MATHEMATICAL BEHAVIORAL FINANCE

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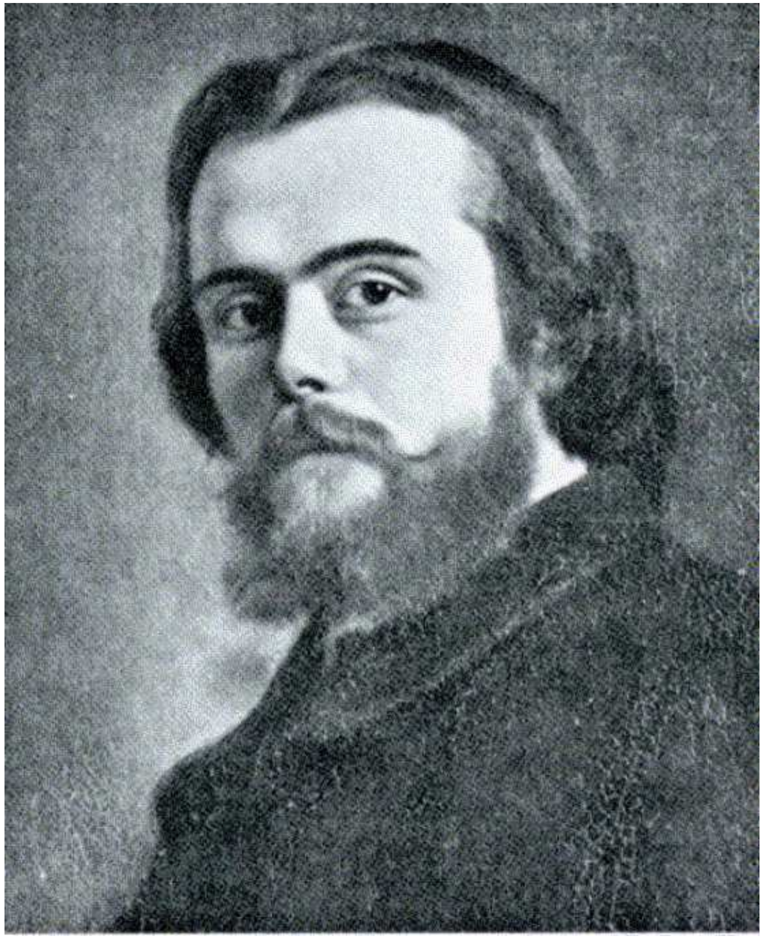
The goal of this work is to develop a new theory of asset market dynamics and equilibrium - **a plausible alternative to the classical General Equilibrium theory** (Walras, Arrow, Debreu, Radner and others).

Walrasian Equilibrium

Conventional models of equilibrium and dynamics of asset markets are based on the principles of **Walrasian General Equilibrium theory**.

This theory typically assumes that market participants are **fully rational** and act so as to **maximize utilities** of consumption subject to budget constraints.

Leon Walras (1834 – 1910)



Mathematical General Equilibrium models:

Arrow, **Debreu**

Introducing stochastics and dynamics: **temporary equilibrium**
Grandmont, **Hildenbrand**

Radner: **equilibrium in incomplete asset markets**

Standard texts: **Duffie**, **Magill and Quinzii**

Fundamental drawbacks of the Walrasian General Equilibrium theory for financial markets

hypothesis of “**perfect foresight**”

indeterminacy of temporary equilibrium

coordination of plans of market participants (which contradicts the very idea of decentralization)

the use of **unobservable agent**’**s characteristics** – individual utilities and beliefs

BEHAVIOURAL EQUILIBRIUM

We develop an alternative equilibrium concept – **behavioural equilibrium**, admitting that market actors may have different patterns of behaviour determined by their individual psychology, which are **not necessarily describable in terms of individual utility maximization**.

Strategies may involve, for example, mimicking, satisficing, rules of thumb based on experience, etc. Strategies might be **interactive** – depending on the behaviour of the others.

Objectives might be of an **evolutionary** nature: **survival** (especially in crisis environments), **domination** in a market segment, fastest capital **growth**, etc. They might be **relative** – taking into account the performance of the others.

Sources:

Behavioural economics – studies at the interface of psychology and economics: Tversky, Kahneman, Smith, Shleifer (1990s). The 2002 Nobel Prize in Economics: Kahneman and Smith

Behavioural finance: Shiller, Thaler, Hens. The 2013 Nobel Prize in Economics: Shiller.

Modelling framework:

synthesis of **stochastic dynamic games** and **evolutionary game theory**

THE BASIC MODEL

Randomness.

S space of "states of the world" (a measurable space);

$s_t \in S$ ($t = 1, 2, \dots$) state of the world at date t;

$s_1, s_2, \dots$ an exogenous stochastic process.

Assets. There are K *assets* $k = 1, \dots, K$.

Dividends. At each date t, assets pay *dividends* $D_{t,k}(s^t) \geq 0$ depending on the history

$$s^t := (s_1, \dots, s_t)$$

of the states of the world up to date t.

Asset supply. Total mass (the number of "physical units") of asset k available at each date t is $V_k > 0$.

Investors and their portfolios

N **investors** $i \in \{1, \ldots, N\}$

Investor i's **portfolio** at date $t = 0, 1, 2, \ldots$

$$x_t^i = (x_{t,1}^i, \ldots, x_{t,K}^i)$$

$x_{t,k}^i$ is the number of ("physical") **units** of asset k

x_t^i for $t \geq 1$ depends on the **history of states of the world**:

$$x_t^i = x_t^i(s^t), \; s^t = (s_1, \ldots, s_t)$$

Asset prices

$p_t = (p_{t,1}, \ldots, p_{t,K}) \in \mathbb{R}_+^K$ vector of equilibrium **asset prices** at date t (equilibrium will be defined later)

coordinate $p_{t,k}$ of p_t stands for the **price of one unit of asset** k at date t

Prices **depend on the history** of states of the world:

$$p_{t,k} = p_{t,k}(s^t), \; s^t = (s_1, \ldots, s_t)$$

Market value of the investor i's portfolio x_t^i at date t:

$$\langle p_t, x_t^i \rangle := \sum_{k=1}^{K} p_{t,k} x_{t,k}^i$$

Investors' budgets

At date $t = 0$ investors $i = 1, 2, \ldots, N$ have **initial endowments** $w_0^i > 0$.

Investor i's **budget** at date $t \geq 1$ is

$$B_t^i(p_t, x_{t-1}^i) := \langle D_t + p_t, x_{t-1}^i \rangle,$$

where

$$D_t(s^t) := (D_{t,1}(s^t), \ldots, D_{t,K}(s^t)).$$

The budget consists of two components:

dividends $\langle D_t(s^t), x_{t-1}^i \rangle$ + **market value** $\langle p_t, x_{t-1}^i \rangle$

Investment rate: fraction α of the budget is invested into assets.

Investment proportions

Investor i allocates wealth across the assets according to a vector of **investment proportions**

$$\lambda_t^i = (\lambda_{t,1}^i, \ldots, \lambda_{t,K}^i) \geq 0, \ \sum_k \lambda_{t,k}^i = 1,$$

depending on the history s^t:

$$\lambda_t^i = \lambda_t^i(s^t).$$

Demand function of investor i:

$$X_{t,k}^i(p_t, x_{t-1}^i) = \frac{\alpha \lambda_{t,k}^i B_t^i(p_t, x_{t-1}^i)}{p_{t,k}}$$

(α is the **investment rate**)

SHORT-RUN EQUILIBRIUM

(**Aggregate**) **market demand**:

$$\sum_{i=1}^{N} X_{t,k}^{i}(p_t, x_{t-1}^{i})$$

Short-run equilibrium at date t:

demand = **supply**

$$\Updownarrow$$

$$\sum_{i=1}^{N} X_{t,k}^{i}(p_t, x_{t-1}^{i}) = V_k \; , k = 1, 2, \ldots, K$$

$$\Updownarrow$$

$$\sum_{i=1}^{N} \frac{\alpha \lambda_{t,k}^{i} B_t^{i}(p_t, x_{t-1}^{i})}{p_{t,k}} = V_k, \; k = 1, 2, \ldots K$$

EQUILIBRIUM MARKET DYNAMICS

Given the vectors of investment proportions λ_t^i, $t = 0, 1, \ldots$ of investors $i = 1, 2, \ldots, N$, the equilibrium market dynamics are described by the equations:

Prices:

$$p_{t,k} V_k = \sum_{i=1}^{N} \alpha \lambda_{t,k}^i \langle D_t(s^t) + p_t, x_{t-1}^i \rangle, \; k = 1, \ldots, K.$$

Portfolios:

$$x_{t,k}^i = \frac{\alpha \lambda_{t,k}^i \langle D_t(s^t) + p_t, x_{t-1}^i \rangle}{p_{t,k}}, \; k = 1, \ldots, K, \; i = 1, 2, \ldots, N.$$

DYNAMICS OF MARKET SHARES

We are primarily interested in the dynamics of **relative wealth** (**market share**) of each investor i:

$$r_t^i = \frac{w_t^i}{w_t^1 + \ldots + w_t^N},$$

where $w_t^i := \langle D_t + p_t, x_{t-1}^i \rangle$ is **investor** i**'s wealth**. The dynamics of the vectors $r_t = (r_t^1, \ldots, r_t^N)$ are described by the random dynamical system

$$r_{t+1}^i = \sum_{k=1}^{K} [\alpha \langle \lambda_{t+1,k}, r_{t+1} \rangle + (1 - \alpha) R_{t+1,k}] \frac{\lambda_{t,k}^i r_t^i}{\langle \lambda_{t,k}, r_t \rangle},$$

$i = 1, \ldots, N,\ t \geq 0$, where

$$R_{t,k} = R_{t,k}(s^t) := \frac{D_{t,k}(s^t) V_k}{\sum_{m=1}^{K} D_{t,m}(s^t) V_m}$$ (**relative dividends**).

STRATEGIC FRAMEWORK

How do investors select their investment proportions?

To describe this we use a **game-theoretic approach**: decisions of players are specified by their **strategies**.

> ***A strategy in a stochastic dynamic game is a rule prescribing how to act based on information about all the previous actions of the player and his rivals, as well as information about the observed random states of the world.***

Definition of a strategy

Players are investors.

Player i's **action** or **decision** at date t is the vector of his investment proportions λ_t^i.

Definition. A **strategy** (**portfolio rule**) Λ^i of investor i is a sequence of measurable mappings

$$\Lambda_t^i(s^t, H^{t-1}),\ t = 0, 1, \ldots,$$

assigning to each history $s^t := (s_1, \ldots, s_t)$ of states of the world and each history of the game

$$H^{t-1} := \{\lambda_m^i : i = 1, \ldots, N,\ m = 0, \ldots, t-1\}$$

the vector of investment proportions $\Lambda_t^i(s^t, H^{t-1})$.

Basic strategies

Basic strategies are those for which $\Lambda_t^i = \Lambda_t^i(s^t)$ ***does not depend on*** H^{t-1}.

They require substantially less information than general strategies!

PATH OF THE GAME

Suppose all the investors $i = 1, 2, \ldots, N$ selected and fixed their strategies (portfolio rules) $\Lambda_t^i(s^t, H^{t-1})$, $t = 0, 1, \ldots$

Suppose they apply the strategies step by step at $t = 0, 1, 2, \ldots$.

The unfolding dynamic game generates:

♦ sequence of **investment proportions** $\lambda_0^i, \lambda_1^i, \ldots$ for each investor i;

♦ sequence of **equilibrium prices** $p_0, p_1, \ldots$;

♦ sequence of **portfolios** $x_0^i, x_1^i, \ldots$ for each investor i;

♦ sequence of **market shares**

$$r_t^i = \frac{w_t^i}{w_t^1 + \ldots + w_t^N},\ t = 0, 1, \ldots,$$

for each investor i. [Recall: $w_t^i := \langle D_t(s^t) + p_t, x_{t-1}^i \rangle$.]

SURVIVAL STRATEGIES

Central question:

> ***Does there exist a strategy that guarantees survival (in the market selection process) of the investor using it, irrespective of what strategies are used by the other investors?***

Survival of investor i (or the strategy used by i) means keeping a.s. a strictly positive, bounded away from zero share of market wealth over an infinite time horizon:

$$\inf_{t \geq 0} r_t^i > 0 \text{ (a.s.).}$$

A survival strategy exists and it is asymptotically unique !

DESCRIPTION OF THE SURVIVAL STRATEGY

Relative dividends. Define the *relative dividends* of the assets $k = 1, \ldots, K$ by

$$R_{t,k} = R_{t,k}(s^t) = \frac{D_{t,k}(s^t)V_k}{\sum_{m=1}^K D_{t,m}(s^t)V_m}, \; k = 1, \ldots, K, \; t \geq 1,$$

and define $R_t(s^t) = (R_{t,1}(s^t), \ldots, R_{t,K}(s^t))$.

Survival strategy Λ^*. Put

$$\alpha_l = \alpha^{l-1}(1-\alpha),$$

$$\lambda^*_{t,k} = E_t \sum_{l=1}^{\infty} \alpha_l R_{t+l,k}\,. \quad [\, E_t(\bullet) \; = E(\bullet | s^t). \,]$$

The strategy Λ^* is the basic strategy $\lambda^*_0, \lambda^*_1(s^1), \lambda^*_2(s^2), \ldots$ defined by

$$\lambda^*_t(s^t) = (\lambda^*_{t,1}(s^t), \ldots, \lambda^*_{t,K}(s^t)).$$

Assume $\lambda^*_{t,k} > 0$ (a.s.)

Theorem 1. ***The portfolio rule Λ^* is a survival strategy***.

We emphasize that the strategy Λ^* is basic, and it survives in competition with **any** (**not necessarily basic**) strategies!

In the class of ***basic*** strategies, the survival strategy Λ^* is **asymptotically unique**:

Theorem 2. *If $\Lambda = (\lambda_t)$ is a basic survival strategy, then*

$$\sum_{t=0}^{\infty} \|\lambda^*_t - \lambda_t\|^2 < \infty \text{ (a.s.).}$$

The meaning of Λ^*. The portfolio rule Λ^* defined by

$$\lambda^*_{t,k} = E_t \sum_{l=1}^{\infty} \alpha_l R_{t+l,k} \ ,$$

combines three general principles in Financial Economics.

(a) Λ^* prescribes the allocation of wealth among assets in the proportions of their **fundamental values** – the expectations of the flows of the discounted future dividends.

(b) The strategy Λ^*, defined in terms of the **relative** (**weighted**) **dividends**, is analogous to the CAPM strategy involving investment in the **market portfolio**.

(c) The portfolio rule Λ^* is closely related (and in some special cases reduces) to the **Kelly portfolio rule** prescribing to maximize the expected logarithm of the portfolio return.

The i.i.d. case

If $s_t \in S$ are **independent and identically distributed** (i.i.d.) and

$$R_{t,k}(s^t) = R_k(s_t),$$

then

$$\lambda^*_{t,k} = \lambda^*_k = ER_k(s_t),$$

does not depend on t, and so Λ^* is a **fixed-mix** (**constant proportions**) **strategy**.

It is independent of the investment rate α !

Global evolutionary stability of Λ^*

Consider the i.i.d. case in more detail. It is important for quantitative applications and admits a deeper analysis of the model. Let us concentrate on fixed-mix (constant proportions) strategies. In the class of such strategies, Λ^* is **globally evolutionarily stable**:

Theorem 3. *If among the N investors, there is a group using Λ^*, then those who use Λ^* survive, while all the others are driven out of the market (their market shares tend to zero a.s.).*

COMMENTS ON THE MODEL

Marshallian temporary equilibrium.

We use the Marshallian “moving equilibrium method,” to model the dynamics of the asset market as a sequence of consecutive temporary equilibria.

To employ this method one needs to distinguish between at least two sets of economic variables changing with different speeds.

Then the set of variables changing slower (in our case, the set of vectors of investment proportions) can be temporarily fixed, while the other (in our case, the asset prices) can be assumed to rapidly reach the unique state of partial equilibrium.

Alfred Marshall, Principles of Economics, 1920.

Samuelson (1947), describing the Marshallian approach, writes:

> I, myself, find it convenient to visualize equilibrium processes of quite different speed, some very slow compared to others. Within each long run there is a shorter run, and within each shorter run there is a still shorter run, and so forth in an infinite regression. For analytic purposes it is often convenient to treat slow processes as data and concentrate upon the processes of interest. For example, in a short run study of the level of investment, income, and employment, it is often convenient to assume that the stock of capital is perfectly or sensibly fixed.

Samuelson thinks about a *hierarchy* of various equilibrium processes with different speeds. In our model, it is sufficient to deal with only two levels of such a hierarchy.

GAME-THEORETIC ASPECTS

Synthesis of evolutionary and dynamic games

The notion of a survival strategy is the **solution concept** we adopt in the analysis of the market game.

This is a solution concept of a ***purely evolutionary nature.***

No utility maximization or Nash equilibrium is involved.

On the other hand, the strategic framework we consider is the one characteristic for **stochastic dynamic games** (Shapley 1953).

IN ORDER TO SURVIVE YOU HAVE TO WIN!

Equivalence of Survival and Unbeatable Strategies

One might think that the focus on survival substantially restricts the scope of the analysis: "one should care of survival only if things go wrong".

It turns out, however, that the class of survival strategies coincides with the class of **unbeatable** strategies performing not worse in the long run – **in terms of wealth accumulation** – than any other strategies competing in the market.

Thus,

in order to survive you have to win!

Unbeatable strategies of capital accumulation

Definition. A strategy Λ is called **unbeatable** if it has the following property:

Suppose investor i uses the strategy Λ, while all the others $j \neq i$ use **any** strategies. Then the wealth process w_t^j of every investor $j \neq i$ cannot grow asymptotically faster than the wealth process w_t^i of investor i: $w_t^j \leq H w_t^i$ (a.s.) for some random constant H.

It is quite easy to show that **a strategy is a survival strategy if and only if it is unbeatable**.

Unbeatable strategies: **a general definition**

An abstract game of N players $i = 1, \ldots, N$ selecting strategies $\Lambda^i \in \mathrm{L}^i$.

$w^i = w^i(\Lambda^1, \ldots, \Lambda^N) \in \mathrm{W}$ the outcome of the game for player i given the strategy profile $(\Lambda^1, \ldots, \Lambda^N)$.

A preference relation

$$w^i \succcurlyeq w^j, w^i, w^j \in \mathrm{W},\ \ i \neq j,$$

is given, comparing relative performance of players i and j.

Definition. A strategy Λ of player i is **unbeatable** if for any admissible strategy profile $(\Lambda^1, \Lambda^2, \ldots, \Lambda^N)$ in which $\Lambda^i = \Lambda$, we have

$$w^i(\Lambda^1, \Lambda^2, \ldots, \Lambda^N) \succcurlyeq w^j(\Lambda^1, \Lambda^2, \ldots, \Lambda^N) \text{ for all } j \neq i.$$

Thus, if player i uses Λ, he/she **cannot be outperformed by any of the rivals** $j \neq i$, **irrespective of what strategies they employ**.

Unbeatable strategies of capital accumulation

In our model:

♦ **An outcome of the game for player** i: wealth process $w^i = (w_t^i)$.

♦ **Preference relation** $\preccurlyeq$. For two sequences of positive random numbers (w_t^i) and (w_t^j), we define

$$(w_t^i) \preccurlyeq (w_t^j) \text{ iff } w_t^i \leq H w_t^j \text{ (a.s.)}$$

for some random $H > 0$.

The relation $w^i \preccurlyeq w^j$ means that (w_t^i) **does not grow asymptotically faster than** (w_t^j) (**a.s.**).

Comment on the **infinite time horizon modelling approach**

Most common in economic modelling, especially in Macroeconomics. *David Gale* (Rev. Econ. Stud., 1967) wrote:

In this connection there is one objection which is frequently raised to considering an infinite time horizon. Even the most omniscient planner, it is argued, cannot know what the technological situation will be after ten years. What nonsense it is then to be planning hundreds and thousands of years into the future. The answer is, perhaps somewhat paradoxically, that infinite horizon plans do not, as one might think, involve making sensitive decisions about what to do hundreds of years hence. On the contrary, all reasonably good plans will look approximately the same for large values of time, approaching some fixed " balanced " path arbitrarily closely, as we shall see. On the other hand, the choice of infinite plan will affect very crucially what one does, say during the first five years, and even more sensitively what one does tomorrow. To describe the situation figuratively, one is guiding a ship on a long journey by keeping it lined up with a point on the horizon even though one knows that long before that point is reached the weather will change (but in an unpredictable way) and it will be necessary to pick a new course with a new reference point, again on the horizon rather than just a short distance ahead.

Pre-von Neumann / Pre-Nash game theory

The notion of a **winning** or **unbeatable** strategy was a central solution concept in the pre-von Neumann and pre-Nash game theory (as a branch of mathematics, pioneered by **Bouton**, **Zermelo**, **Borel**, 1900s - 1920s).

The question of **determinacy** of a game (existence of a winning strategy for one of the players) was among the key topics in game theory until 1950s: **Gale**, **Stewart**, **Martin** ("Martin’s axiom").

The first mathematical paper in game theory "solving" a game (= finding a winning strategy) was:

Bouton, **C**. **L**. (1901-2) Nim, a game with a complete mathematical theory, **Annals of Mathematics**, 3, 35–39.

Unbeatable strategies and evolutionary game theory

The basic solution concepts in evolutionary game theory – **evolutionary stable strategies** (Maynard Smith & Price, Schaffer) – may be regarded as "**conditionally**" **unbeatable strategies** (the number of mutants is small enough, or they are identical).

Unconditional versions: Kojima (2006).

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The model described was developed in:

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The results presented were obtained (in a more general form) in:

R. Amir, I.V Evstigneev, T. Hens, L. Xu, Evolutionary finance and dynamic games, **Mathematics and Financial Economics** (2011).

Versions of the model

Short-lived assets

Assets live one period, yield payoffs, and then are identically reborn at the beginning of the next period. A simplified version of the basic model (reduces to it when $\alpha \to 0$).

I.V. Evstigneev, T. Hens, K.R. Schenk-Hoppé, Market selection of financial trading strategies: Global stability, **Mathematical Finance** (2002).

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