GEOMETRY OF CYCLIC COMPLEX NUMBERS

ВЛАДИМИРОВ-90- МИАН

- *A L G E B R A*
- *OF*
- *PROJECTIVE*
- *REFLEXIVE NUMBERS*

Dedicated to the memory
of the outstanding Russian mathematician-
to Vasilij Sergeevich Vladimirov
(09.01.1923-03.11.2012)

The Geometry of C_n - Normed - Division Algebras

VLADIMIROV-90 M 2013

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15.11.2013

1 Theories of Numbers in Geometry and in Physics.I

The further progress in modern models of elementary particles and cosmology is related to the searching for new Riemann and tensor structures in multidimensional spaces $D \geq 4$ based on the theories of new hyper-numbers, new algebras and new symmetries. The traditional geometry of Riemann and pseudo-Riemann symmetric homogeneous compact and non-compact spaces was associated to the classification of the Killing-Cartan-Lie algebras, according to the theories of binary hyper-numbers,- the well-known real-$\mathbb{R}$, complex- $\mathbb{C}$, quaternions- $\mathbb{H}$, octonions-$\mathbb{O}$- normed division algebras.

2 Theories of Numbers in Geometry and in Physics.II

A further prominent step was based on the construction of an alternative geometry of non-symmetric spaces using the holonomy group classification what has been suggested by Marceille Berger in 1955 [15]. Since that time there has been done enough big progress in the study the Riemannian manifolds over different normed division algebras. The possible holonomy groups are directly linked to the manifols defined over binary-normed division algebras, *i.e.* $\mathbb{R} \mapsto O(n)$, $SO(n)$, $\mathbb{C} \mapsto U(n)$, $SU(n)$, quaternions-$\mathbb{H} \mapsto Sp(n)Sp(1)$, $Sp(n)$, octonions-$\mathbb{O} \mapsto Spin(7)$, $G(2)$, what are non-locally symmetric spaces. Remarkably, that the infinite series of the compact complex spaces with $SU(n)$-holonomy $(n = 1, 2, 3, ...)$ is also connected to the algebra of reflexive projective numbers [18]. These numbers are the generalizations of the well-known ancient Egyptian numbers. In this approach our accent was related to the generalization of the binary hyper-numbers and searching for the n-ary algebras through the "n-dimensional" Berger graphs.

3 Theories of Numbers in Geometry and in Physics.III

We started this process from the study the norm-division algebras of the plural-hyper-complex numbers to construct the geometrical hyper-surfaces in R^n-Euclidean spaces for any dimensions $n = 4, 5, 6, ...$ Due to properties of the cyclic group the basis of the plural hyper-complex numbers these geometrical objects are described by homogeneous n-dimensional polynomials of n - variables what can be factorized, what allow to determine the Abelian transitive group symmetries of such spaces. The latter circumstance gives us the way to study the properties of geometrical objects in any dimension in detail. It can be also done the hyper-complex analysis in $\mathbb{M}C^n$-spaces and, in particularly, one can construct the hyper-complex analysis for hyper-holomorphic functions, construct the hyper-Laplace equations for "plural-harmonic" functions. On this way we can construct the extension of the Pythagoras theorem for any n-dimensional simplex in R^n-Euclidean space. As the plural hyper-complex division algebras can be realized in terms of the $n \times n$ matrices one can find the corresponding hyper-Dirac equations for the n-dimensional spinors. The next steps of the consideration the n-ary generalization of the $\mathbb{H}$- and $\mathbb{O}$-normed division algebras will be considered later/

4 Physical ideas. Cosmology and SM.IV.

Recently for solving the neutrino problems there have been suggested the possible extensions of the $4 = (3+1)$-dimensional space-time by adding one or two additional non-compact large dimensions. The one basic idea of such possible approach has been associated to the proposal that neutrino due its exceptional sterile properties can penetrate into the extended space-time with one or two extra dimensions $D = (4+1)$ or $D = (4+2)$. The second idea was initiated by the contrast between the spatial and temporal properties of neutrinos from one side, and quarks/ charged leptons from the other side, what leaves a room to consider the observed three neutrino states as a quantum 6-spinor field in the 6-dimensional space-time, and, in accordance to the "ternary complexity" to consider three implementations of neutrinos as the "particle-" - "anti-particle-" and "anti-anti-particle"- states, in analogy with the 4-dimensional Dirac theory of the electron-positron.

PHYSICAL IDEAS. V.

The third idea has been based on the non-trivial $D > 4$ -extension of Lorentz-metrics what could conserve the validity all principles of the $D = (3+1)$-special theory of relativity for electromagnetic- charged matter. As a result, it was proposed that the problems of neutrino physics is clearly directed to the search for the new multi-dimensional geometric structures and symmetries. Thus, in such approach in addition to the observed electromagnetic-charged matter there could exist a hypothetical sterile Matter (Dark Matter or pra-matter?) with his "invisible" radiation, and for the description of which it is necessary to generalize $D = (3+1)$ special theory of relativity.

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INCOMPLETENESS OF SM

In addition to the geometrical questions to discussed models there are some reasons why we are looking for new symmetries beyond Lie algebras/groups. One of the main of them is related with conjecture that the known models of quarks and leptons cannot just be solved in terms of binary Lie groups. This problem can be formulated as incompleteness of the "Standard Model" in terms of the binary Lie groups. The same one can say about GUT-constructions in terms of ordinary $SU(5)$,$SO(10$,$E(6)$, ...groups. The main fundamental questions of the quark-lepton quantum numbers origin left without any explanations.

THE PROBLEMS WITH STRING APPROACH

Theories of strings and superstrings are also based on the binary Lie groups and the description of the quark/lepton- world in superstring approaches did not bring us a success. In string formulation there was done "trivial quadratic" assumption about D-dimensional extension of the $D = (1 + 3)$-Lorentz group. Also, if the construction of the quantum theory of superstrings was not yet done it might be useful to know the point-limit of the superstring theories in the spaces with dimensions $4 \leq D \leq 10$. But this point-limit is known just for $D = 4$ where there exist the renormalizable quantum field theories based on the $D = 1 + 3$ Lorentz group symmetry.

6 The Holonomy principle in Riemann and pseudo-Riemann space structure

6.1 Cartan classification of the symmetric homogeneous spaces.

Table 1: *Some examples of symmetric homogeneous Riemannian spaces.*

M	G_{Tran}	H_{Isot}	Dim_R	$metrics$
$SO(n)/SO(n-1)$	$SO(n)$	$SO(n-1)$	$n-1$	S^{n-1}
$O(n)/O(n-1)\times O(1)$	$O(n)$	$O(n-1)\times O(1)$	$n-1$	$\mathbb{R}P^{n-1}$
$SU(n)/SU(n-1)$	$SU(n)$	$SU(n-1)$	$2n-1$	S^{2n-1}
$SU(n)/SU(n-1)\times U(1)$	$SU(n)$	$SU(n-1)\times U(1)$	$2n-1$	$\mathbb{CP}^{n-1}\cong S^{2n-1}/U(1)$
$Sp(n)/Sp(n-1)$	$Sp(n)$	$Sp(n-1)$	$4n-1$	S^{4n-1}
$Sp(n)/Sp(n-1)\times Sp(1)$	$Sp(n)$	$Sp(n-1)\times Sp(1)$	$4n-1$	$\mathbb{HP}^{n-1}\cong S^{4n-1}/Sp(1)$
$SO(p,q)/SO(p)\times SO(q)$	$SO(p,q)$	$SO(p)\times SO(q)$	$(p+q-1)$	$Hyperbolic$

6 Berger classification of non-symmetric Riemann spaces

Firstly, in 1955, Berger presented the classification of irreducibly acting matrix Lie groups occur as the holonomy of a torsion free affine connection. The Berger list of non-symmetric irreducible Riemann manifolds with the list of holonomy groups H of M one can see.

Table 2: *The list of Berger classification for non-symmetric Riemannian spaces.*

M	G_{ISOM}	H_{Hol}	Dim_R	$metrics$
$General$	$-$	$SO(n)$	n	
$Kahler$	$-$	$U(n) \subset O(2n)$	n	
$Calabi-Yau$	$-$	$SU(n) \subset SO(2n)$	$2n$	
$Hyper-Kahler$	$-$	$Sp(n) \subset SO(4n)$	$4n$	
$quaternion--Kahler$	$-$	$Sp(n) \times Sp(1) \subset SO(4n)$	$4n$	
$exceptional$	$-$	$G(2) \subset SO(7)$	7	
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9 The n-ary Algebra of Reflexive Projective Numbers and CY-classification

A CY_n space can be realized as an algebraic variety $\mathcal{M}$ in a weighted projective space $\mathrm{CP}^{n-1}(\overrightarrow{k})$ where the weight vector reads $\overrightarrow{k} = (k_1, \ldots, k_n)$. This variety is defined by

$$\mathcal{M} \equiv (\{x_1, \ldots, x_n\} \in \mathrm{CP}^{n-1}(\overrightarrow{k}) : \mathcal{P}(x_1, \ldots, x_n) \equiv \sum_{\overrightarrow{m}} c_{\overrightarrow{m}} x^{\overrightarrow{m}} = 0), \tag{6}$$

i.e., as the zero locus of a quasi–homogeneous polynomial of degree

$$d_k = \sum_{i=1}^{n} k_i, \tag{7}$$

with the monomials being

$$x^{\overrightarrow{m}} \equiv x_1^{m_1} \cdots x_n^{m_n}. \tag{8}$$

The points in CP^{n-1} satisfy the property of projective invariance

$$\{x_1, \ldots, x_n\} \approx \{\lambda^{k_1} x_1, \ldots, \lambda^{k_n} x_n\} \tag{9}$$

leading to the constraint

$$\overrightarrow{m} \cdot \overrightarrow{k} = d_k. \tag{10}$$

One way of constructing at least some Calabi-Yau manifolds is as solutions of homogeneous polynomial equations in weighted projective spaces, *i.e.*, polynomial equations in n variables whose monomial exponents m_i satisfy the relation $k_i m_i = d$, where the k_i are n positive weights, and d is given by the sum over i of the k_i. The algebraic Calabi-Yau spaces can be obtained from weighted projective spaces, including the Calabi-Yau folds as the intersections of two or more such spaces.

A d-dimensional Calabi-Yau space X_d can be given by the locus of zeroes of a transversal quasihomogeneous polynomial $\wp$ of degree $deg(\wp) = [d] : [d] = \sum_{j=1}^{n+1} k_j$ in a complex projective space $CP^n(\vec{k}) \equiv CP^n(k_1, ..., k_{n+1})$ [?]:

$$X \equiv X^{(n-1)}(k) \equiv \{\vec{x} = (x_1, ..., x_{n+1}) \in CP^n(k) | \wp(\vec{x}) = 0\}. \tag{12}$$

The general quasihomogeneous polynomial of degree $[d]$ is a linear combination

$$\wp = \sum_{\vec{\mu}_\alpha} c_{\vec{\mu}_\alpha} x^{\vec{\mu}_\alpha} \tag{13}$$

of monomials $x^{\vec{\mu}_\alpha} = x_1^{\mu_{1\alpha}} x_2^{\mu_{2\alpha}} ... x_{r+1}^{\mu_{(r+1)\alpha}}$ with the condition:

$$\vec{\mu}_\alpha \cdot \vec{k} = [d]. \tag{14}$$

This algebraic projective variety is irreducible if and only if its polynomial is irreducible. A hypersurface will be smooth for almost all choices of polynomials. To obtain Calabi-Yau d-folds one should choose reflexive weight vectors (RWVs), related to Batyrev's reflexive polyhedra or to the set of IMs. Other examples of compact complex manifolds can be obtained as the complete intersections (CICY) of such quasihomogeneous polynomial constraints:

$$X^{(n-r)}_{CICY} = \{\vec{x} = (x_1, \ldots x_{n+1}) \in CP^n \,|\, \wp_1(\vec{x}) = \ldots = \wp_r(\vec{x}) = 0\}, \tag{15}$$

where each polynomial $\wp_i$ is determined by some weight vector $\vec{k}_i$, $i = 1, \ldots, r$.

K3-POLYHEDRON: (1011)+(0123)=(1134)[9]

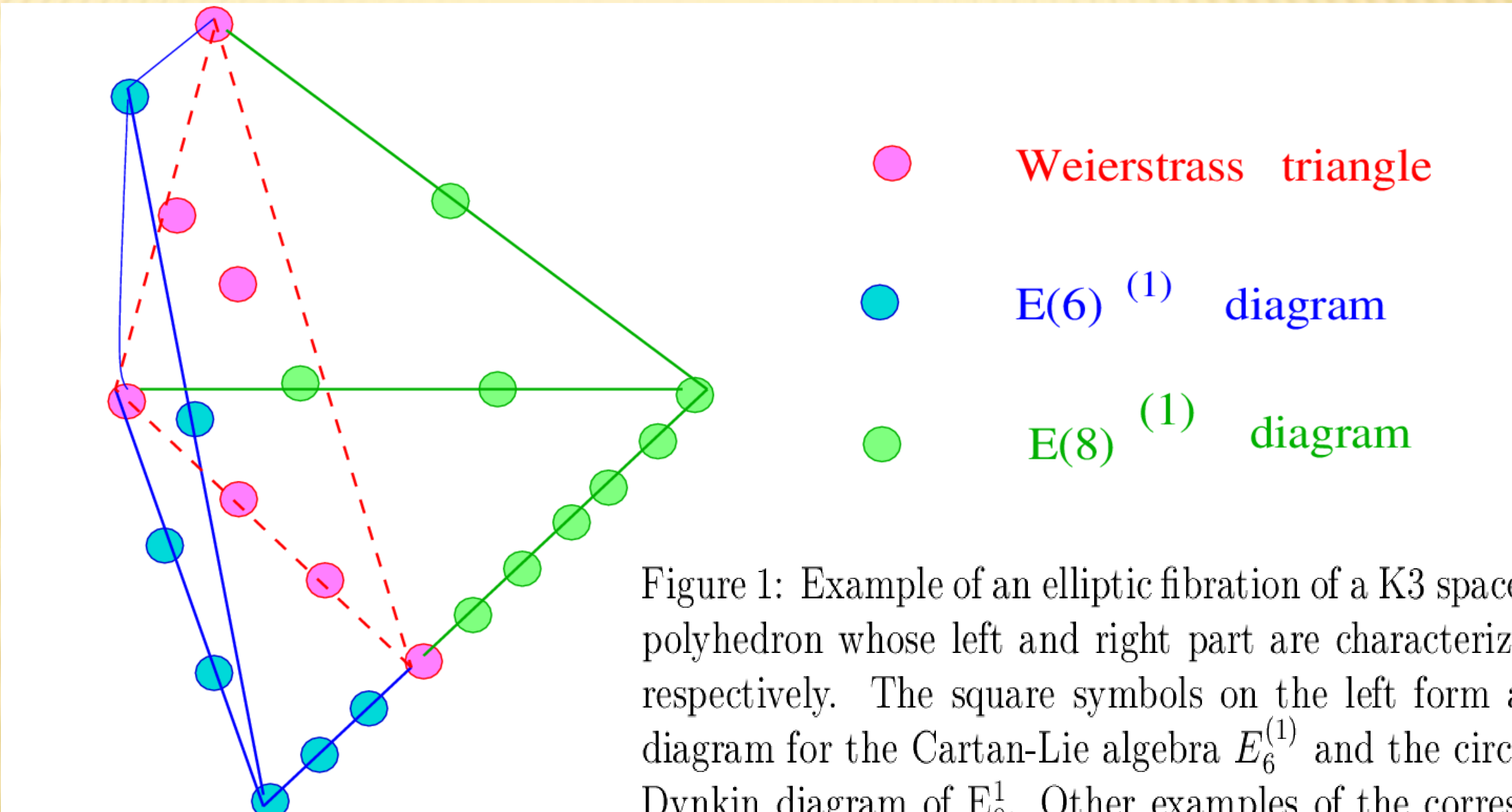

Figure 1: Example of an elliptic fibration of a K3 space, re
polyhedron whose left and right part are characterized b
respectively. The square symbols on the left form a gr
diagram for the Cartan-Lie algebra $E_6^{(1)}$ and the circular
Dynkin diagram of E_8^1. Other examples of the correspon
vectors and the Dynkin diagrams for the ADE series appe
spaces are listed below.

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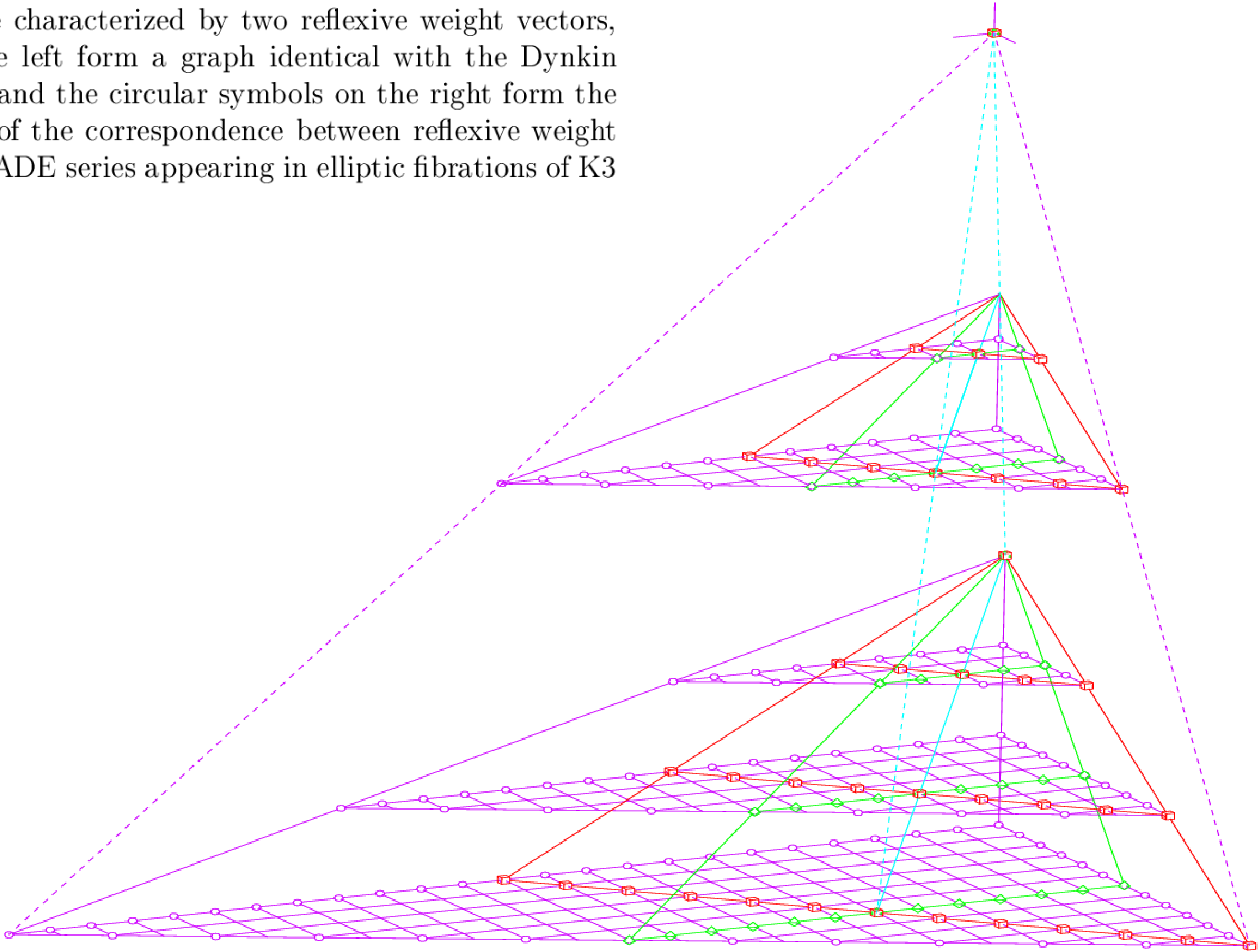
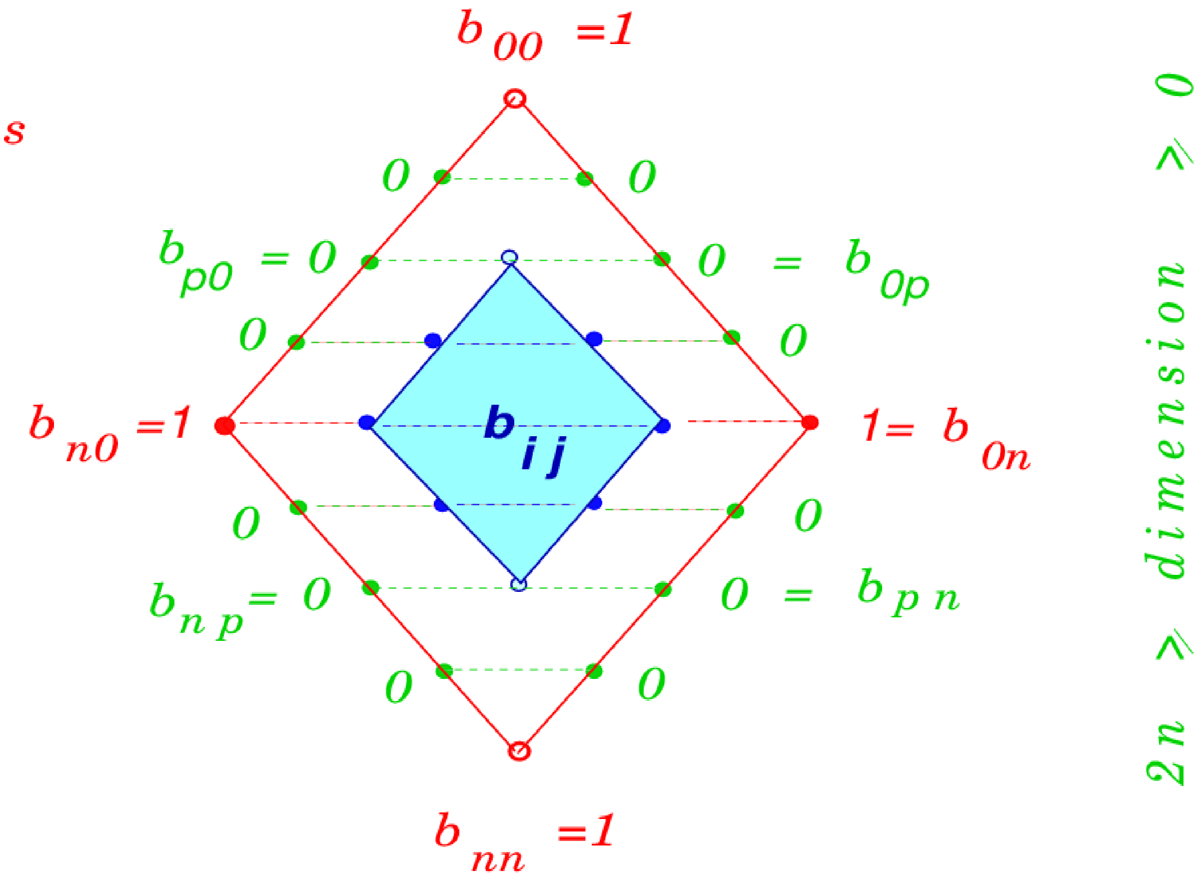


BETTI – HODGE DIAMOND

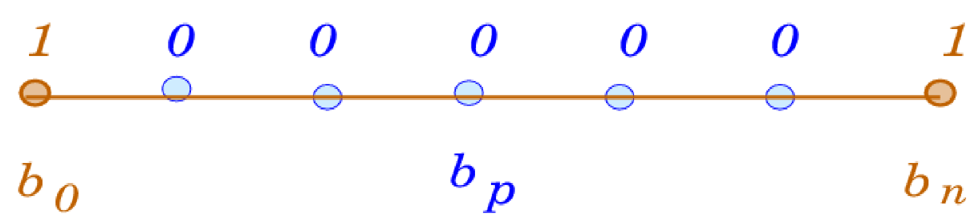
Infinite

CY(n) -series



SU(n) holonomy

$$S^n = \frac{SO(n+1)}{SO(n)}$$



SO(n) holonomy

MIRROR SYMMETRY

This method gives hypersurfaces in toric varieties and unlike previous constructions it is manifestly mirror symmetric, *i.e.*, it gives a pair of C-Y spaces, M_{CY} and M^*_{CY}, with hodge numbers, satisfying the mirror symmetry relation [?, ?, ?, ?]:

$$\begin{aligned} h_{1,1}(M) &= h_{d-1,1}(M^*), \\ h_{d-1,1}(M) &= h_{1,1}(M^*) \end{aligned} \qquad (16)$$

This means that the *Hodge diamond* of M^*_{CY} is a mirror reflection through a diagonal axis of the Hodge diamond of M_{CY}. The existence of the mirror symmetry is the consequence of the dual properties of the CY manifolds.

HODGE NUMBERS

A pair of reflexive polyhedra $(\triangle, \triangle^*)$ gives a pair of mirror C-Y families and the following identities for the hodge numbers $(n \geq 4)$:

$$\begin{aligned} h_{1,1}(\triangle) &= h_{d-1,1}(\triangle^*) = \\ &= l(\triangle^*) - (d+2) - \sum_{codim\Theta^*=1} l^{'}(\Theta^*) \\ &+ \sum_{codim\Theta^*=2} l'(\Theta^*) l^{'}\Theta, \end{aligned} \tag{17}$$

$$\begin{aligned} h_{1,1}(\triangle^*) &= h_{d-1,1}(\triangle) = \\ &= l(\triangle) - (d+2) - \sum_{codim\Theta=1} l'(\Theta) \\ &+ \sum_{codim\Theta=2} l'(\Theta) l'(\Theta^*), \end{aligned} \tag{18}$$

$$h_{p,1} = \sum_{codim\Theta^*=p+1} l^{'}(\Theta) \cdot l^{'}(\Theta^*), \;\; 1 < p < d-1. \tag{19}$$

9.1 The Arity-Dimension Stucture of UCYA and Dynkin Graphs in K3- Polyhedra.

One of the main results in the Universal Calabi-Yau Algebra (UCYA) is that the reflexive weight vectors (RWVs) $\vec{k_n}$ of dimension n can obtained directly from lower-dimensional RWVs $\vec{k_1}, \ldots, \vec{k_{n-r+1}}$ by algebraic constructions of arity r [?, ?, ?, ?]. As an example of an arity $r = 2$ construction, first two $(n-1)$-dimensional RWVs $\vec{k}_{n-1}$ and $\vec{l}_{n-1}$ (which can be taken the same) can be used to obtain two new extended n-dimensional vectors,

$$\begin{aligned} \vec{k}_n^{(ex)} &= (k_1, 0|k_2, \ldots, k_{n-1}), \\ \vec{l}_n^{(ex)} &= (0, l_1|l_2, \ldots, l_{n-1}). \end{aligned} \tag{24}$$

Then, using the composition rule of arity $r = 2$, one can obtain from these two good extended vectors a new n-dimensional RWV:

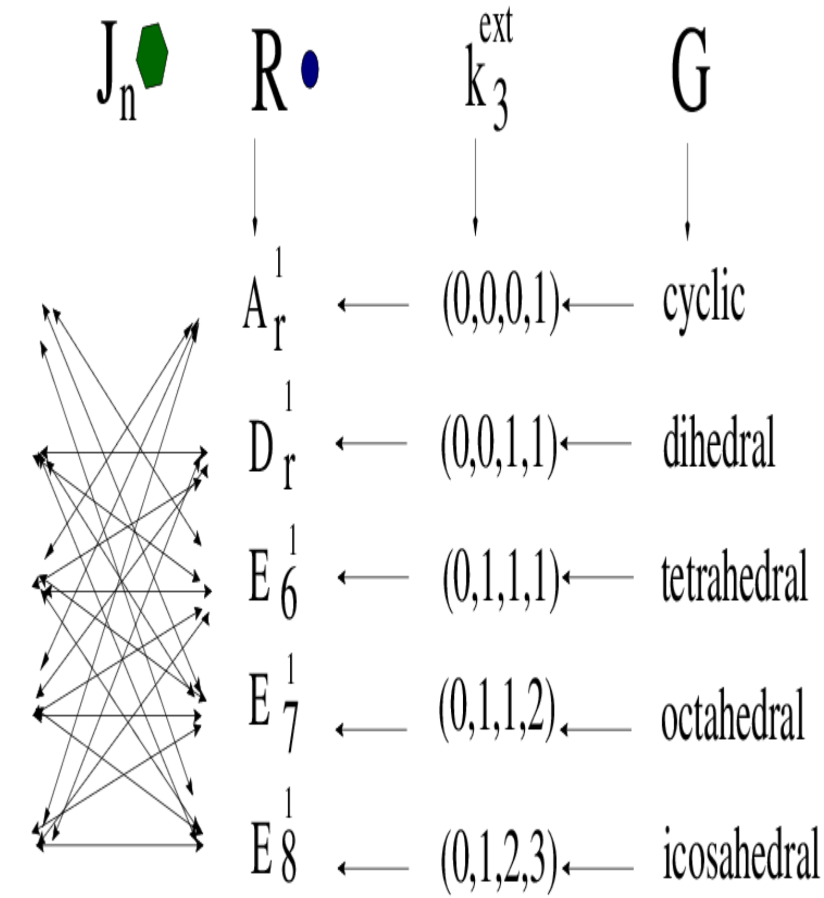
$$\vec{p}_n = \vec{k}_n^{(ex)} + \vec{l}_n^{(ex)} = (k_1, l_1|k_2 + l_2, \ldots, k_{n-1} + l_{n-1}), \tag{25}$$

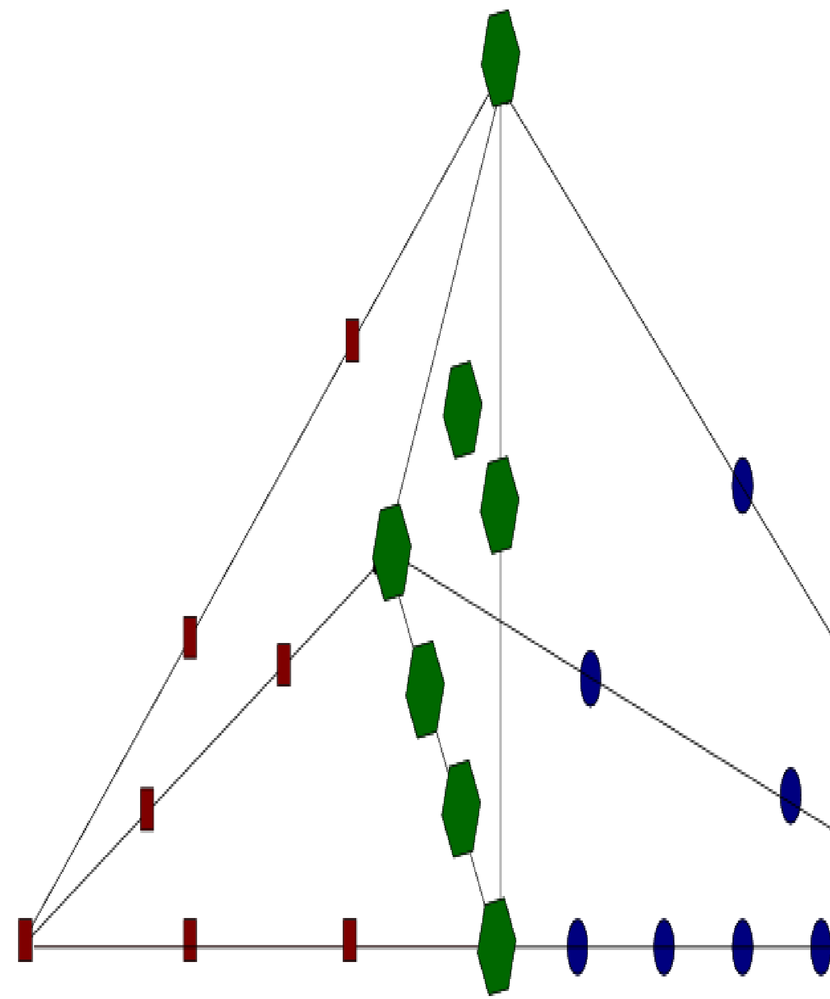
which originates a chain of n-dimensional RWVs (Compare UCYA to theory of operads [2]).

REFLEXIVE NUMBERS AND ADE-GROUPS

This arity-2 composition rule of the UCYA gives complete information about the $(d-1)$-dimensional fibre structure of CY_d spaces, where $d = n-2$. For example, in the K3 case, 91 of the total of 95 RWVs $\vec{k_4}$ can be obtained by such arity-2 constructions out of just five RWVs of dimensions 1,2 and 3, namely

$$
\begin{aligned}
\vec{k_1} &= (1)[1], & \rightarrow \quad & A_r^{(1)} \\
\vec{k_2} &= (1,1)[2], & \rightarrow \quad & D_r^{(1)} \\
\vec{k_3} &= (1,1,1)[3], & \rightarrow \quad & E_6^{(1)} \\
\vec{k_3} &= (1,1,2)[4], & \rightarrow \quad & E_7^{(1)} \\
\vec{k_3} &= (1,2,3)[6] & \rightarrow \quad & E_8^{(1)}
\end{aligned}
\tag{26}
$$





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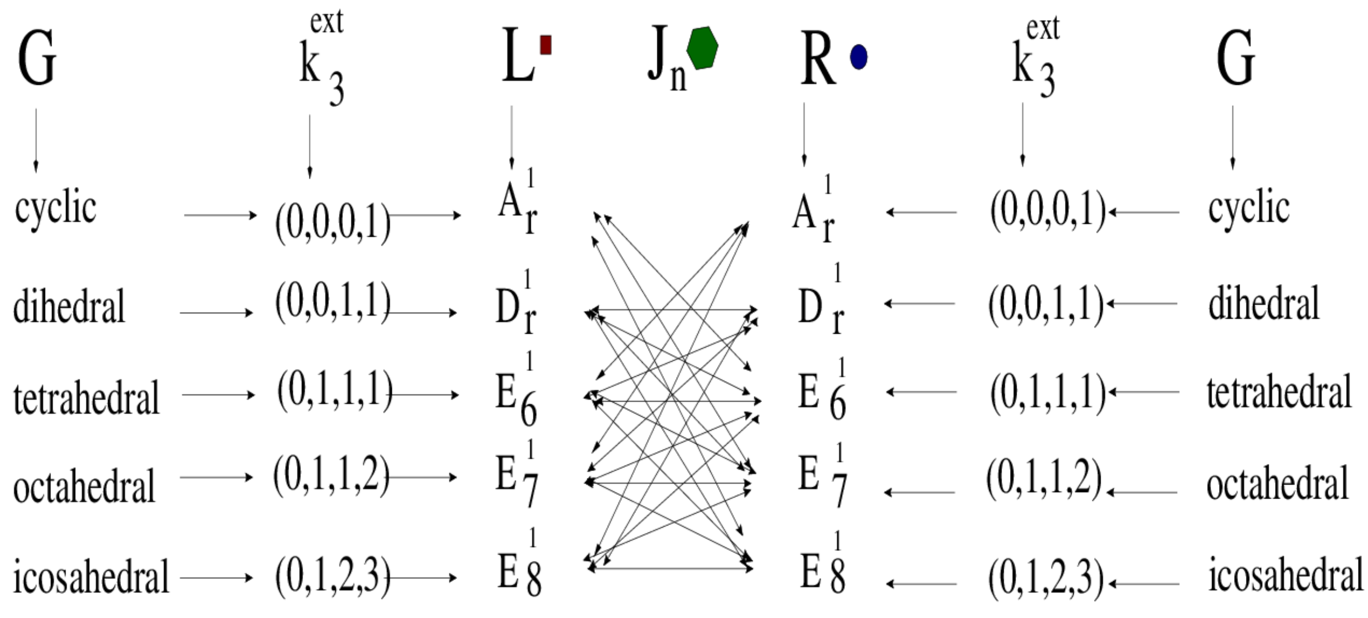

Figure 4: Example of an elliptic fibration of a K3 space, represented by a reflexive Newton polyhedron whose left and right part are characterized by two reflexive weight vectors, respectively. The square symbols on the left form a graph identical with the Dynkin diagram for the Cartan-Lie algebra $E_6^{(1)}$ and the circular symbols on the right form the Dynkin diagram of E_8^1. Other examples of the correspondence between reflexive weight vectors and the Dynkin diagrams for the ADE series appearing in elliptic fibrations of K3

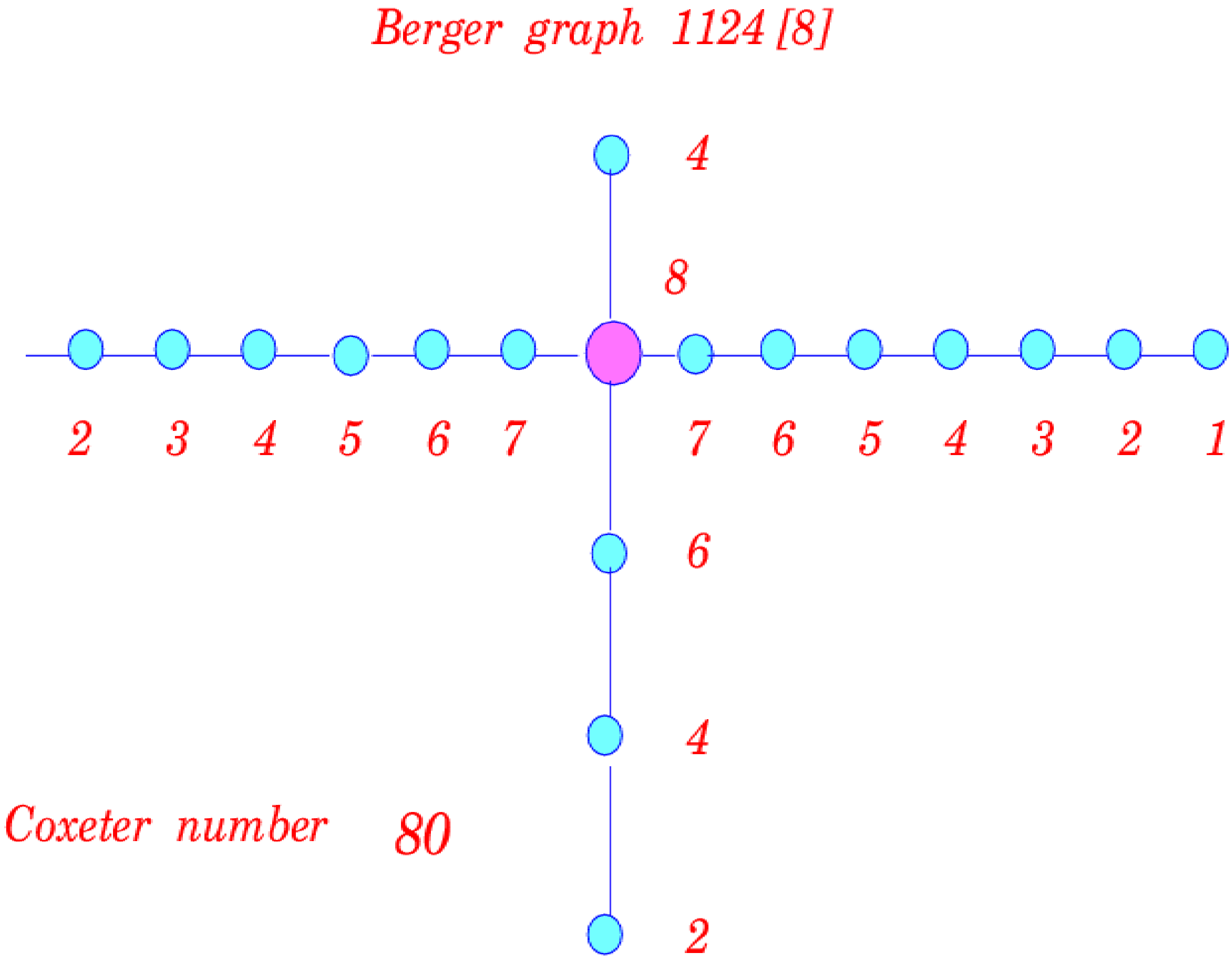

Figure 6: *Berger affine Cross graph in CY_3 polyhedron* $\vec{k}_4 = (1,1,2,2,5)[11] = (1,0,0,0,0) + (0,1,2,2,5)$ *with (1,2,2,5)- intersection. One can see the labels which correspond to the Coxeter labels.*

SIMPLY - LACED REFLEXIVE NUMBERS

An interesting subclass of the reflexive numbers is the so–called "simply–laced" numbers =Egyptian numbers. A simply–laced number $\overrightarrow{k} = (k_1, \cdots, k_n)$ with degree $d = \sum_{i=1}^{n} k_i$ is defined such that

$$\frac{d}{k_i} \in \mathbb{Z}^+ \text{ and } d > k_i. \tag{22}$$

For these numbers there is a simple way of constructing the corresponding affine Berger graphs together with the corresponding Berger matrices. In dimensions $n = 1, 2, 3$ the Egyptian numbers are

$D = 1$	$(1)[1]$				1
$D = 2$	$(1, 1)[2]$				1
$D = 3$	$(1, 1, 1)[3]$	$(1, 1, 2)[4]$	$(1, 2, 3)[6]$		3
$D = 4$	$1, 1, 1, 1)[4]$	$(1, 1, 1, 3)[6]$	$(1, 1, 2, 2)[6]$	$(1, 1, 2, 4)[8]$	–
–		$(1, 2, 2, 5)[10]$	$(1, 1, 4, 6)[12]$	$(1, 2, 3, 6)[12]$	–
–		$(1, 3, 3, 4)[12]$	$(2, 3, 3, 4)[12]$	$(1, 2, 6, 9)[18]$	–
–		$(1, 4, 5, 10)[20]$	$(1, 3, 8, 12)[24]$	$(2, 3, 10, 5)[30]$	–
–		$(1, 6, 14, 21)[42]$			14
$D = 5$	$(1, 1, 1, 1, 1)[5]$	$(1, 1, 1, 1, 4)[8]$	......		146

(23)

11 Berger graphs from the lattice of reflexive CY_d ($d \geq 3$) polyhedra.

Our reflexive polyhedra allow us to consider new graphs, which we will call Berger graphs, and for corresponding Berger matrices we suggest the folowing rules:

$$
\begin{aligned}
\mathbb{B}_{ii} &= 2 \qquad or \qquad 3, \\
\mathbb{B}_{ij} &\leq 0, \\
\mathbb{B}_{ij} = 0 &\mapsto \mathbb{B}_{ji} = 0, \\
\mathbb{B}_{ij} &\in \mathbb{Z}, \\
Det\mathbb{B} &= 0, \\
Det\mathbb{B}_{\{(i)\}} &> 0.
\end{aligned}
\tag{29}
$$

We call the last two restriction the *affine condition*. In these new rules comparing with the generalized affine Cartan matrices we relaxed the restriction on the diagonal element $\mathbb{B}_{ii}$, *i.e.* to satisfy the affine conditions we allow also to be

$$\mathbb{B}_{ii} = 3\, for\, CY_3, \qquad \mathbb{B}_{ii} = 4\, for\, CY_4, \qquad and\ etc. \tag{30}$$

ENVA-BERGER GRAPH

$$Det(AENV4^1) \quad = $$

$$\left(\begin{array}{ccc|ccc|ccc|ccc|c}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
\hline
0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 3
\end{array}\right) \tag{31}$$

Note, that the determinant is equal zero. If we remove the zero node (label =1), the $Det(ENV4) = 4^2$. In general case for CY_d, $d + 2 = n$, which corresponded to the RWV $\vec{k}_n = (1, \ldots, 1)[n]$, the determinant of the corresponding non-affine matrix is equal n^{n-2} ($n \geq 3$).

11 Berger graphs from the lattice of reflexive CY_d $(d \geq 3)$ polyhedra.

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$$\begin{array}{rcl} \mathbb{B}_{ii} & = & 2 \quad or \quad 3, \\ \mathbb{B}_{ij} & \leq & 0, \\ \mathbb{B}_{ij} = 0 & \mapsto & \mathbb{B}_{ji} = 0, \\ \mathbb{B}_{ij} & \in & \mathbb{Z}, \\ Det\mathbb{B} & = & 0, \\ Det\mathbb{B}_{\{(i)\}} & > & 0. \end{array} \tag{29}$$

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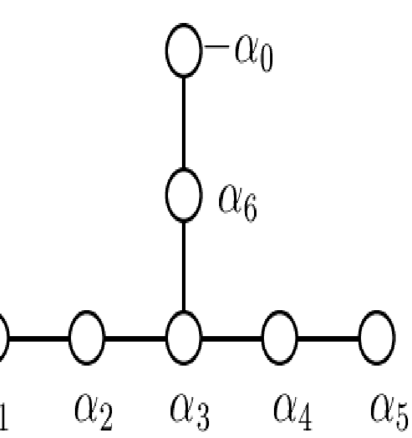
Apart from these rules we will check the coincidence of the graph's labels, which we

Table 5: Rank, Coxeter number h, Casimir depending on B_{ii} and determinants for the non–affine exceptional Berger graphs. The maximal Coxeter labels coincide with the degree of the corresponding reflexive simply–laced vector. The determinants in the last column for the infinite series (0,0,1,1,1)[3], (0,0,1,1,2)[4] and (0,0,1,2,3)[6] are independent from the number l of internal binary $B_{ii} = 2$ nodes. The numbers 1_3 and 2_3 denote the number of nodes with $B_{ii} = 3$.

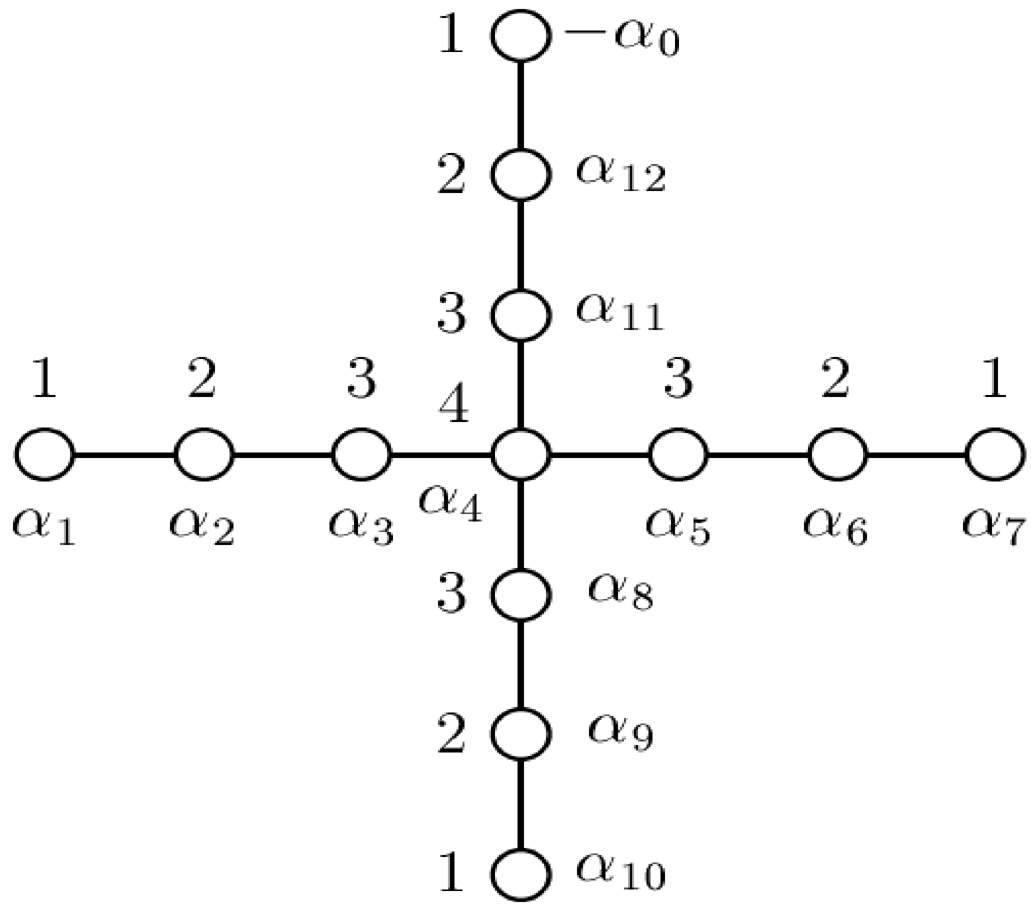
$\vec{k}_{3,4}^{\text{ext}}$	Rank	h	Casimir(B_{ii})	Determinant
$(0,1,1,1)[3]$	$6(E_6)$	12	6	3
$(0,1,1,2)[4]$	$7(E_7)$	18	8	2
$(0,1,2,3)[6]$	$8(E_8)$	30	12	1
$(0,0,1,1,1)[3]$	$2_3 + 10 + l$	$18 + 3(l+1)$	9	3^4
$(0,0,1,1,2)[4]$	$2_3 + 13 + l$	$32 + 4(l+1)$	12	4^3
$(0,0,1,2,3)[6]$	$2_3 + 15l$	$60 + 6(l-1)$	18	6^2
$(0,1,1,1,1)[4]$	$1_3 + 11$	28	12	16
$(0,2,3,3,4)[12]$	$1_3 + 12$	90	36	8
$(0,1,1,2,2)[6]$	$1_3 + 13$	48	18	9
$(0,1,1,1,3)[6]$	$1_3 + 15$	54	18	12
$(0,1,1,2,4)[8]$	$1_3 + 17$	80	24	8
$(0,1,2,2,5)[10]$	$1_3 + 17$	100	30	5
$(0,1,3,4,4)[12]$	$1_3 + 17$	120	36	3
$(0,1,2,3,6)[12]$	$1_3 + 19$	132	36	6
$(0,1,4,5,10)[20]$	$1_3 + 26$	290	60	2

$(0,1,1,4,6)[12]$	1_3+24	162	36	6
$(0,1,2,6,9)[18]$	1_3+27	270	54	3
$(0,1,3,8,12)[24]$	1_3+32	420	72	2
$(0,2,3,10,15)[30]$	1_3+25	420	90	4
$(0,1,6,14,21)[42]$	1_3+49	1092	126	1

LA $\mathcal{E}_6$ and Berger algebra defined by reflexive number



$\mathcal{BER}$



$(0,1,2,2,5)[10]$	1_3+17	100	30	5
$(0,1,3,4,4)[12]$	1_3+17	120	36	3
$(0,1,2,3,6)[12]$	1_3+19	132	36	6
$(0,1,4,5,10)[20]$	1_3+26	290	60	2
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An interesting subclass of the reflexive numbers is the so–called "simply–laced" numbers (Egyptian numbers). A simply–laced number $\vec{k}=(k_1,\cdots,k_n)$ with degree $d=\sum_{i=1}^{n}k_i$ is defined such that

$$\frac{d}{k_i}\in Z^+ \text{ and } d>k_i. \qquad (36)$$

SIMPLE ROOTS OF BERGER GRAPH

$$
\begin{aligned}
\alpha_1 &= e_1 - e_2 \\
\alpha_2 &= e_2 - e_3 \\
\alpha_3 &= e_3 - e_4 \\
\alpha_4 &= e_4 - e_5 - e_9 \\
\alpha_5 &= e_5 - e_6 \\
\alpha_6 &= e_6 - e_7 \\
\alpha_7 &= e_7 - e_8 \\
\alpha_8 &= e_9 - e_{10} \\
\alpha_9 &= -\frac{1}{2}(e_9 - e_{10} + e_1 + e_2 + e_3 + e_4 + e_{11} - e_{12}) \\
\alpha_{10} &= e_{11} - e_{12} \\
\alpha_{11} &= e_9 + e_{10} \\
\alpha_{12} &= -\frac{1}{2}(e_9 + e_{10} - e_5 - e_6 - e_7 - e_8 + e_{11} + e_{12}) \\
\alpha_{13} &= e_{11} + e_{12} = -\alpha_0
\end{aligned}
$$

$$4\alpha_4 + 3(\alpha_3 + \alpha_5 + \alpha_8 + \alpha_{11}) + 2(\alpha_2 + \alpha_6 + \alpha_9 + \alpha_{12})$$

BERGER MATRIX

$$
\left(\begin{array}{ccc|c|ccc|ccc|ccc}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\
\hline
\end{array}\right)
$$

We will discuss the following themes:

- C^n- hyper-plural division numbers
- C^n- complexification of R^n- spaces, $n = 3, 4, 5, 6, ...,$
- C^n- structure and the invariant surfaces, $n = 3, 4, 5, 6, ...$
- C^n- hyper-holomorphic and hyper-harmonic functions
- The link between C^n-holomorphism and the origin of n-spinors

13 C_N-division numbers in Riemannian geometry

Historically the discovery of Killing-Cartan-Lie algebras was closely related to the four norm division C_2 algebras- $\mathbb{R}$,$\mathbb{C}$,$\mathbb{H}$,$\mathbb{O}$ in dimensions 2^n, $n = 0, 1, 2, 3$, i.e. real numbers, complex numbers, quaternions and octonions, respectively [26, 27, 12, 29]. For example, it is well known that binary complex numbers of module 1 are related to Abelian $U(1) = S^1$ group. The imaginary quaternion units are related to the $su(2)$ algebra and the unit quaternions are related to the $SU(2) = S^3$ group. At last, octonions are related to the exceptional $G(2)$ algebra. So, our way is based on the $(n > 2)$ generalization of binary $(n = 2)$ division algebras (real, C^n-complex, C^n n^2-onions, C^n n^3-onions).

An algebra A will be a vector space that is equipped with a bilinear map $f : A \times A \to A$ called by multiplication and a non zero element $1 \in A$ called the unit such that $f(1, a) = f(a, 1) = a$. These algebras admit an anti-involution (or conjugation) $(a^*)^* = a$ and $(ab)^* = b^* a^*$. A norm division algebra is an algebra A that is also a norm vector space with $|ab| = |a||b|$. For such algebras the following identities can be obtained:

$$(x_1^2 + ... + x_n^2)(y_1^2 + ... + y^2) = (z_1^2 + ... + z_n^2) \tag{39}$$

The doubling process, which is known as the Cayley-Dickson process, forms the sequence of division algebras

$$\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}. \tag{40}$$

Note that next algebra is not a division algebra. So $n = 1$ $\mathbb{R}$ and $n = 2$ $\mathbb{C}$ these algebras are the commutative associative norm division algebras. The quaternions, $\mathbb{H}$, $n = 4$ form the non-commutative and associative norm division algebra. The octonion algebra $n = 8, \mathbb{O}$ is an non-associative alternative algebra. If the discovery of complex numbers took a long period about some centuries years, the discovery of quaternions and octonions was made in a short time, in the middle of the XIX century by W. Hamilton [26], and by J. Graves and A.Cayley [27]. The complex numbers, quaternions and octonions can be presented in the general form:

$$\hat{q} = x_0 e_0 + x_p e_p, \qquad \{x_0,\, x_p\} \in \mathbb{R}, \tag{41}$$

where $p = 1$ and $e_1 \equiv \mathbf{i}$ for complex numbers C, $p = 1, 2, 3$ for quaternions $\mathbb{H}$, and $p = 1, 2, ..., 7$ for O. The e_0 is as unit and all e_p are imaginary units with conjugation

GEOMETRY OF BINARY HYPER NUMBERS

$$x^2 = 1, x \in R, \tag{44}$$

$$|\hat{Z}| = x_0^2 + x_1^2 = 1 \in \mathbb{C}, \tag{45}$$

$$|\hat{Q}| = x_0^2 + x_1^2 + x_2^2 + x_3^2 = |Z_1|^2 + |Z_2|^2 = 1 \in \mathbb{H}, \tag{46}$$

$$|\hat{O}| = x_0^2 + x_1^2 + ... + x_7^2 = |Q_1|^2 + |Q|^2 = 1 \in \mathbb{O}, \tag{47}$$

(48)

N-ARY HYPER NUMBERS

If the binary alternative division algebras (real numbers, complex numbers, quaternions, octonions) over the real numbers have the dimensions 2^p, $p = 0, 1, 2, 3, 4, ...$, the n=3-ary and n=4-ary norm division algebras have the following dimensions n^p, $p = 0, 1, 2, 3$, respectively:

$$
\begin{array}{lc|lc}
\mathbb{R}: & 2^0 = 1 & \mathbb{R}: & 3^0 = 1 \\
\mathbb{C}: & 2^1 = 1+1 & \mathbb{TC}: & 3^1 = 1+1+1 \\
\mathbb{Q}: & 2^2 = 1+2+1 & \mathbb{TQ}: & 3^2 = 1+2+3+2+1 \\
\mathbb{O}: & 2^3 = 1+3+3+1 & \mathbb{TO}: & 3^3 = 1+3+6+7+6+3+1 \\
\mathbb{S}: & 2^4 = 1+4+6+4+1 & \mathbb{TS}: & 3^4 = 1+4+10+16+19+16+10+4+1
\end{array}
\tag{49}
$$

$$
\begin{array}{lc}
\mathbb{R}: & 4^0 = 1 \\
\mathbb{N}_4\mathbb{C}: & 4^1 = 1+1+1+1 \\
\mathbb{N}_4\mathbb{Q}: & 4^2 = 1+2+3+4+3+2+1 \\
\mathbb{N}_4\mathbb{O}: & 4^3 = 1+3+6+10+12+12+10+6+3+1 \\
\mathbb{N}_4\mathbb{S}: & 4^4 = 1+4+10+20+31+40+44+40+31+20+10+4+1
\end{array}
\tag{50}
$$

CALLEY DIXON METHOD

In the last lines one can see the sedenions which do not produce division algebra. For both cases we have the unit element e_0 and the n basis elements:

$$\mathbb{R} \to \mathbb{TC} \to \mathbb{TQ} \to \mathbb{T}O \to \mathbb{T}S$$
$$\mathbb{R} \to \mathbb{N}_4\mathbb{C} \to \mathbb{N}_4\mathbb{Q} \to \mathbb{N}_4\mathbb{O} \to \mathbb{N}_4\mathbb{S} \tag{51}$$

ABELIAN CYCLIC GROUPS

A representation of the group G is a homomorphism of this group into the multiplicative group $GL_m(\Lambda)$ of nonsingular matrices over the field Λ, where $\Lambda = \mathbb{R}, \mathbb{C}$ or etc. The degree of representation is defined by the size of the ring of matrices. If degree is equal one the representation is linear. For Abelian cyclic group C_n one can easily find the character table, which is $n \times n$ square matrix whose rows correspond to the different characters for a particular conjugation class, q^α, $\alpha = 0, 1, .., n-1$. For cyclic groups C_n the n irreducible representations are one dimensional (see Table):

$$\left(\begin{array}{c|cccccc} - & 1 & q & ... & q^{\alpha} & ... & q^{n-1} \\ \hline \xi^{(1)} & 1 & 1 & ... & 1 & ... & 1 \\ \xi^{(2)} & 1 & \xi_2^{(2)} & ... & \xi_\alpha^{(2)} & ... & \xi_n^{(2)} \\ ... & .... & ... & ... & ... & ... & \\ \xi^{(k)} & 1 & \xi_2^{(k)} & ... & \xi_\alpha^{(k)} & ... & \xi_n^{(k)} \\ ... & .... & ... & ... & ... & ... & \\ \xi^{(N)} & 1 & \xi_2^{(n)} & ... & \xi_\alpha^{(n)} & ... & \xi_n^{(n)} \end{array}\right) \tag{66}$$

where the characters can be defined through n-th root of unity. For example, if the character table for C_n can be summarized as

$$\xi^{\alpha} ... \xi^{\alpha} \exp[(2\pi i(k-1)(\alpha-1))/n] \quad (k, \alpha = 1, 1, 2, ..., n) \tag{67}$$

CONJUGATIONS CLASSESS AND ONE DIMENSIONAL REPRESENTATIONS

C_2	1	i	
$R^{(1)}$	1	1	z
$R^{(2)}$	1	-1	$\bar{z}$

.

The cyclic group C_3 has three conjugation classes, q_0, q and q^2, and, respectively, three one dimensional irreducible representations, $R^{(i)}$, $i = 1, 2, 3$. We write down the table of their characters, $\xi_l^{(i)}$:

$$\left(\begin{array}{c|ccc} - & 1 & q & q^2 \\ \hline \xi^{(1)} & 1 & 1 & 1 \\ \xi^{2)} & 1 & j_3 & j_3^2 \\ \xi^{(3)} & 1 & j_3^2 & j_3 \end{array}\right) \tag{68}$$

for C_3 ($j_3 =\equiv j = \exp\{2\pi\mathbf{i}/3\}$).

14 Ternary C_3- hyper-numbers

For the ternary complexification of the vector space, $\mathbb{R}^3$, one uses its cyclic symmetry subgroup C_3. In the physical context the elements of the group C_3 are actually spatial rotations through a restricted set of angles, $0, 2\pi/3, 4\pi/3$ around, for example, the x_0-axis. After such rotations the coordinates, x_0, x_1, x_2, of the point in $\mathbb{R}^3$ are linearly related with the new coordinates, x'_0, x'_1, x'_2 which can be realized by the 3×3 matrices corresponding to the C_3-group transformations.

The algebra of ternary complex numbers is defined over the field of real numbers generating by one element q with condition $q^3 = q_0 \equiv \hat{1}$ []. The class of ternary complex numbers

$$z = x_0\hat{q}_0 + x_1 q + x_2 q^2 \tag{73}$$

is denoting by $\mathbb{T}_3C$. Let note that for odd n the cases $q^n = 1$ and $q^n = -1$ are equivalent.

TERNARY HYPER-NUMBERS

One can introduce the following basis forms:

$$\begin{aligned} K &= \frac{1}{3}(1 + q + q^2), \\ E &= \frac{1}{3}(2 - q - q^2), \\ I &= \frac{1}{\sqrt{3}}(q - q^2). \end{aligned} \qquad (74)$$

Reversely

$$\begin{aligned} q_0 &= K + E, \\ q &= K - \frac{1}{2}E + \frac{\sqrt{3}}{2}I, \\ q^2 &= K - \frac{1}{2}E - \frac{\sqrt{3}}{2}I. \end{aligned} \qquad (75)$$

TERNARY HYPER-NUMBERS

This basis one can separate on two commutative subalgebras generated by $\{K\}$ and $\{E, I\}$,

$$K^n = K, \qquad K \cdot E = 0, \qquad K \cdot I = 0, \tag{76}$$

$$\tag{77}$$

and

$$\begin{aligned} E^n &= E & E \cdot I &= I \\ I^{2n} &= (-1)^n E, & I^{2n+1} &= (-1)^n I \end{aligned} \tag{78}$$

TERNARY HYPER-NUMBERS

Accordance this basis the ternary complex number can be represented in the next form:

$$\begin{aligned} z &= x_0(K+E) + x_1(K - \frac{1}{2}E + \frac{\sqrt{3}}{2}I) + x_2(K - \frac{1}{2}E - \frac{\sqrt{3}}{2}I) \\ &= \{(x_0 + x_1 + x_2)\,K\} + \{[x_0 - \frac{1}{2}(x_1 + x_2)]\,E + \frac{\sqrt{3}}{2}(x_1 - x_2)\,I\}. \end{aligned} \tag{79}$$

The singular regions for definition of these numbers are the plane $x_0 + x_1 + x_2 = 0$ and the line "trisectrice" $x_1 = x_2 = x_3$. Based on the multiplications rules of generators of two subalgebra, $\{K\}$ and $\{E, I\}$, one can define the pseudo-norm:

$$\begin{aligned} \langle z \rangle^3 &= (x_0 + x_1 + x_2)_{\{K\}} \cdot [(x_0 - \frac{1}{2}(x_1 + x_2))^2 + (\frac{\sqrt{3}}{2}(x_1 - x_2))^2]_{\{E,I\}}. \\ &= x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2 \end{aligned} \tag{80}$$

TERNARY HYPER-NUMBERS

The surface

$$x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2 = \rho^3 \quad (82)$$

(see figure ??) is a ternary analogue of the S^1 circle and it is related with the ternary Abelian group, $TU(1)$.

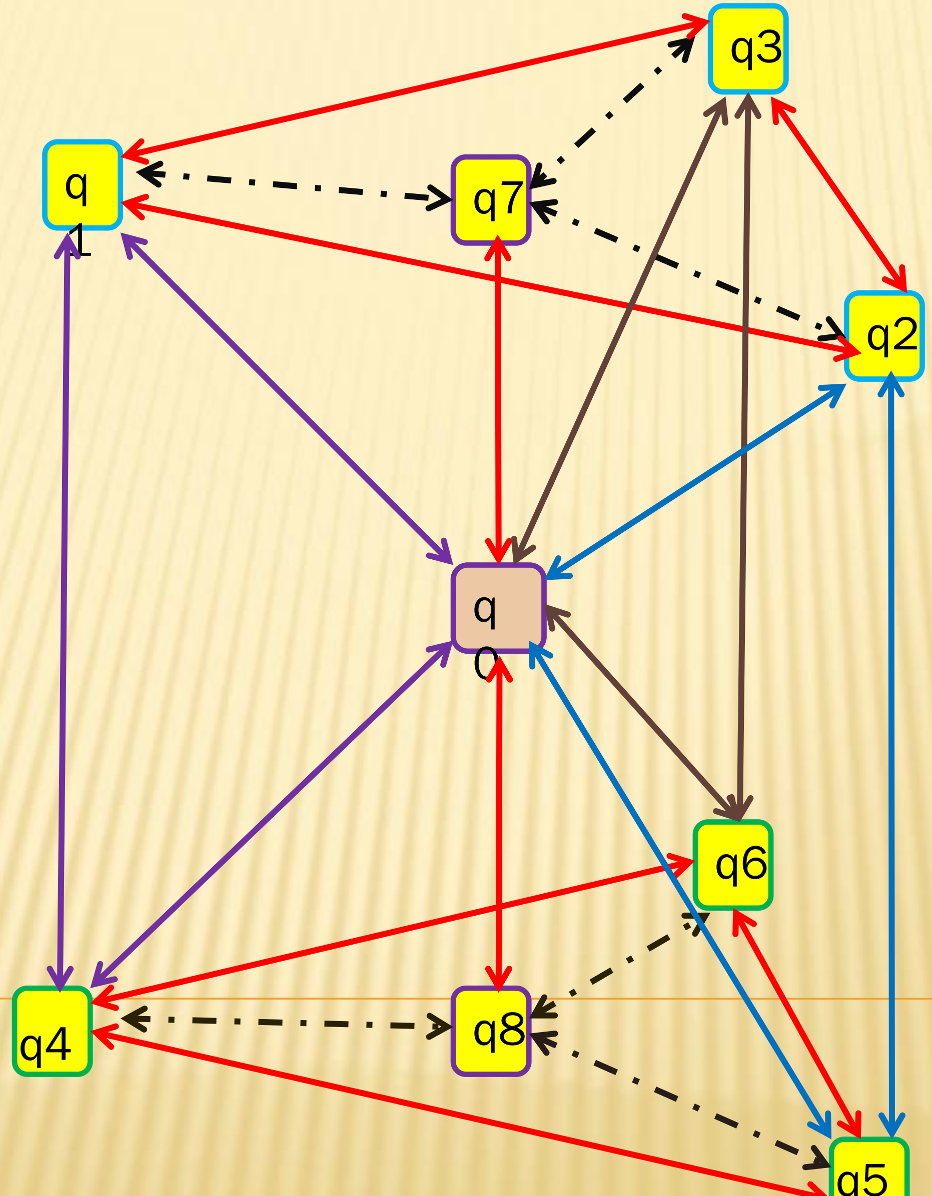
From the above figure one can see, that this surface approaches asymptotically the plane $x_0+x_1+x_2=0$ and the line $x_0 = x_1 = x_2$ orthogonal to it. In $\mathbb{T}_3\mathbb{C}$ they correspond to the ideals I_2 and I_1, respectively. The latter line will be called the "trisectrice".

One can compare this cubic surface to the quadratic surface -cylinder- what can be consider as equation $x_1^2 + x_2^2 = 1$ in the $\mathbb{C} \otimes \mathbb{R}$-space. This fact could be accepted as the first success of the ternary complex numbers of binary complex numbers what can be embedded only into even dimensional $\mathbb{R}^{2n}$ space.

TERNARY HYPER-NUMBERS

TERNARY HYPER-NUMBERS

NONETRION ALGEBRA



q1q4=q0
q2q5=q0
q3q6=q0
q7q8=q0