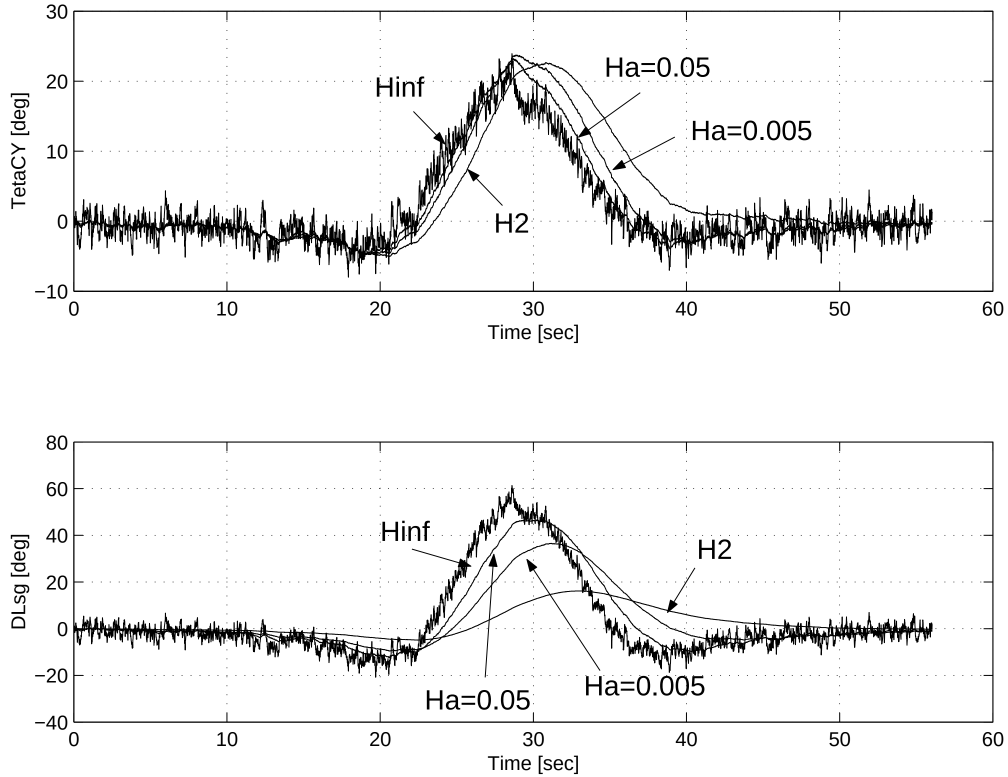


Рис. 8: Control for different types of feedback controllers

Anisotropic-based optimal control problem for the systems with parametric uncertainties.

Problem 2 *For system F, given by*

$$\begin{cases} x_{k+1} &= (A + F_1\Omega_k E_1)\, x_k + (B_0 + F_2\Phi_k E_2)w_k + (B_2 + F_3\Psi_k E_3)u_k, \\ z_k &= C_1 x_k + D_{12}u_k, \\ y_k &= C_2 x_k + D_{21}w_k, \end{cases} \tag{44}$$

where Ω_k, Φ_k, Ψ_k are unknown with conditions:

$$\Omega_k^\top \Omega_k \le I, \quad \Phi_k^\top \Phi_k \le I, \quad \Psi_k^\top \Psi_k \le I, \quad -\infty < k < +\infty, \tag{45}$$

and for given level of mean anisotropy to find the controller, that minimized

$$J_0(K) = \sup_{\Omega_k, \Phi_k, \Psi_k} \left\| \mathcal{F}_l(F, K) \right\|_a . \tag{46}$$

Solution of anisotropic-based design problem with parametric uncertainties

The solution of the problem is reduced to the solving of four algebraic matrix Riccati equations, Lyapynov equation and one algebraic equation of special type.

End of the Past Period

Present of the theory

- Anisotropy-based theory for descriptor systems.
  Analysis and synthesis problems.
- Suboptimal anisotropy- based problem.
- KYP lemma for suboptimal problem.
- LMI methods in Anisotropic theory.
  Semidefinite programming in Anisotropic theory.

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Anisotropic-based optimal problem for descriptor systems

$$\begin{cases} Ex(k+1) = Ax(k) + B_1 w(k) + B_2 u(k) \\ z(k) = C_1 x(k) + D_{11} w(k) + D_{12} u(k) \\ y(k) = C_2 x(k) + D_{21} w(k) + D_{22} u(k) \end{cases} \tag{47}$$

$rank(E) = r < n.$

Problem 3 *For given system (57) and mean anisotropy level $a \geq 0$ of W find K minimizing a- anisotropy norm of closed loop system :*

$$|||\mathcal{F}_l(F,K)|||_a = \sup\left\{ \frac{\|\mathcal{F}_l(F,K)G\|_2}{\|G\|_2} : G \in \mathbb{G}_a \right\} \to \inf, \tag{48}$$

$\mathcal{F}_l(F,K)$ *is low linear fractional transformation.*

Anisotropic-based suboptimization problem

Let F be describe by

$$\left\{\begin{array}{lllllll} x_{k+1} & = & Ax_k & + & B_1 w_k & + & B_2 u_k \\ z_k & = & C_1 x_k & + & D_{11} w_k & + & D_{12} u_k \\ y_k & = & C_2 x_k & + & D_{21} w_k & & \end{array}\right. , \quad -\infty < k < +\infty\,, \quad (49)$$

Problem 4 *For given system F and mean anisotropy level $a \geq 0$ of input disturbance W find the controller $K \in \mathcal{K}$, that minimizes the a-anisotropic norm of closed loop system $\mathcal{F}_l(F, K)$:*

$$\|\mathcal{F}_l(F,K)\|_a \equiv \sup\left\{\frac{\|\mathcal{F}_l(F,K)G\|_2}{\|G\|_2} : \; G \in \mathbb{G}_a\right\} \leq \gamma,, \quad K \in \mathcal{K}\,. \quad (50)$$

Anisotropic-based suboptimization analysis

Given $a \geq 0$, $\gamma > 0$, verify if (50) holds true

Bounded real lemma for anisotropic norm

Lemma 3

$$F \in \mathcal{H}_\infty^{p\times m} : \quad \begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \tag{51}$$

Given $a \geq 0$, $\gamma > 0$,

$$|||F|||_a < \gamma$$

if and only if there exists $\eta > \gamma^2$ *such that*

$$\eta - \left(e^{-2a} \det\left(\eta I_m - \mathcal{B}^\top \Phi \mathcal{B} - \mathcal{D}^\top \mathcal{D}\right)\right)^{1/m} < \gamma^2 \tag{52}$$

holds true for a real $(n \times n)$*-matrix* $\Phi = \Phi^\top \succ 0$,

$$\begin{bmatrix} \mathcal{A}^\top \Phi \mathcal{A} - \Phi + \mathcal{C}^\top \mathcal{C} & \mathcal{A}^\top \Phi \mathcal{B} + \mathcal{C}^\top \mathcal{D} \\ \mathcal{B}^\top \Phi \mathcal{A} + \mathcal{D}^\top \mathcal{C} & \mathcal{B}^\top \Phi \mathcal{B} + \mathcal{D}^\top \mathcal{D} - \eta I_{m_w} \end{bmatrix} \prec 0 \tag{53}$$

Lemma 4 *(reciprocal matrices) [Tchaikovsky et al., 2011] Given* $F \in \mathcal{H}_\infty^{p\times m}$ *with realization (51),* $a \geq 0$, $\gamma > 0$,

$$\|F\|_a < \gamma$$

if there exists $\eta > \gamma^2$ *such that*

$$\eta - (e^{-2a} \det \Psi)^{1/m} < \gamma^2 \tag{54}$$

holds true for real $(m \times m)$*-matrix* $\Psi = \Psi^\top \succ 0$ *and* $(n \times n)$ *reciprocal matrices* $\Phi = \Phi^\top \succ 0$, $\Pi = \Pi^\top \succ 0$ *that satisfy*

$$\begin{bmatrix} \Psi - \eta I_m & B^\top & D^\top \\ B & -\Pi & 0 \\ D & 0 & -I_p \end{bmatrix} \prec 0 \quad \begin{bmatrix} -\Phi & 0 & A^\top & C^\top \\ 0 & -\eta I_m & B^\top & D^\top \\ A & B & -\Pi & 0 \\ C & D & 0 & -I_p \end{bmatrix} \prec 0 \tag{55}$$

$$\Phi\Pi = I_n \tag{56}$$

Future of the theory

- Suboptimal problem for descriptor systems.
- How to find generating filter for concrete mean anisotropy level. Signal processing problem.
- How to extend anisotropic theory to some non linear systems. Absolute stability.
- Adaptive anisotropy-based control.
- Anisotropy based theory for the systems with non-zero mean of input disturbance.

Anisotropic-based suboptimal problem for descriptor system

$$\begin{cases} Ex(k+1) = Ax(k) + B_1 w(k) + B_2 u(k) \\ z(k) = C_1 x(k) + D_{11} w(k) + D_{12} u(k) \\ y(k) = C_2 x(k) + D_{21} w(k) + D_{22} u(k) \end{cases} \tag{57}$$

$rank(E) = r < n.$

Problem 5 *For given system (57) and mean anisotropy level $a \geq 0$ of W find K minimizing a- anisotropy norm of closed loop system :*

$$|||\mathcal{F}_l(F,K)|||_a = \sup\left\{\frac{\|\mathcal{F}_l(F,K)G\|_2}{\|G\|_2} : G \in \mathbb{G}_a\right\} \leq \gamma, \tag{58}$$

$\mathcal{F}_l(F,K)$ is low linear fractional transformation.

Creation of stochastic sequence with given property

Problem 6 *Let a level of mean anisotropy a of sequence $\{w_k\}$ be given. Sequence $\{w_k\}$ is received from white noise by filter*

$$\begin{cases} x_{k+1} = Ax_k + Bv_k, \\ w_k = Cx_k + Dv_k, \end{cases} \qquad (59)$$

где $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$.

Matrix A is stable and D is not singular, e.g. $\rho(A) < 1, \quad \det D \neq 0$.

Find the matrixes A, B, C, D.

Adaptive anisotropy-based control

Problem 7 *Let a level of mean anisotropy a of sequence* $\{w_k\}$ *be given. Sequence* $\{w_k\}$ *is received from white noise by filter*

$$\begin{cases} x_{k+1} = Ax_k + Bv_k, \\ w_k = Cx_k + Dv_k, \end{cases} \tag{60}$$

где $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$.

Matrix A is stable, $\rho(A) < 1$.

We have found the solution of anisotropy-based optimal or suboptimal problem.

The mean anisotropy level was changed. We can the opportunity to have information about new level of mean anisotropy. Find the law for modification the control.

VERY DIFFICULT PROBLEM

How to extend anisotropic theory for continue - time systems.