Discrete universality of the Riemann zeta-function and Hurwitz zeta-function

Antanas Laurinčikas, Vilnius University, Lithuania

A. A. Karatsuba memorial conference

Let $s = \sigma + it$ be a complex variable. The Riemann zeta-function $\zeta(s)$ and the Hurwitz zeta-function $\zeta(s, \alpha)$ with parameter α, $0 < \alpha \leq 1$, are defined, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

and

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

where the infinite product is taken over all prime numbers p. Moreover, the functions $\zeta(s)$ and $\zeta(s, \alpha)$ have analytic continuations to the whole complex plane, except for simple poles at the point $s = 1$ with residue 1.

It is well known that the functions $\zeta(s)$ and $\zeta(s, \alpha)$ for some classes of the parameter α are universal in the sense that their shifts $\zeta(s + i\tau)$ and $\zeta(s + i\tau, \alpha)$, $\tau \in \mathbb{R}$, approximate any analytic function uniformly on compact sets of the right-hand side of the critical strip. Universality of the Riemann zeta-function was discovered by S. M. Voronin in 1975 [4]. We remind the last version of the Voronin theorem. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$, $K \in \mathcal{K}$, the class of continuous non-vanishing functions on K which are analytic in the interior of K. Then we have, see, for example, [5], [12], the following statement.

Theorem 1

Suppose that $K \in \mathcal{K}$ *and* $f(s) \in H_0(K)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Here and in the sequel, $\mathrm{meas}A$ denotes the Lebesque measure of a measurable set $A \subset \mathbb{R}$.
From the definition of the function $\zeta(s, \alpha)$, we have that $\zeta(s, 1) = \zeta(s)$, and

$$\zeta(s, \tfrac{1}{2}) = (2^s - 1)\zeta(s).$$

The universality property of $\zeta(s, \alpha)$ depends on the parameter α, and differs a bit from Theorem 1. Denote by $H(K)$, $K \in \mathcal{K}$, the class of continuous functions on K which are analytic in the interior of K. Then we have

Theorem 2

Suppose that the number α is transcendental or rational $\neq 1, \frac{1}{2}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon\right\} > 0.$$

Theorem 2 for rational $\alpha \neq 1, \frac{1}{2}$ was proved independently by S. M. Voronin [13], S. M. Gonek [4] and B. Bagchi [1]. The case of transcendental α can be found in [6]. The function $\zeta(s, \alpha)$ is also universal with $\alpha = 1$ (Theorem 1) and $\alpha = \frac{1}{2}$, however, in this case, the approximated function belongs to the class $H_0(K)$. The case of algebraic irrational α is an open problem.
In [8], H. Mishou obtained an interesting joint universality theorem on the approximation of a pair of analytic functions by shifts $(\zeta(s + i\tau), \zeta(s + i\tau, \alpha))$.

Theorem 3 [8]

Suppose that the number α is transcendental. Let $K_1, K_2 \in \mathcal{K}$, and $f_1(s) \in H_0(K_1)$ and $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon, \sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon \right\} > 0.$$

Theorems 1-3 show that the sets of shifts $\zeta(s+i\tau)$, $\zeta(s+i\tau,\alpha)$ and $(\zeta(s+i\tau), \zeta(s+i\tau,\alpha))$ approximating given analytic functions are infinite and even have a positive lower density.

Theorems 1-3 are of the so-called continuous type. In shifts $\zeta(s+i\tau)$ and $\zeta(s+i\tau,\alpha)$ τ varies continuously in the interval $[0, T]$. Also, the discrete universality is considered. In this case, analytic functions are approximated by shifts $\zeta(s+ikh)$ and $\zeta(s+ikh,\alpha)$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $h > 0$ is a fixed number. A discrete analogue of Theorem 1 is of the form.

Theorem 4

Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon \right\} > 0.$$

The discrete universality of $\zeta(s)$ was proved by A. Reich in [10]. Theorem 4 under slightly different conditions on the set K was also obtained by B. Bagchi [1].
The discrete universality for the Hurwitz zeta-function is more complicated because in this case two parameters α and h occur, and a connection between them plays an important role.

Theorem 5

Suppose that the number α is transcendental or rational $\neq 1, \frac{1}{2}$, $K \in \mathcal{K}$ and $f(s) \in H(K)$. In the case of rational α, let the number $h > 0$ be arbitrary, while in the case of transcendental α, let the number $h > 0$ be such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational. Then, for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leq m \leq N : \sup_{s\in K} |\zeta(s+imh, \alpha) - f(s)| < \varepsilon \right\} > 0.$$

Theorem 5 for rational α was obtained in [1]. J. Sander and J. Steuding gave [11] a different proof. In the case of transcendental α, Theorem 5 is a particular case of a theorem from [7].

Universality of zeta-functions has a series theoretical and practical applications. For practical applications, the discrete universality is more convenient. For example, a discrete universality theorem was applied [2] for estimation of complicated integrals over analytic curves which are considered in quantum mechanics. This is a motivation together with continuous universality also to investigate the discrete universality of zeta-functions. The aim of this report is a discrete version of Theorem 3. Denote by $\mathcal{P}$ the set of all prime numbers, an define the set

$$L(\mathcal{P}, \alpha, h) = \left\{ (\log p : p \in \mathcal{P}), (\log(m+\alpha) : m \in \mathbb{N}_0), \tfrac{2\pi}{h} \right\}$$

Theorem 6

Suppose that the set $L(\mathcal{P}, \alpha, h)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K, K_1 \in \mathcal{K}$, and $f(s) \in H_0(K)$, $f_1(s) \in H(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \Big\{ 0 \leq k \leq N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon, \sup_{s\in K_1} |\zeta(s+ikh,\alpha) - f_1(s)| < \varepsilon \Big\} > 0.$$

We give some examples of the numbers α and h satisfying the hypothesis of Theorem 6. Suppose that the numbers α and $\exp\{\frac{2\pi}{h}\}$ are algebraically independent over $\mathbb{Q}$. Then the set $L(\mathcal{P}, \alpha, h)$ is linearly independent over $\mathbb{Q}$. Really, it is well known that the set $\{\log p : p \in \mathcal{P}\}$ is linearly independent over $\mathbb{Q}$. Since the numbers α and $\exp\{\frac{2\pi}{h}\}$ are algebraically independent, they are transcendental. Therefore, if we have that

$$k_1 \log p_1 + \cdots + k_m \log p_m + l_1 \log(m_1 + \alpha) + \cdots + l_r \log(m_r + \alpha) = 0,$$

where not all $k_j \in \mathbb{Z}$ and $l_j \in \mathbb{Z}$ are zeros, we obtain that

$$p_1^{k_1} \cdots p_m^{k_m} (m_1 + \alpha)^{l_1} \cdots (m_r + \alpha)^{l_r} = 1,$$

and this contradicts the transcendence of the number α. If

$$k_1 \log p_1 + \cdots + k_m \log p_m + l_1 \log(m_1 + \alpha) + \cdots + l_r \log(m_r + \alpha) + l \frac{2\pi}{h} = 0 \qquad (1)$$

with $l \in \mathbb{Z} \setminus \{0\}$ and at least one $l_j \in \mathbb{Z} \setminus \{0\}$, then

$$p_1^{k_1} \cdots p_m^{k_m} (m_1 + \alpha)^{l_1} \cdots (m_r + \alpha)^{l_r} \left(\exp\{\tfrac{2\pi}{h}\}\right)^l = 1,$$

and this contradicts the algebraic independence of the numbers α and $\exp\{\frac{2\pi}{h}\}$. If, in equality (1), $l \in \mathbb{Z} \setminus \{0\}$, all $l_j = 0$ and at last one $k_j \in \mathbb{Z} \setminus \{0\}$, then the equality

$$p_1^{k_1} \cdots p_m^{k_m} \left(\exp\{\tfrac{2\pi}{h}\}\right)^l = 1$$

contradicts the transcendence of the number $\exp\{\frac{2\pi}{h}\}$. For example, by the Nesterenko theorem [9], the numbers π and e^{π} are algebraically independent over $\mathbb{Q}$. Thus, the numbers $\alpha = \pi^{-1}$ and $h = 2$ satisfy the hypothesis of Theorem 6. Similarly, since the numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent over $\mathbb{Q}$ [3], we may take $\alpha = \left(2^{\sqrt[3]{2}}\right)^{-1}$ and $h = \frac{2\pi}{\sqrt[3]{4} \log 2}$.

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