Random symmetrizations of convex bodies

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October 23, 2013

Steiner symmetrization

Let A be a convex body (i.e. a convex compact set with non empty interior) of $\mathbb{R}^d$. Let $u \in \mathbb{S}^{d-1}$ be a unit vector.

Definition

The Steiner symmetral S_uA of A with direction u is obtained as follows. For each straight line L parallel to u and s.t. $L \cap A \neq \emptyset$, shift the line segment $L \cap A$ along L until its midpoint is in $u^\perp$.

- S_uA is still a convex body.
- The Steiner symmetrization S_u conserves the volume and reduces the surface area:

"S_uA is more round than A in the isoperimetric sense."

State of the Art

- Steiner (1796 - 1863).

Let D be the unit (closed) ball and A be a convex body s.t. $vol(A) = vol(D)$.

Theorem (Gross -1917)

There exists a sequence $(A_n)_{n\geq 1}$ of convex bodies, each obtained from A by finitely many successive Steiner symmetrizations, s.t. $A_n \to D$.

$\rightsquigarrow$ A direct proof of the Isoperimetric Inequality.

State of the Art: more recently...

→ Deterministic directions

- Klain -2011.
- Bianchi, Lutwak, Klain, Yang and Zhang -2011.

→ Random directions

- Mani-Levitska -1986.
- Volčič -2009.
- Burchard and Fortier -2011.

→ Rate of convergence

- Bourgain, Lindenstrauss and Milman -1989.
- Klartag -2004.

State of the Art: Random Steiner symmetrizations

Theorem A (Mani-Levitska, 1986)

Let A be a convex body in $\mathbb{R}^d$, $vol(A) = vol(D)$, $D = B(0,1)$.
Let $\{u_k\}$ be i.i.d. uniformly distributed directions.
Then a.s.

$$d_H(S_n A,\ D) \to 0,$$

where d_H is Hausdorff distance.

Corollary. Solution of isoperimeter problem.

Minkowski symmetrization

Definition

The Minkowski symmetral B_uA of the convex body A with direction u is defined by:

$$B_uA = \frac{1}{2}(A + \pi_u(A)) ,$$

where π_u denotes the orthogonal reflection operator with respect to $u^\perp$.

- B_uA is still a convex body.
- Let f_A be the support function of A

$$f_A(\theta) = \sup_{x \in A} \langle x, \theta \rangle, \quad \text{for any } \theta \in S^{d-1} .$$

and $L(A) = \int_{\mathbb{S}^{d-1}} f_A d\sigma$ be the mean radius of A. Then,

$$L(B_uA) = L(A) .$$

Rate of convergence for Minkowski

We set

$$B_n A = B_{U_n} \circ \ldots \circ B_{U_2} \circ B_{U_1} A \ ,$$

where $U_k \in \mathbb{S}^{d-1}$, $k \geq 1$ are independent random directions.

Theorem

Assume that, for any $k \geq 1$, the distribution ν_k of U_k satisfies

$$\frac{d\nu_k}{d\sigma}(u) \leq \alpha < \frac{d}{d-1}$$

for some $\alpha > 0$ and $\sigma-$a.e. $u \in \mathbb{S}^{d-1}$. Then, there exists a constant $c > 0$ s.t. with probability 1,

$$\exists n_0(\omega), \ \forall n \geq n_0, \ d_H\left(B_n A, L(A)D\right) \leq e^{-cn} \ .$$

Contraction for Minkowski

Let h_A be the centered support function of A: $h_A = f_A - L(A)$.

Remark; $h_A \equiv 0 \Leftrightarrow A = L(A)D$.

Proposition

Let U be a random variable of $\mathbb{S}^{d-1}$ with distribution σ. Then,

$$\mathbb{E}\|h_{B_U A}\|_2^2 \leq \frac{d-1}{d}\|h_A\|_2^2 .$$

It implies exponential rate of convergence for Minkowski:

- $\rightarrow$ $\mathbb{E}\|h_{B_n A}\|_2 \searrow 0$ at rate exponential.
- $\rightarrow$ Idem for $\mathbb{E}\|h_{B_n A}\|_\infty$ since $h_{B_n A}$ is lipschitz.
- $\rightarrow$ Borel-Cantelli lemma.
- $\rightarrow$ $d_H(B_n A, L(A)D) = \|f_{B_n A} - f_{L(A)D}\|_\infty = \|f_{B_n A} - L(A)\|_\infty = \|h_{B_n A}\|_\infty$.

Proof of contraction when $d = 2$

Writing $h_{B_u A} = \frac{1}{2}(h_A + h_{\pi_u A})$, we develop

$$\|h_{B_u A}\|_2^2 = \frac{1}{4}\|h_A\|_2^2 + \frac{1}{2}\langle h_A, h_{\pi_u A}\rangle + \frac{1}{4}\|h_{\pi_u A}\|_2^2 \, .$$

As σ is invariant under $v \mapsto \pi_u(v)$, $\|h_{\pi_u A}\|_2 = \|h_A\|_2$.

Moreover, <u>when $d = 2$</u>, σ is also invariant under $u \mapsto \pi_u(v)$, so

$$\int_{\mathbb{S}^1} h_A(\pi_u v) d\sigma(u) = \int_{\mathbb{S}^1} h_A(u) d\sigma(u) = 0$$

and

$$\mathbb{E}\langle h_A, h_{\pi_U A}\rangle = \int_{\mathbb{S}^1} h_A(v) \left(\int_{\mathbb{S}^1} h_A(\pi_u v) d\sigma(u) \right) d\sigma(v) = 0 \, .$$

Therefore $\mathbb{E}\|h_{B_u A}\|_2^2 = \frac{1}{2}\|h_A\|_2^2$.

But false when $d > 2$!!!

Proof of contraction when $d > 2$

We follow ideas of Klartag based on spherical harmonics. Let

$$\mathcal{S}_k = \{P_{|\mathbb{S}^{d-1}},\ P \text{ is a harmonic, homogeneous polynomial of degree } k\}\,.$$

Using the orthogonal direct sum decomposition $L_2(\mathbb{S}^{d-1}) = \bigoplus_{k\geq 0} \mathcal{S}_k$ we write,

$$h_A = \sum g_k \ \text{ and thus } \ h_{B_u A} = \sum B_u g_k$$

where $B_u g_k = \frac{1}{2}\left(g_k + g_k \circ \pi_u\right) \in \mathcal{S}_k$.

The orthogonal projection Theorem leads to

$$\mathbb{E}\|B_u g_k\|_2^2 = c(k)\|g_k\|_2^2$$

and Pythagoras' Theorem provides

$$\mathbb{E}\|h_{B_u A}\|_2^2 \leq \frac{d-1}{d}\|h_A\|_2^2$$

where $\frac{d-1}{d} = c(1) = \min c(k)$.

Rate of convergence for Steiner

We set

$$S_n A = S_{U_n} \circ \ldots \circ S_{U_2} \circ S_{U_1} A$$

where $U_k \in \mathbb{S}^{d-1}$, $k \geq 1$, are independent random directions, and A is s.t. $vol(A) = vol(D)$.

Theorem

Assume that, for any $k \geq 1$, the distribution ν_k of U_k satisfies

$$\frac{d\nu_k}{d\sigma}(u) \leq \alpha < \frac{d}{d-1}$$

for some $\alpha > 0$ and $\sigma-$a.e. $u \in \mathbb{S}^{d-1}$. Then, there exist $c, c' > 0$ (depending on d, A and α) s.t. with probability 1,

$$\exists n_0(\omega),\ \forall n \geq n_0,\ d_H(S_n A, D) \leq c e^{-c'\sqrt{n}}\ .$$

Comparison with other results

- The convergence occurs at rate $ce^{-c'\sqrt{n}}$ (as Klartag) and is almost sure (as Mani-Levitska, Volčič. . .).
- The first random integer n_0 satisfies

$$\mathbb{P}(n_0 > n) \leq ce^{-c'\sqrt{n}} .$$

- The random directions U_k may be non identically distributed and their distributions ν_k may avoid some open sets of $\mathbb{S}^{d-1}$ (unlike Volčič).
- The independence hypothesis between random directions U_k can be slightly weakened.

Proof

The proof is based on 3 ingredients:

- The only link between Steiner and Minkowki used here:

$$S_u A \subset B_u A \ .$$

- The exponential rate of convergence for Minkowki.
- A result of Bokowski and Heil (1986).
  Let $\varepsilon > 0$ and $K \subset (1+\varepsilon)D$ be a convex body having the same volume as D. Then,

$$L(K) \leq 1 + \left(1 - \frac{1}{d^2}\right)\varepsilon \ .$$

Theorem of equivalence

We say that (u_k) **strongly** (S)-rounds the set A if for each k

$$S_{k,k+n}A \to r(A)D, \quad n \to \infty,$$

and **strongly** (M)-rounds A if for each k

$$B_{k,k+n}A \to L(A)D, \quad n \to \infty.$$

The sequence (u_k) is called (S)-**unversal** (respectively (M)-**unversal**) if it is strongly (S)-rounds (respectively strongly (M)-rounds) all sets A from $\mathcal{K}^d$.

Theorem

The sequence (u_k) is (S)-universal if and only if it is (M)-universal.

Convergence to a random limit

We set

$$BS_nA = B_{U_{2n}} \circ S_{U_{2n-1}} \circ \ldots \circ B_{U_2} \circ S_{U_1} A$$

where $U_k \in \mathbb{S}^{d-1}$, $k \geq 1$, are independent random directions, and A is s.t. $vol(A) = vol(D)$.

From the inclusion $S_uA \subset B_uA$, it follows:

- $vol(A) = vol(S_uA) \leq vol(B_uA)$.
- $L(S_uA) \leq L(B_uA) = L(A)$.

Proposition

There exists a random real number $\rho \in [1; L(A)]$ s.t. with probability 1,

$$BS_nA \longrightarrow \rho D \ .$$

An idea: to use the contraction for Minkowski to prove exponential rate for the mixed symmetrization, and thus for Steiner...

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