Russian Academy of Sciences,

Steklov Mathematical Institute

V. V. Kozlov

Euler–Jacobi–Lie theorem

2013

Tensor invariants

$$\dot{x}_i = v^i(x_1, \ldots, x_n), \quad 1 \leqslant i \leqslant n,$$

$M = \{x_1, \ldots, x_n\}$ is the phase space.

Functions (first integrals), vector fields (symmetry fields), differential k-forms (generating integral invariants).

$v = (v^1, \ldots, v^n)$ is a trivial invariant.

It is useful to know $n-1$ nontrivial tensor invariants.

Examples:

1°. $f_1, \ldots, f_{n-1} \implies$ effective straightening out of the trajectories.

2°. Euler–Jacobi:

$$f_1, \ldots, f_{n-2} \ \text{ and } \ \Omega = \rho(x)\, dx_1 \wedge \cdots \wedge dx_n \ \ (\operatorname{div} \rho v = 0).$$

$$\dot{c}_1 = \cdots = \dot{c}_{n-2} = 0, \quad \dot{\varphi}_1 = \frac{\lambda_1}{\Phi}, \quad \dot{\varphi}_2 = \frac{\lambda_2}{\Phi};$$

$$\lambda_k = \text{const}, \quad \Phi\colon D \times \mathbb{T}^2 \to \mathbb{R}.$$

Euler ($n = 2$): the 1-form $i_v\Omega$ is closed, $= df$, f is a first integral.

3°. S. Lie: $u_1 = v$, u_2, ..., u_n generate a solvable Lie algebra (with respect to the bracket $[\cdot,\cdot]$).
An important example: $[u_i, u_j] = 0$, the fields $u_1, \dots, u_n$ are unconstrained on M and linearly independent at each point. Then $M \simeq \mathbb{T}^k \times \mathbb{R}^{n-k}$, and there exist coordinates $\varphi_1, \dots, \varphi_k \mod 2\pi$ and $y_{k+1}, \dots, y_n$ such that

$$\dot{\varphi}_j = \omega_j, \quad \dot{y}_s = l_s \quad (\omega, l = \text{const}).$$

A. Einstein (1917)
See: Ph. Frank and R. von Mises Die Differential- und Integralgleichungen der Mechanik und Physik. Braunschweig: Vieweg u, Sohn, 1930.
(From p. 90 of the Russian translation on.)

The main theorem

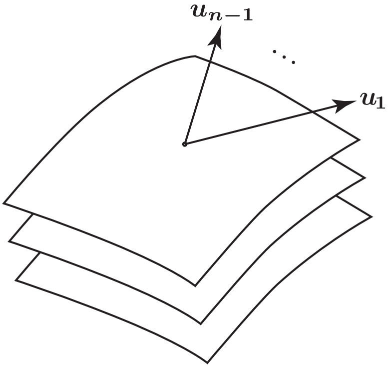
$n-2$ symmetry fields $u_2,\dots,u_{n-1}$: $[u_1,u_j]=0$, (here $u_1=v$)
$u_1,\dots,u_{n-1}$ are linearly independent (on an invariant subset).
$L_v\Omega=0$, where $L_v=i_vd+di_v$ is the Lie derivative

Теорема 1. *Suppose that*
1) $L_{u_s}\Omega=0$, $1\leqslant s\leqslant n-1$,
2) $u_1,\dots,u_{n-1}$ *generate a nilpotent Lie algebra.*
Then the original system of differential equations is integrable in quadratures.

An algebra g is nilpotent if there exists a chain of ideals

$$g=g_n\supset g_{n-1}\supset\cdots\supset g_0=\{0\}$$

such that $[g,g_i]\subset g_{i-1}$.
If $[u_i,u_j]=\sum c_{ij}^k u_k$, then $c_{ij}^k=0$ for $k\geqslant j$.
For $n=2$ we obtain Euler's result.
If $n=3$, then there is one symmetry field and $L_u\Omega=0$.



Leaves are locally level sets of a first integral. How can we find this integral?

Lemma 1. ***Let*** $d\Omega = 0$ ***and let*** $u_1, \dots, u_m \quad (m \leqslant n)$ ***be a system of vector fields such that*** $L_{u_s}\Omega = 0 \quad (1 \leqslant s \leqslant m)$***. Then***

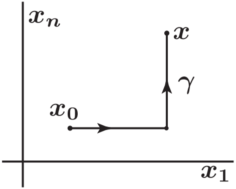
$$d(i_{u_m} \dots i_{u_2} i_{u_1} \Omega) = \sum_{i<j} \pm i_{[u_i, u_j]} i_{u_p} \dots i_{u_r} \Omega,$$

where $p, \dots, r$ ***is the subset of indices*** $1, \dots, m$ ***distinct from*** i ***and*** j***.***

Corollary 1. *If $[u_i, u_j] = \sum c_{ij}^k u_k$, where $c_{kj}^k = c_{ik}^k = 0$, then the differential form*

$$i_{u_m} \dots i_{u_1} \Omega$$

is closed.



Corollary 2. *If the fields $u_1, \dots, u_{n-1}$ generate a nilpotent algebra, then (locally)*

$$i_{u_{n-1}} \dots i_{u_1} \Omega = df$$

and f is a first integral.
If $H^1(M, \mathbb{R}) = 0$, then f is a single-valued function on the whole of M.

“The Euler–Jacobi–Lie theorem”. *Assume that the system has*

1) *k independent first integrals*

$$f_1, \dots, f_k,$$

2) *$n-k-2$ independent symmetry fields*

$$u_{k+1}, \dots, u_{n-2},$$

which together with $u_{n-1} = v$ generate a nilpotent algebra,

3) *an invariant volume form Ω, such that*

$$L_{u_j} f = 0, \qquad L_{u_j} \Omega = 0$$

for each j, $k+1 \leqslant j \leqslant n-2$.

Then the system is integrable in quadratures.

Remark. Here $0 \leqslant k \leqslant n-2$. For $k = n-2$ we obtain the Euler–Jacobi theorem; for $k = 0$ we obtain Theorem 1.

Regular systems

Again, $u_2, \dots, u_{n-1}$, $[u_1, u_j] = 0$ (where $u_1 = v$), $L_v\Omega = 0$, where Ω is a volume form on M. The fields $u_1, u_2, \dots, u_{n-1}$ generate an $(n-1)$-dimensional Lie algebra g; let G be the corresponding group, an $(n-1)$-dimensional transformation group of M.

Theorem 3. ***Assume that M is closed and the symmetry group G does not preserve the n-form Ω. Then the original system has a nontrivial first integral.***

◁ $L_{u_j}\Omega \neq 0$ for some $j \geqslant 2$. Then

$$L_{u_j}\Omega = \lambda\Omega, \qquad \lambda\colon M \to \mathbb{R}. \tag{$*$}$$

$L_v(\lambda\Omega) = \dot\lambda\Omega + \lambda L_v\Omega = \dot\lambda\Omega = L_v L_{u_j}\Omega = L_{u_j} L_v\Omega = 0$. Thus $\dot\lambda = 0$. If $\lambda \neq$ const, then the proof is complete. Assume that $\lambda =$ const. Then $d(i_{u_j}\Omega) = \lambda\Omega$ by $(*)$. Stokes' formula shows that

$$0 = \int_{\partial M} i_{u_j}\Omega = \int_M d(i_{u_j}\Omega) = \lambda \int_M \Omega.$$

Hence $\lambda = 0$, but then G preserves Ω. ▷

$n = 3$

S. V. Bolotin, V. V. Kozlov // Rus. J. Math. Phys., 1995, v. 3, no. 3, p. 279–295.

Connected regular level sets of the function λ are 2-tori, and there exist coordinates $\Psi_1, \Psi_2 \pmod{2\pi}$ and I in a neighbouhood of these tori such that the original system takes the form

$$\dot\Psi_1 = \frac{\omega_1(I)}{\rho}, \quad \dot\Psi_2 = \frac{\omega_2(I)}{\rho}, \quad \dot I = 0, \tag{$**$}$$

where $\rho(\Psi, I) > 0$ is a smooth function.

This is the Poincaré–Siegel–Kolmogorov theorem.

$\nu = \omega_1/\omega_2$ gives the rotation numbers $\Omega = \rho\, dI \wedge d\Psi_1 \wedge d\Psi_2$.

Theorem 4. ***Assume that there exists a symmetry field not preserving Ω and let C be the set of critical values of $\lambda\colon M \to \mathbb{R}$. Then***

1) ***$\nu = \omega_1/\omega_2$ is constant on each connected component D of the set of regular points $M \setminus \lambda^{-1}(C)$;***

2) ***if $\nu|_D$ is irrational, then ω_1 and ω_2 are constant on D.***

In other words, $(**)$ is a *degenerate system*.

$n = 3$ (continued)

Let $\rho = \sum \rho_{p_1 p_2}(I)\, e^{i(p_1 \Psi_1 + p_2 \Psi_2)}$ be the Fourier expansion of the density ρ.

Let $P = \{I\colon p_1\omega_1(I) + p_2\omega_2(I) = 0$ for some integers p_1 and p_2 such that $p_1^2 + p_2^2 \neq 0$, $\rho_{p_1 p_2}(I) \neq 0\}$.

Theorem 5. *Assume that*

1) $\omega_2 \neq 0$,

2) $\dfrac{d}{dI}\left(\dfrac{\omega_1}{\omega_2}\right) \neq 0$,

on some interval $I_1 < I < I_2$, *and*

3) P *is dense in* (I_1, I_2).

If u *is a symmetry field of* $(**)$ *in the domain* $\mathbb{T}^2 \times (I_1, I_2)$, *then* $u = \mu(I)v$, *where* μ *is a smooth function.*

In other words, generically, all the symmetry fields are trivial.

Magnetohydrodynamics

$\frac{\partial u}{\partial t} = \mathrm{rot}(v \times u), \quad v(x,t)$ is the velocity field,

$u(x,t)$ is the magnetic fueld,

$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho v) = 0, \quad \mathrm{div}\, u = 0.$

Stationary "vortex" flows: $u \times v \neq 0$.

$\dot{x} = v(x)$ gives the flow streamlines,

$\Omega = \rho(x)\, dx_1 \wedge dx_2 \wedge dx_3$ is the mass 3-form.

$w = u/\rho$ is a symmetry field (Prikl. Mat. Mech. 1983, v. 47, no. 2, p. 341–342; English transl. in J. Appl. Math. Mech.).

$L_w \Omega = 0$ since $\mathrm{div}(\rho w) = 0$.

Theorem 6. *Assume that the flow domain M is compact, simply connected, and bounded by a regular analytic surface, let v, u, ρ be analytic in M and $u \times v \neq 0$. Then the equation $\dot{x} = v(x)$ has a nonconstant analytic integral $f\colon M \to \mathbb{R}$ and almost all connected surfaces $B_c = \{x\colon f(x) = c\}$ (with a possible exception of finitely many) are diffeomorphic to a 2-torus (with conditionally periodic motion) or an annulus (with closed trajectories with the same period).*

V. I. Arnold. Prikl. Mat. Mech. 1966, v. 30, no. 1, p. 183–185; English transl. in J. Appl. Math. Mech.

In the case of a non-simply connected domain trajectories of some particles can be dense in the whole flow domain.