A Tale of Two Models for Random Graphs

Jeong Han Kim

Korea Institute for Advanced Study (KIAS)

- Random Graph $G(n,p)$
  - Component and Branching process
  - Emergence of Giant (connected) Component
  - Motivation for Poisson cloning

a. Poisson Cloning Model (PCM)
b. Cut-Off Line Algorithm (COLA)
c. New Results via PCM and COLA

- Random Graph $G(n,m)$

a. Graph Process $G(n,1), G(n,2), \ldots, G(n,m), \ldots.$
b. Graph Process under constraints
c. Triangle-free Process and Ramsey Number $R(3,t)$

Complete Graph K_n

- When the edge set E of a graph $G = (V, E)$ is the set of all pairs of distinct vertices, then the graph is called the complete graph on V, and denoted by K_V, or simply K_n, where n is the number of all vertices.

Notice that there are

$$\binom{n}{2} := \frac{n(n-1)}{2}$$ edges in K_n.

Random Graph $G(n,p)$

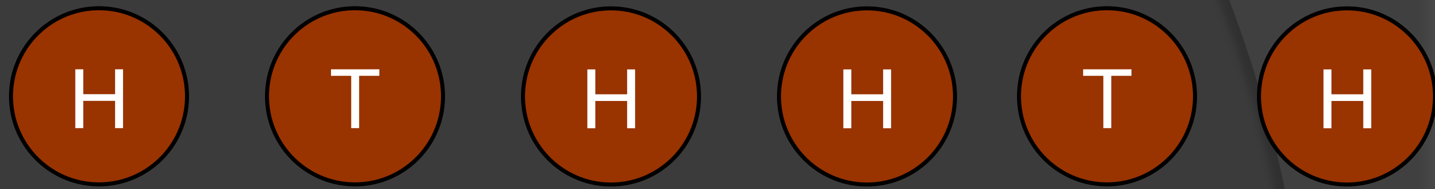
Each of $\binom{n}{2}$ edges is independently in $G(n,p)$ with probability p,

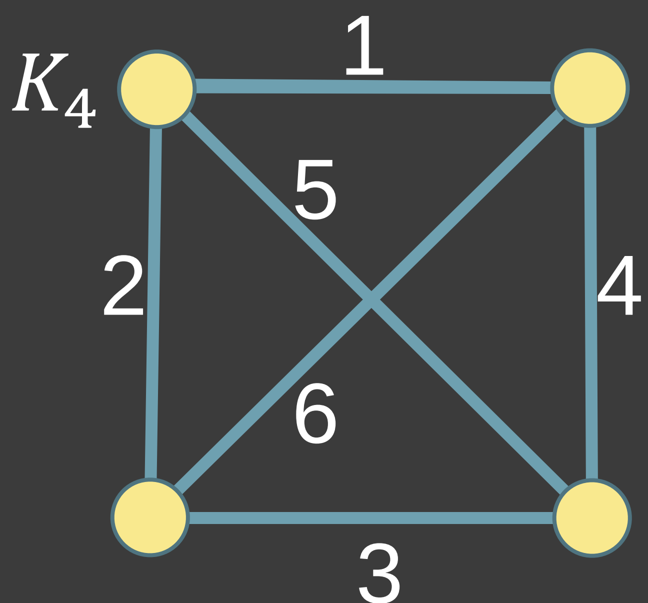
$p=1$: complete graph

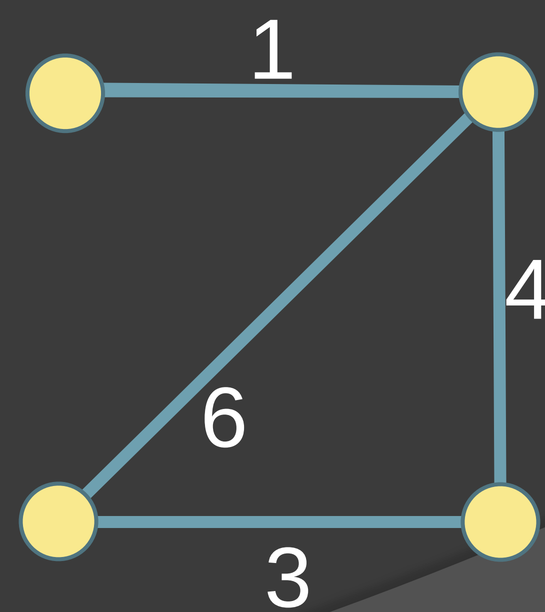
$p=0$: empty graph

- For example, if $n = 4, p = 1/2$, toss a fair coin six times

an instance

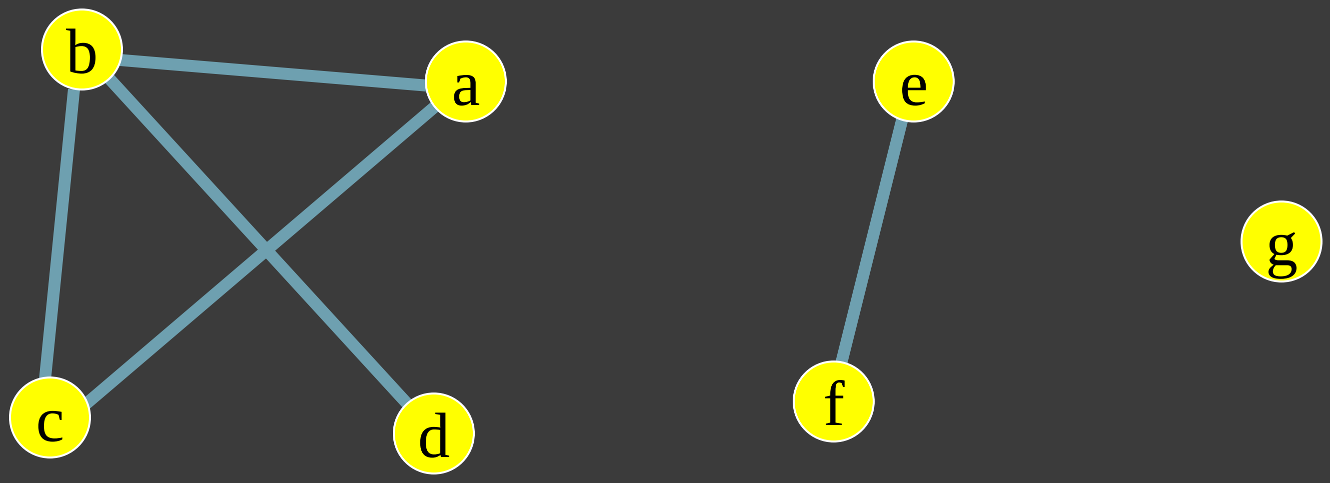






an instance of $G(4,1/2)$

For $n = 7$,



$$\Pr[\, G(7,p) = \ G \,] = p^5(1-p)^{21-5}$$

as $\binom{7}{2} = \frac{7\cdot 6}{2} = 21$

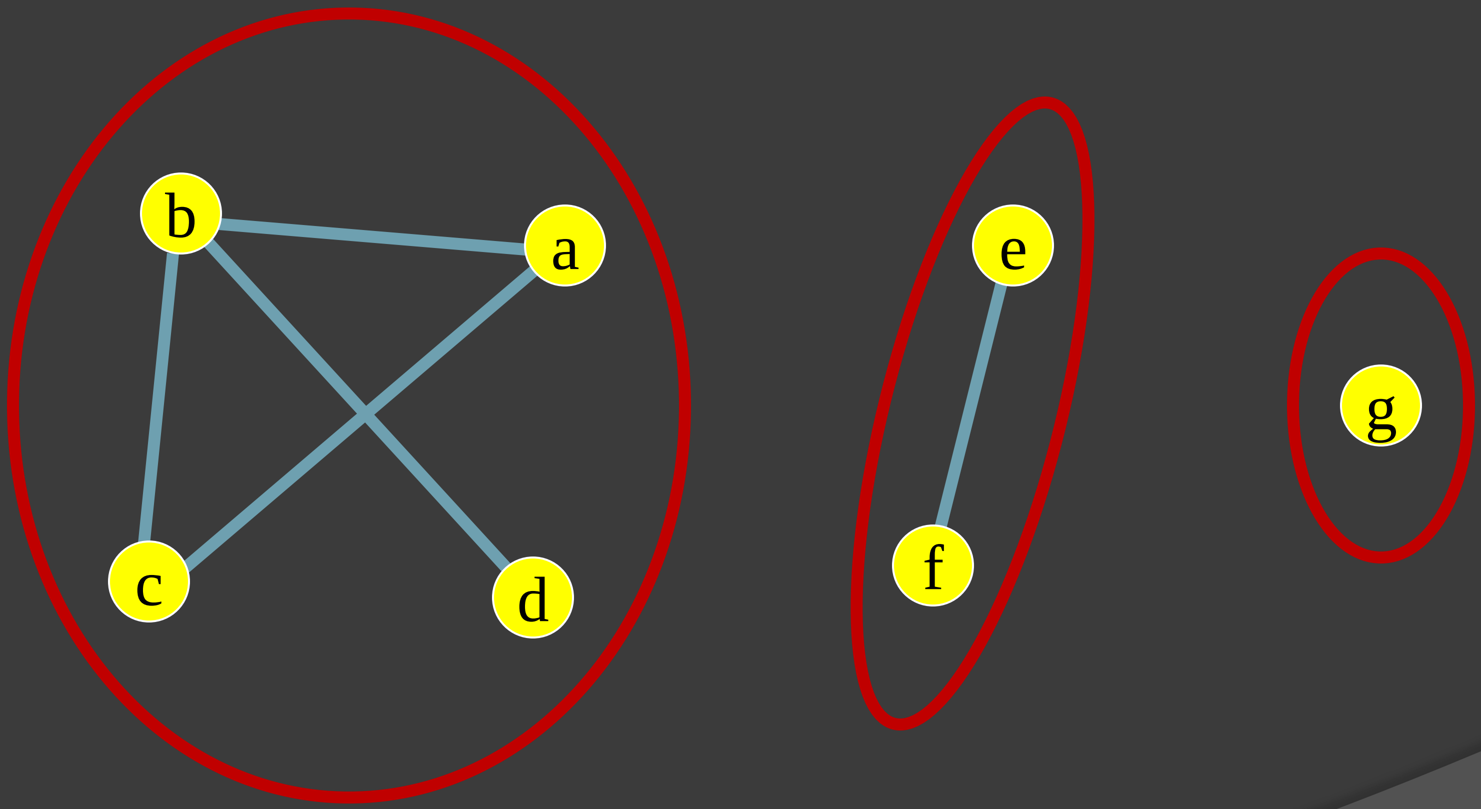
- Expected number of edges in $G(n, p)$:

$$p\binom{n}{2}$$

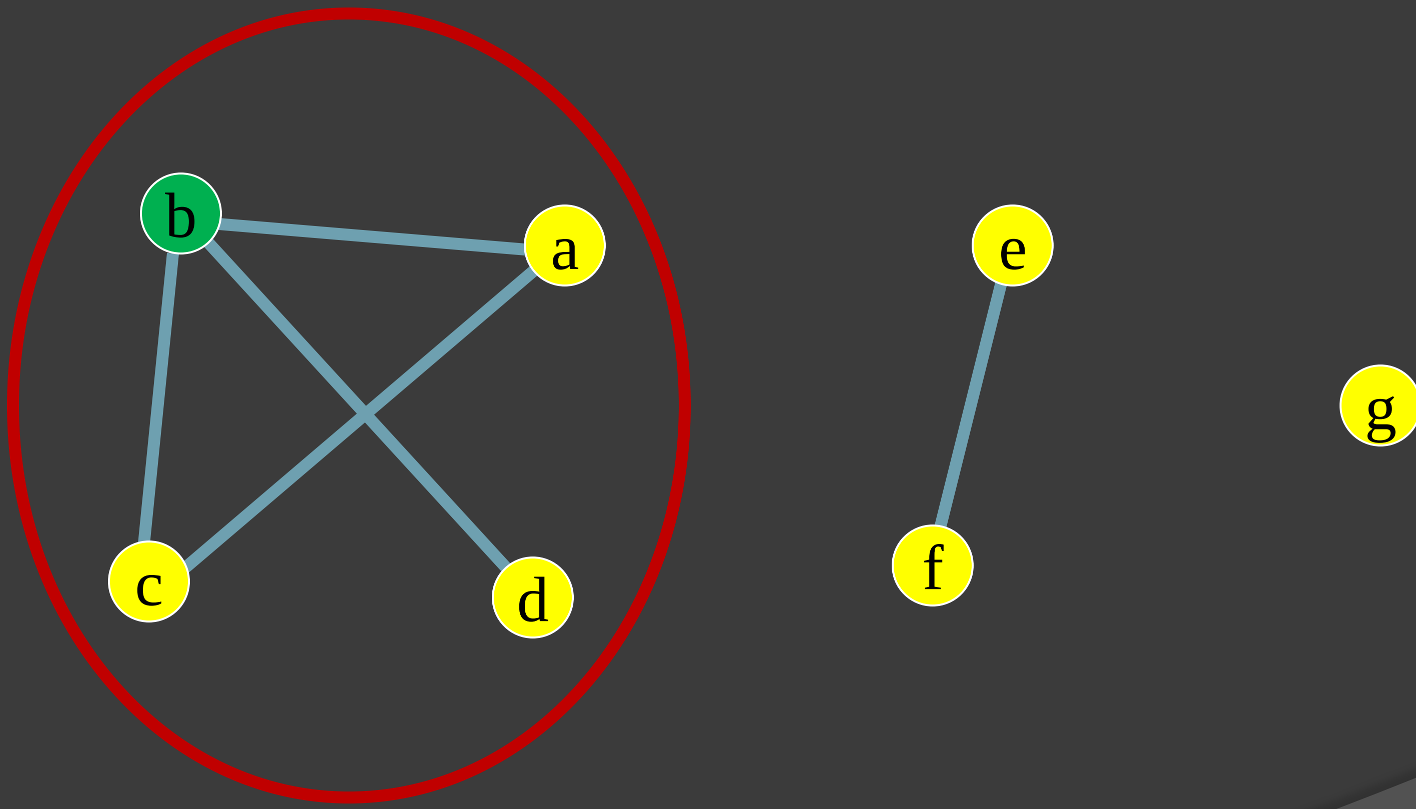
- For a fixed graph G with n vertices and m edges,

$$\Pr[\, G(n,p) \, = G \,] \, = \, p^{m}(1-p)^{\binom{n}{2}-m}$$

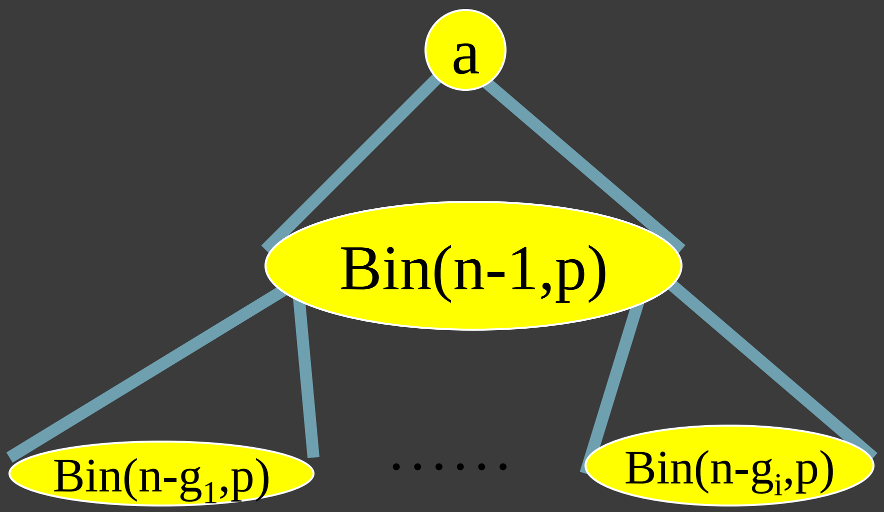
(Connected) Component

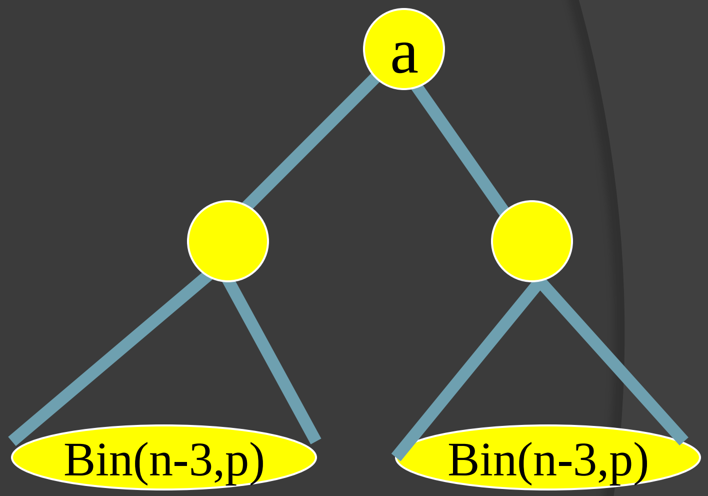


Component containing a vertex



Component containing a vertex in $G(n, p)$





where

$$\Pr[\, Bin(t,p) = k \,] = \binom{t}{k} p^k (1-p)^{t-k}$$

Emergence of Giant Component

- Erdős & Rényi ('60,'61):

In $G(n, p)$ with $pn = \lambda$ for a constant λ

there is a giant component if and only if $\lambda > 1$

Emergence of Giant Component

- Erdős & Rényi ('60,'61):

For the size $W = W(n,p)$ of a largest component of $G(n,p)$,

$$W = \begin{cases} < c_\lambda \log n & if\ pn \to \lambda < 1 \\ \theta_\lambda n & if\ pn \to \lambda > 1, \end{cases}$$

where θ_λ is the positive solution for

$$1 - \theta - e^{-\theta\lambda} = 0$$

- Łuczak ('90)

In $G(n,p)$ with $pn = \lambda$,

the giant component emerges when $\lambda - 1 \gg n^{-1/3}$

improving $\lambda - 1 \gg n^{-\frac{1}{3}} \log n$ due to Bollobás ('84).

- Łuczak ('90)

In $G(n, p)$ with $\lambda = pn$ satisfying $\varepsilon := \lambda - 1 \gg n^{-1/3}$,

$$W = (2 + O(\varepsilon))\, \varepsilon\, n$$

and for the size W' of a second largest component

$$W' = \Theta\left(\varepsilon^{-2} \log(\varepsilon^3 n)\right) \ll \varepsilon\, n$$

- Łuczak ('90)

Theorem (supercritical region) Let $\lambda := pn = 1+\varepsilon$ with $\varepsilon \gg n^{-1/3}$. Then with probability $1-O((\varepsilon^3 n)^{-1/9})$,

$$|W(n,p) - \theta_\lambda n| \le 0.2\, n^{2/3}.$$

Emergence of Giant Component

- Erdős & Rényi ('60,'61):

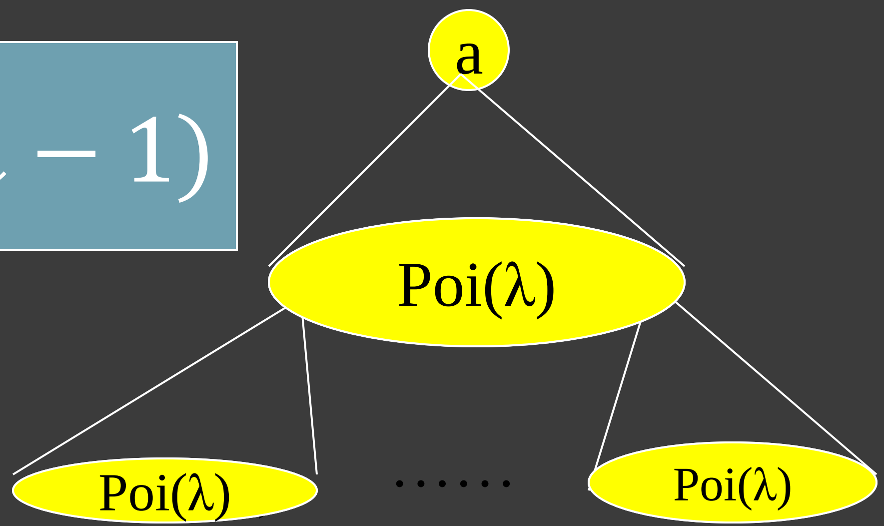
In $G(n,p)$ with $pn = \lambda$ for a constant λ

There is a giant component if and only if $\lambda > 1$.

Why 1 ?

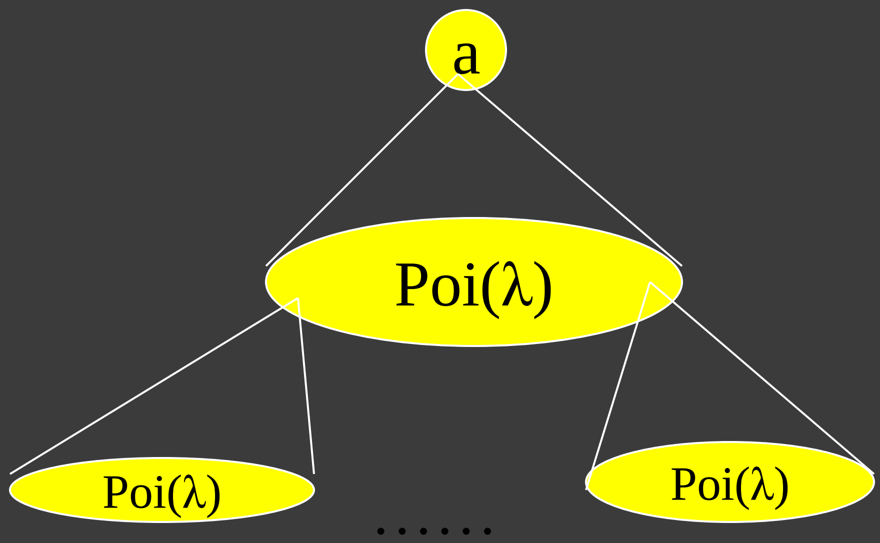
Component containing a vertex in $G(n,p)$

For $\lambda = p(n-1)$



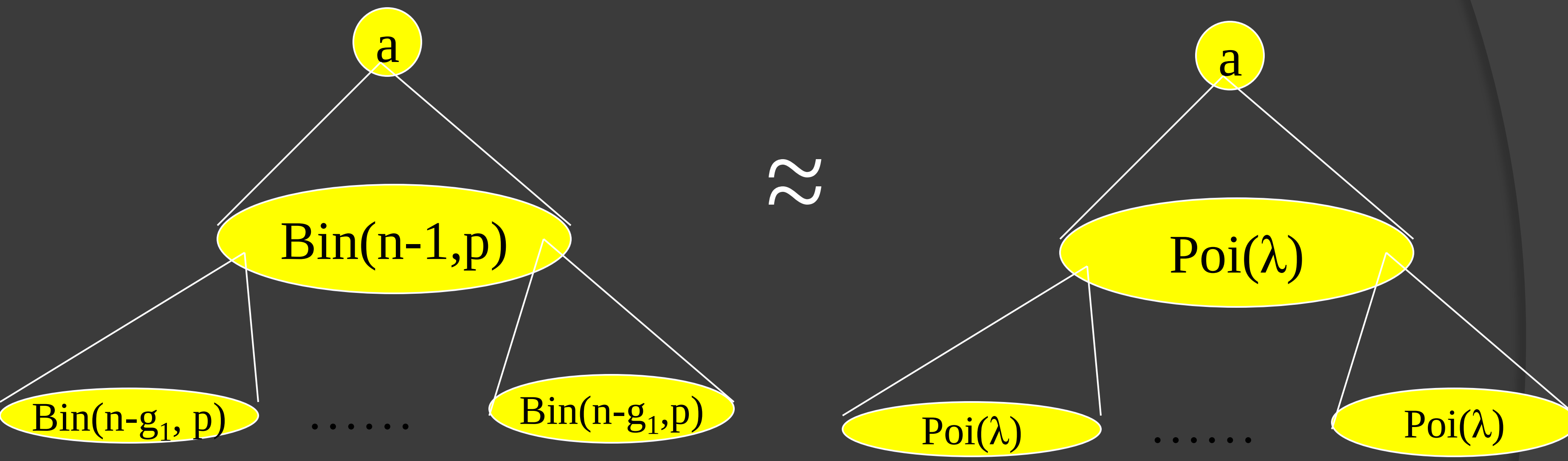
$$\Pr[\ \mathrm{Poi}(\lambda) = k\ ] = e^{-\lambda} \frac{\lambda^k}{k!}$$

Poisson Branching Process



- If $\lambda < 1$, the process dies out with probability 1
- If $\lambda > 1$, the process survives forever with probability θ_λ (recall θ_λ is the positive solution for $1 - \theta - e^{-\theta\lambda} = 0$).

Two Obstacles



- The difference between $Bin(n-1,p)$ and $Poi(\lambda)$.
- The drift : $p(n-g_i)$ keeps decreasing

Similar but more serious obstacles occur in the analyses of

- The k-core problem of the random graph (Pittel-Spencer-Wormald, …)
- Giant strong component of the random directed graph (Karp,…)
- Pure literal algorithm for the random satisfiability problem (Broder-Frieze-Upfal, …)
- Unit clause algorithm for the random satisfiability problem (Chao-Franco,…)
- Karp-Sipser Algorithm to find a large matching in the random graph (Karp-Sipser, …)

The First Obstacle

- The difference between $Bin(n-1, p)$ and $Poi(\lambda)$.

More generally, is there a random graph model $G_{new}(n,p)$, in which all degrees are i.i.d $Poi(\lambda)$ and

$$G_{new}(n,p) \approx G(n,p)?$$

NO: The sum of degrees must be even.

YES in an asymptotic sense

Is it possible to define a random graph model in which all degrees are i.i.d $Poi(\lambda)$?

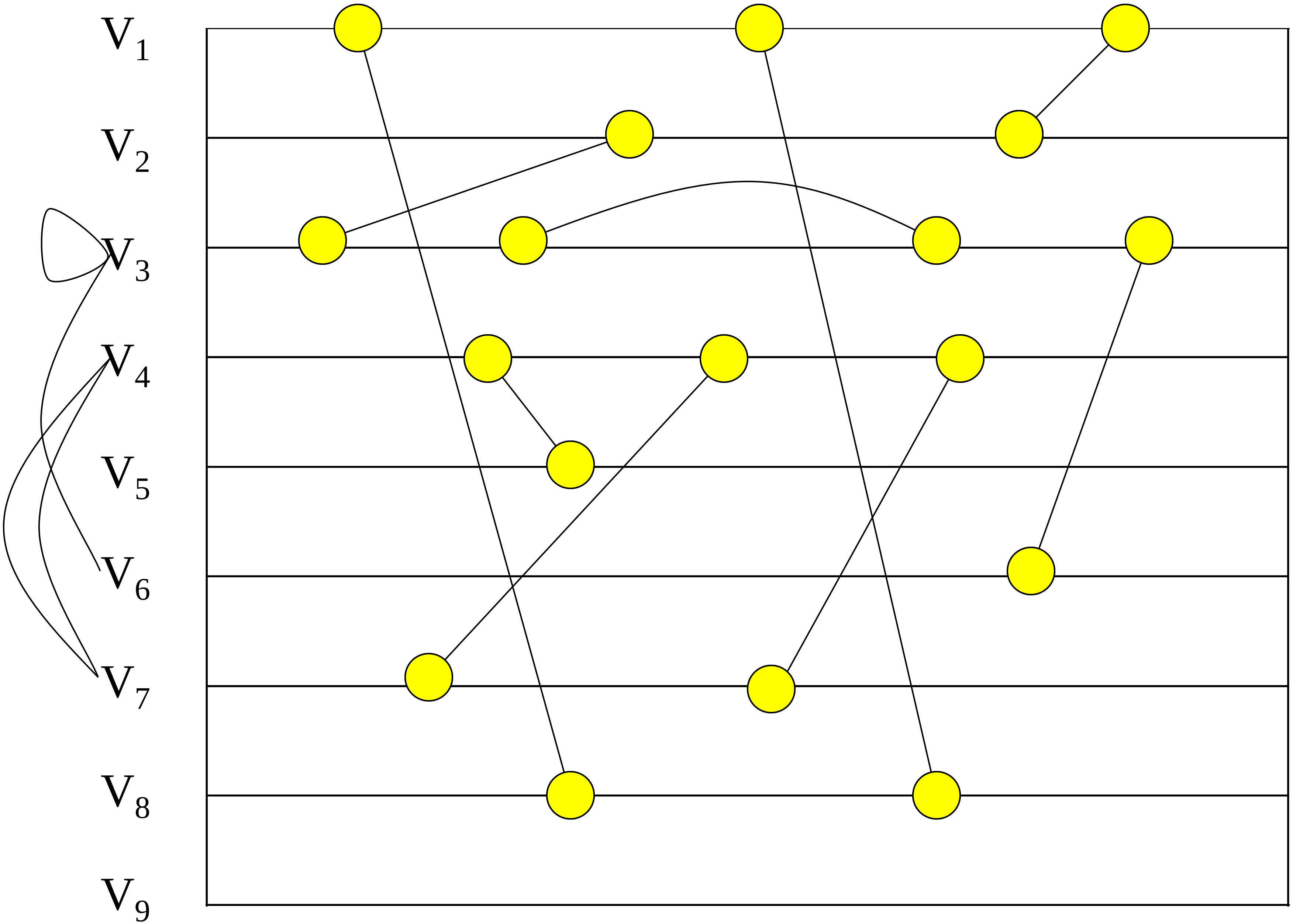
YES

HOW?

Just Do It!

Poisson Cloning Model $G_{PC}(n, p)$: Definition

- Take i.i.d Poisson random variables $d(v)$'s, v in V, with mean $\lambda = p(n-1)$.
- For each vertex v in V, take $d(v)$ copies, or clones, of v.
- If $\Sigma\, d(v)$ is even, generate a uniform random perfect matching on all clones and then contract clones of the same vertex.



V_1
V_2
V_3
V_4
V_5
V_6
V_7
V_8
V_9

- If $\Sigma d(v)$ is odd, generate a perfect matching excluding a clone. The excluded clone induces a loop.

(When $d(v) = d$ for all v, this is the configuration model for random regular graphs due to B. Bolloás ('84).)

$G(n,p) \approx G_{PC}(n,p)$

Theorem (K) If $pn = O(1)$, then, for any event A regarding $G(n,p)$,

$$Pr[A] \geq c_1 Pr_{PC}[A] - e^{-(pn^2)}$$

and

$$Pr[A] \leq c_2 Pr_{PC}[A]^{1/2} + e^{-\Omega(pn^2)}$$

In particular,

$$Pr[A] \to 0 \quad iff \quad Pr_{PC}[A] \to 0$$ and

$$Pr[A] \to 1 \quad iff \quad Pr_{PC}[A] \to 1$$

The Second Obstacle

- The drift : $p(n - g_i)$ keeps decreasing

We introduce

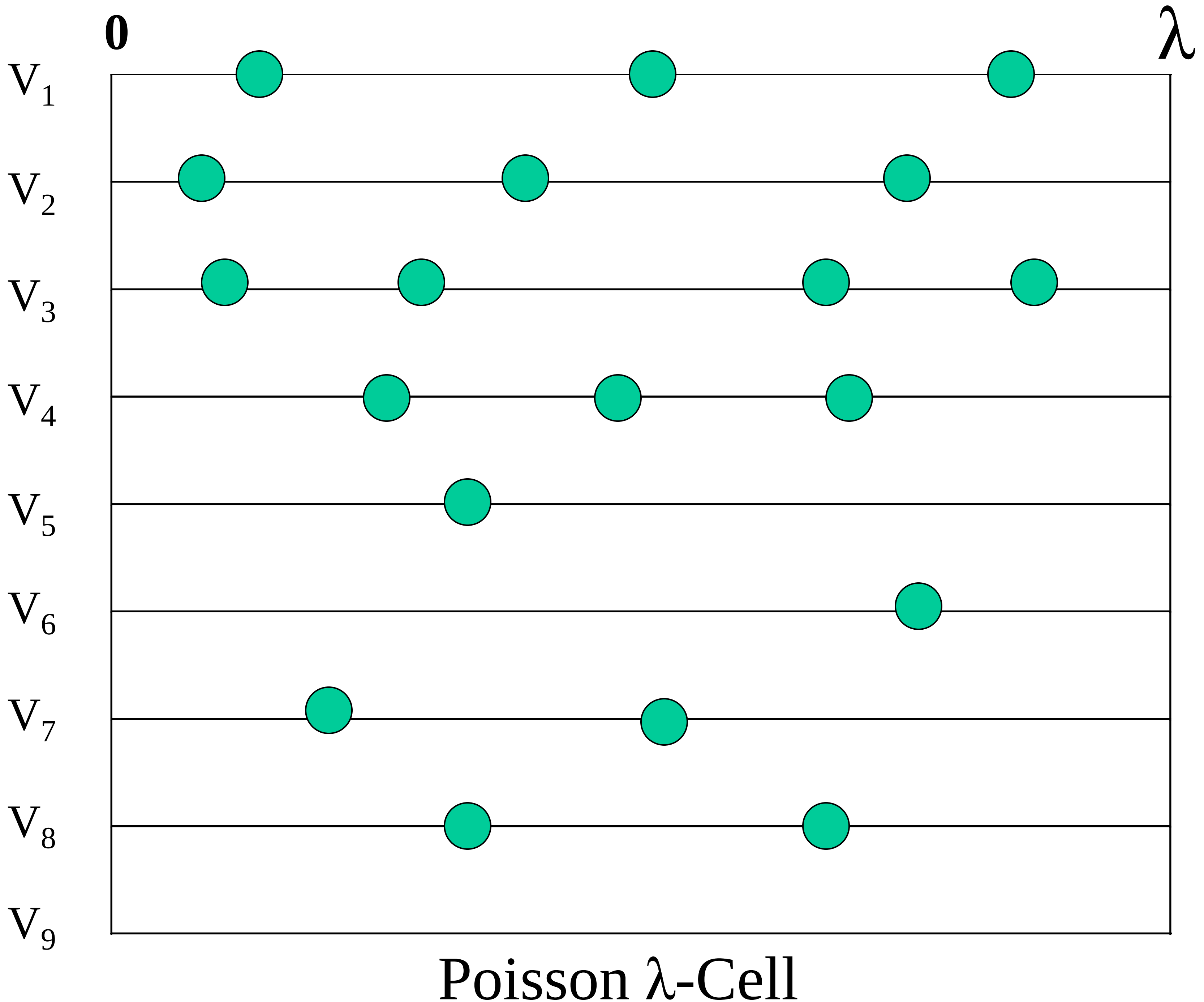
the cut-off line algorithm

(COLA)

Poisson λ-cell

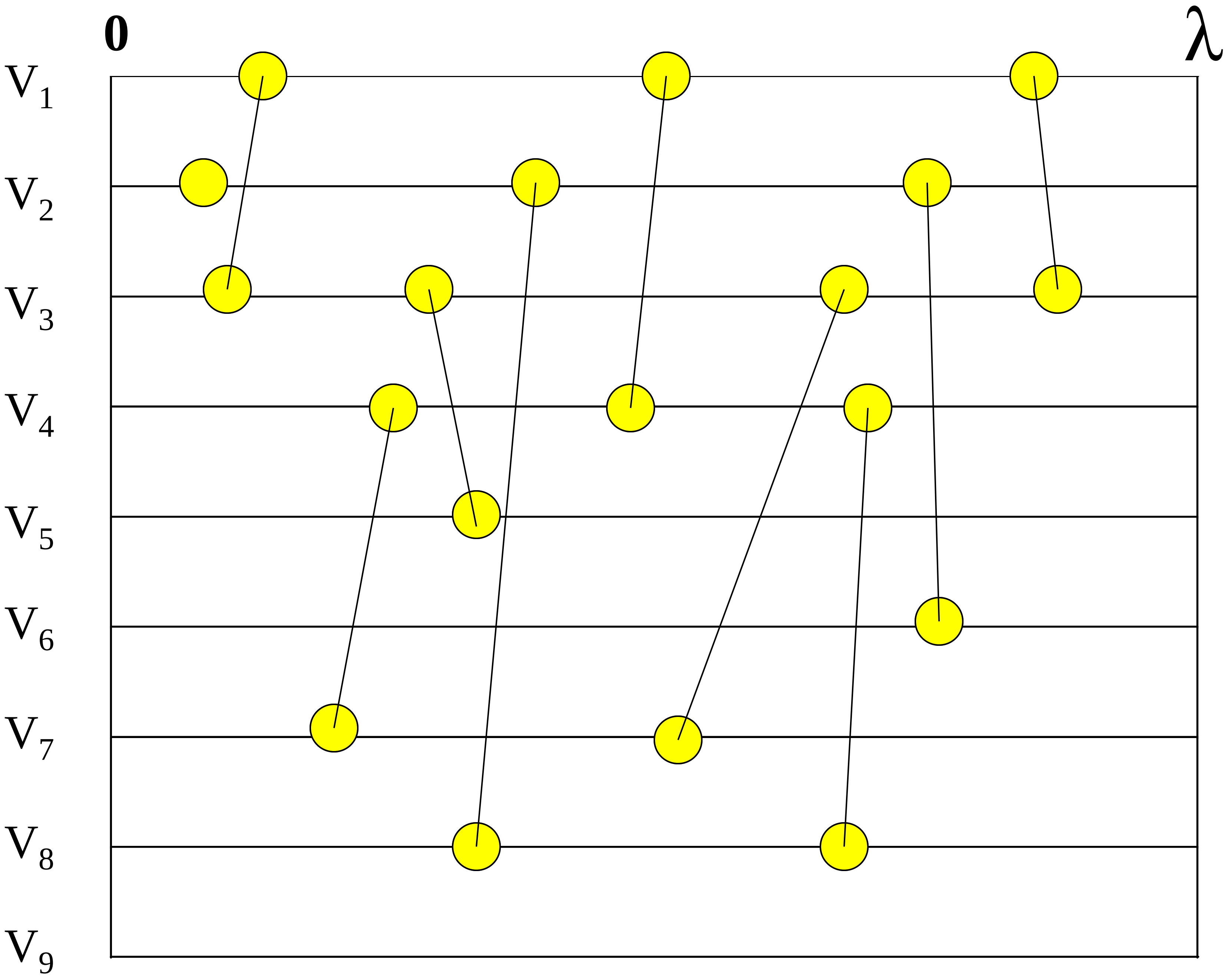
- For each clone w, assign a uniform random (real) number between
  0 and λ, independently of all others.

- A clone is larger than another clone if so are the assigned numbers.


0
λ
V_1
V_2
V_3
V_4
V_5
V_6
V_7
V_8
V_9
Poisson λ-Cell

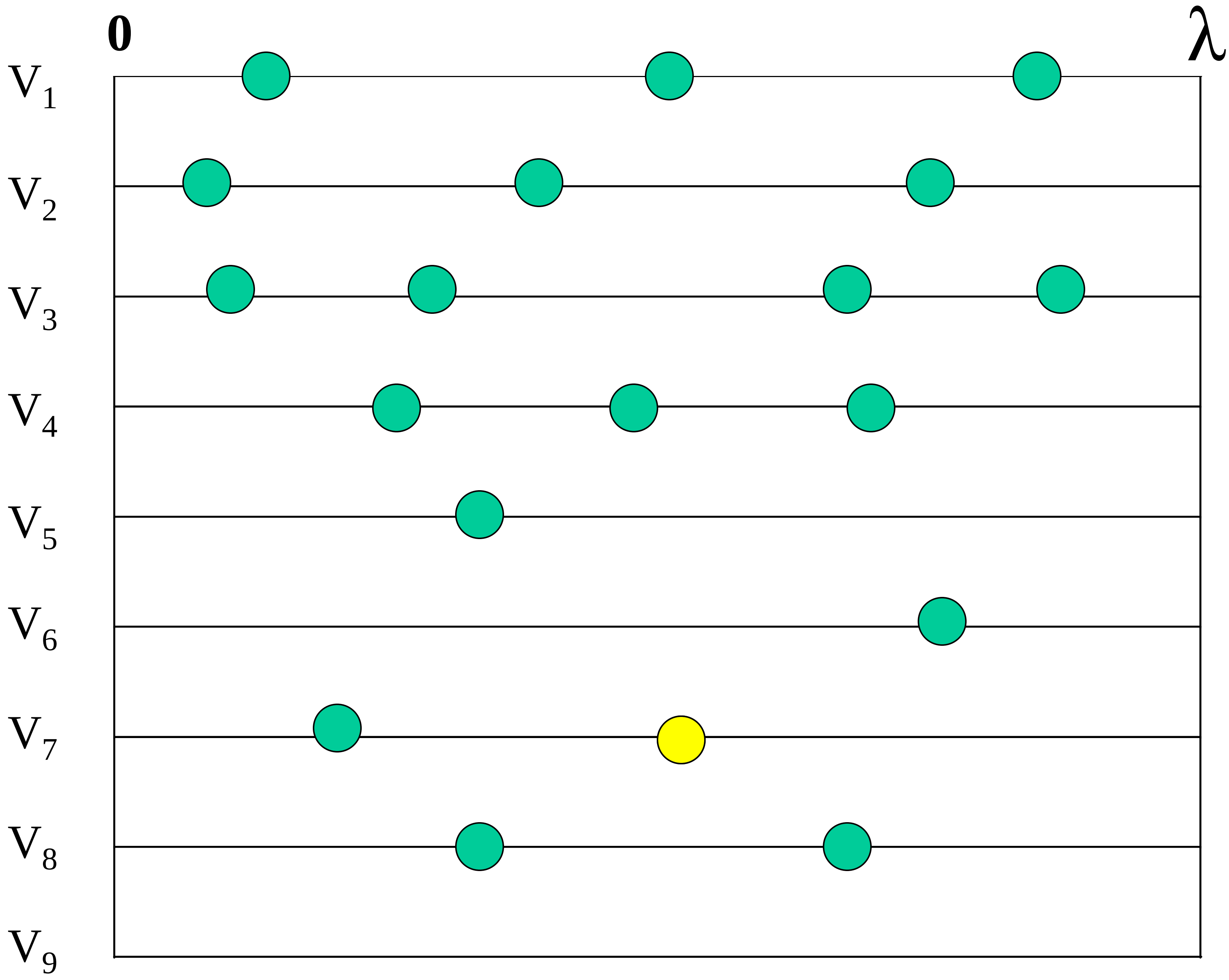
A way to generate the random perfect matching

Take two largest clones and match them. Repeat this for the remaining unmatched clones

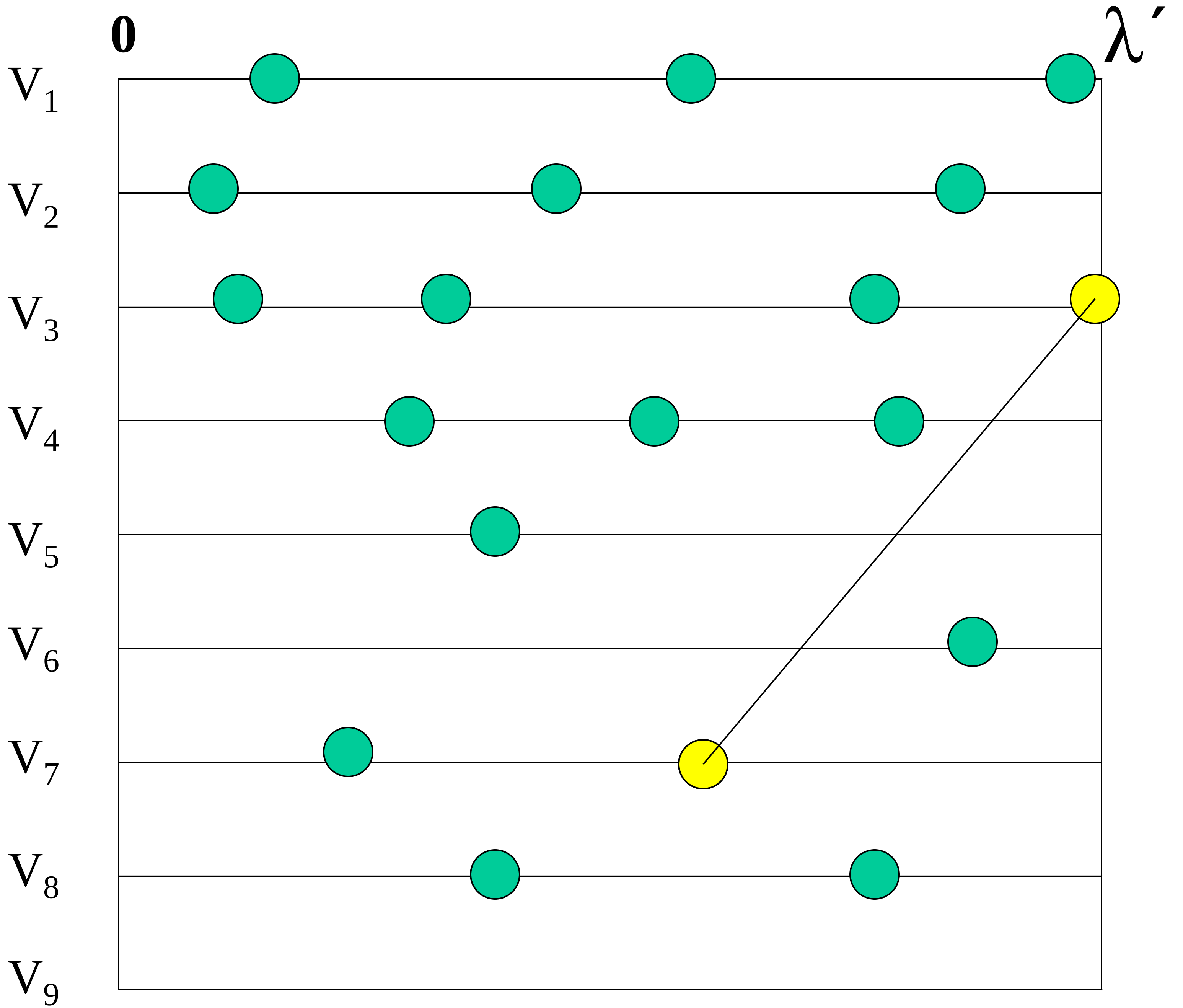

0
λ
V1
V2
V3
V4
V5
V6
V7
V8
V9

Cut-Off Line Algorithm (COLA)

Choose the first unmatched clone
according to a certain selection rule
independently of assigned numbers
and
match it to the largest
unmatched clone



0
λ
V1
V2
V3
V4
V5
V6
V7
V8
V9

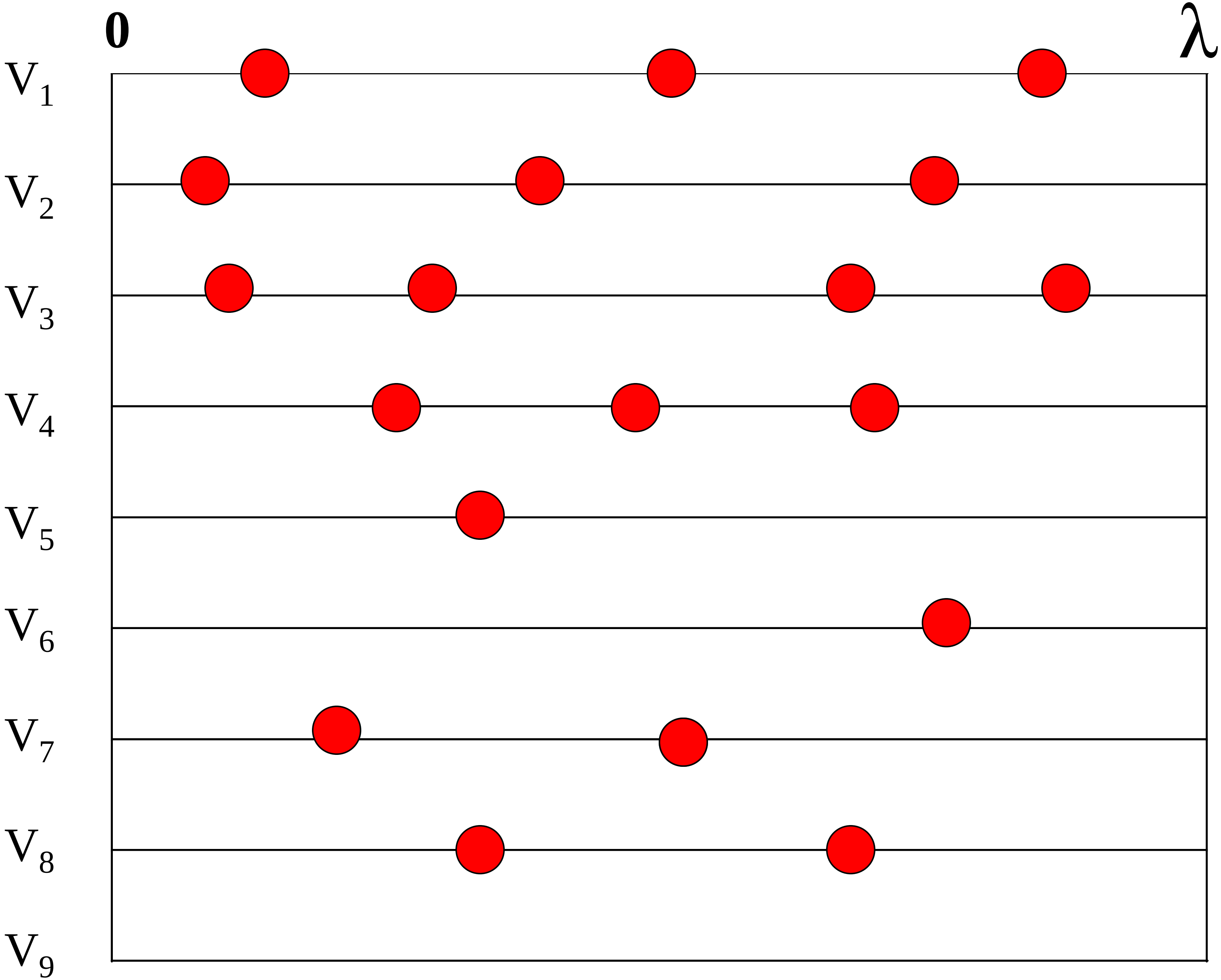

0
λ′
V1
V2
V3
V4
V5
V6
V7
V8
V9

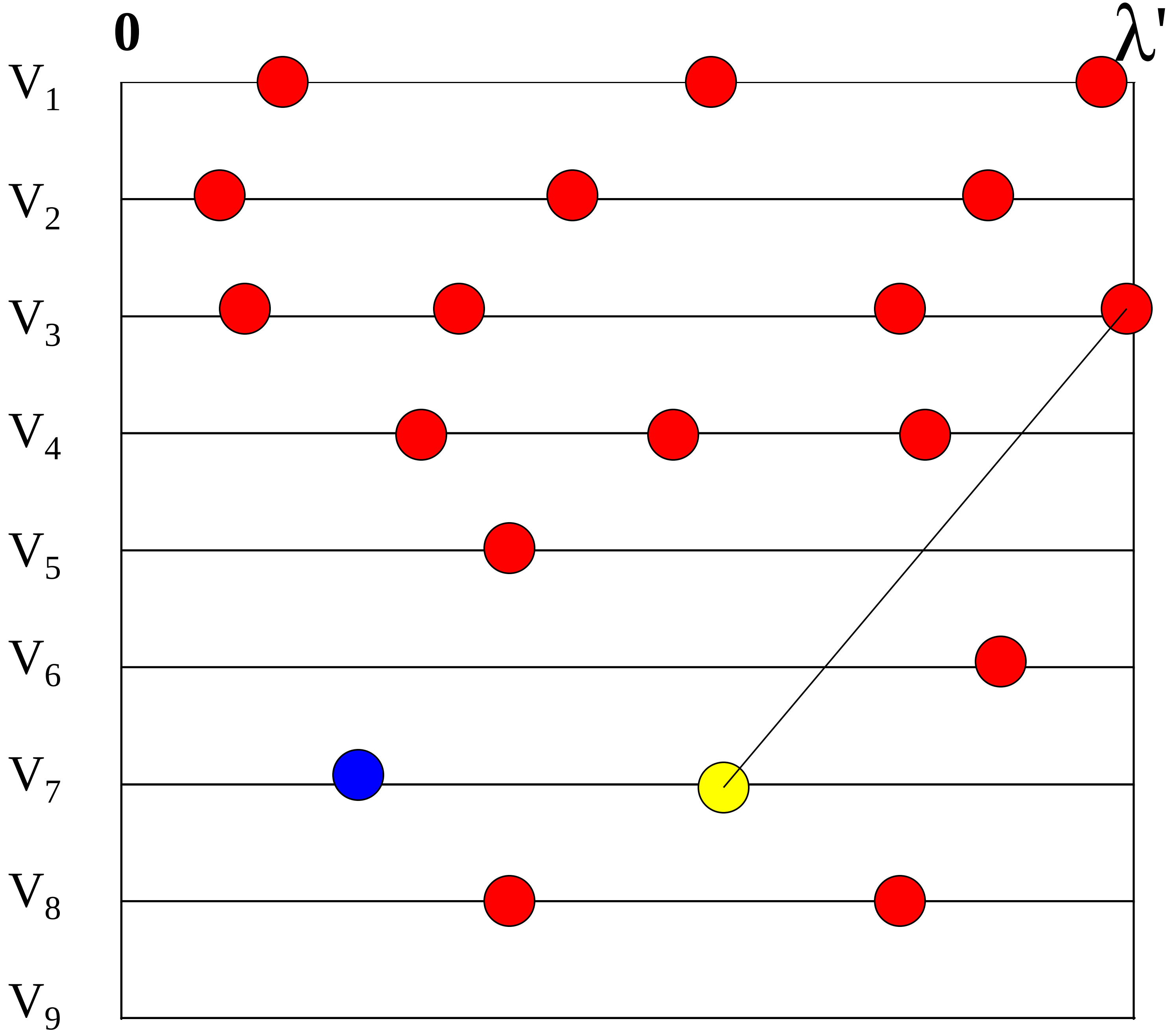
Cut-Off Line Algorithm (COLA)

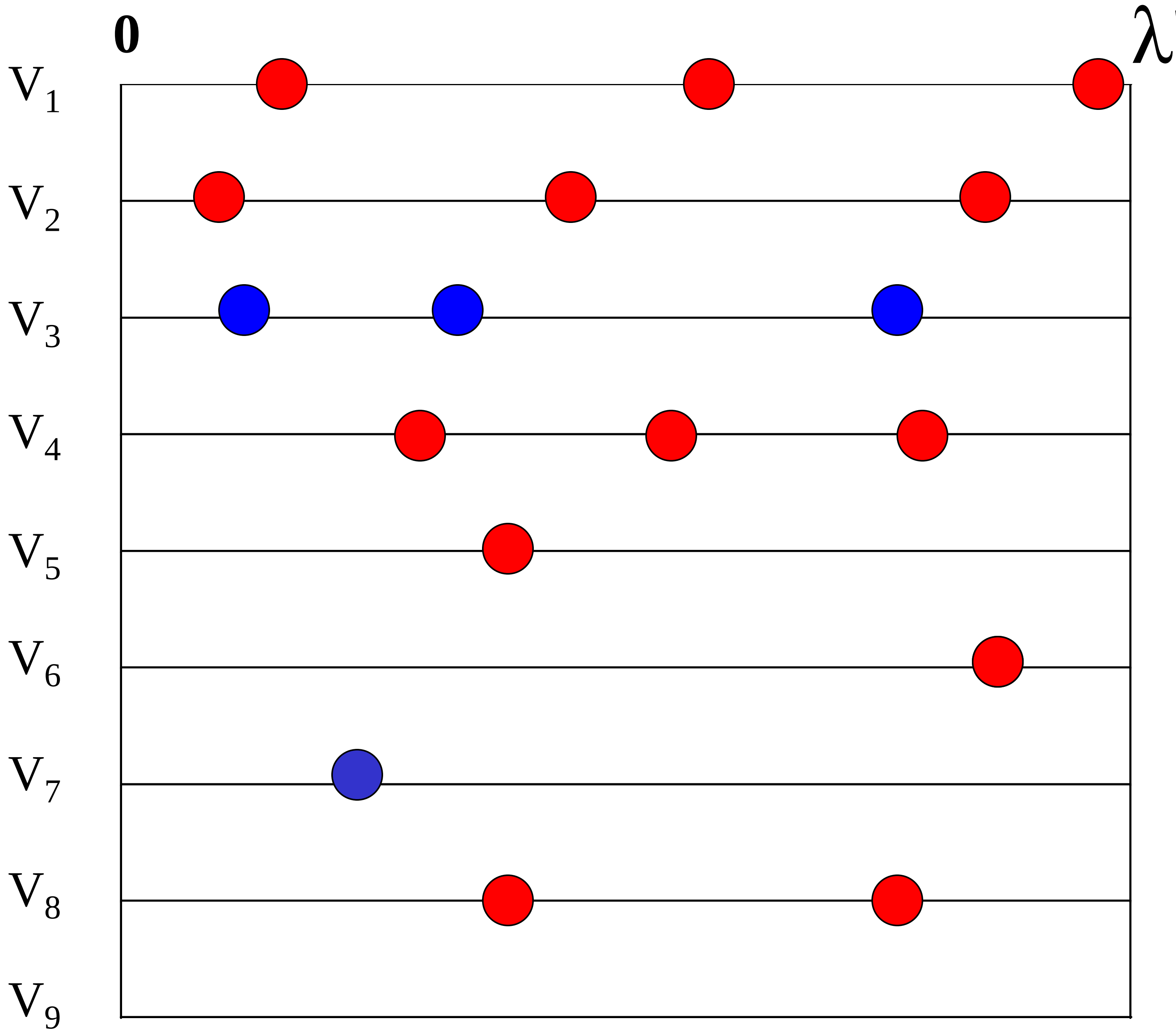
- Initially, the cut-off value $\Lambda = \lambda$. The cut-off line is the vertical line containing $(\Lambda, 0)$.
- After one step, the cut-off value $\Lambda = \lambda'$. The cut-off line is the vertical line containing $(\Lambda, 0)$.

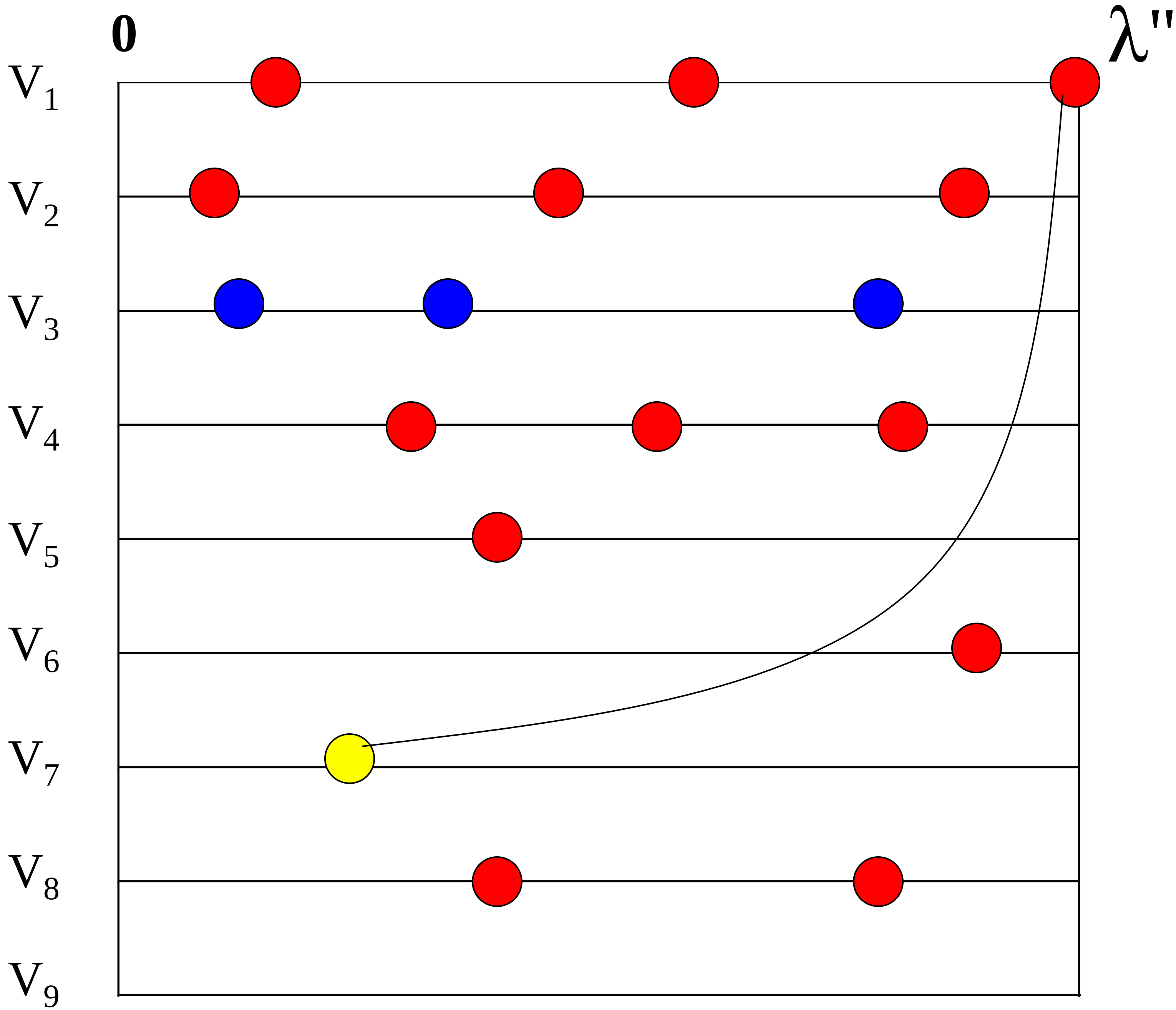
Notice that λ' is the largest number among $N-1$ independent uniform random numbers between 0 and λ.

Cut-Off Line Algorithm for Component


0
λ
V_1
V_2
V_3
V_4
V_5
V_6
V_7
V_8
V_9


0
λ'
V1
V2
V3
V4
V5
V6
V7
V8
V9


0
λ'
V_1
V_2
V_3
V_4
V_5
V_6
V_7
V_8
V_9


0
λ"
V1
V2
V3
V4
V5
V6
V7
V8
V9

New Results

Random Graph $G(n, m)$

- For a fixed vertex set V of n elements, consider all graphs on V with m edges.

  There are

  $$\binom{\binom{n}{2}}{m} \text{ such graphs}$$

- Choose a graph uniformly at random from all such graphs.

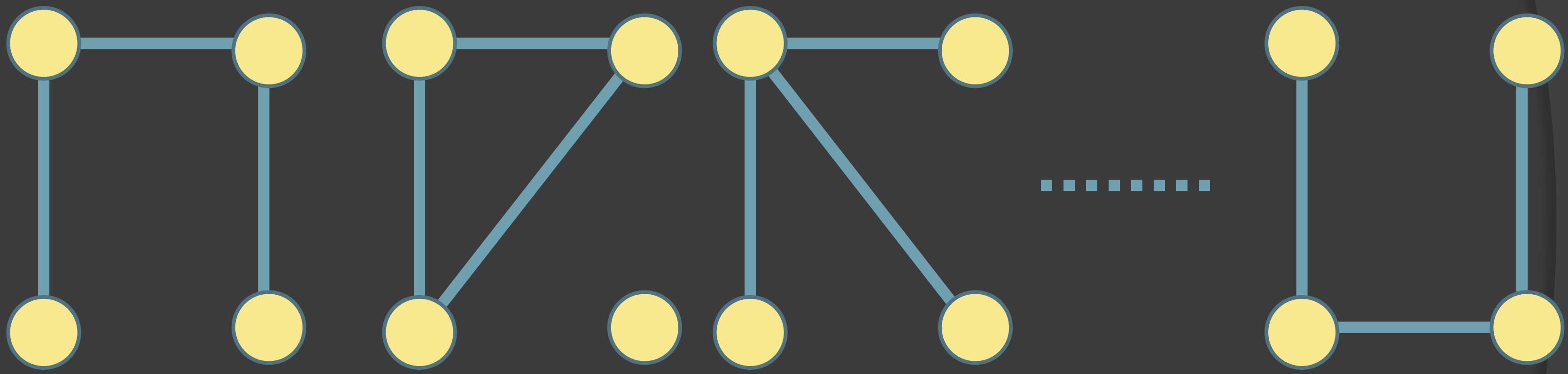
- Easy

  $$G(n, m) \approx G(n, p)$$

provided $p\binom{n}{2} = m$.

- For example, if $n = 4\ m = 3$, then

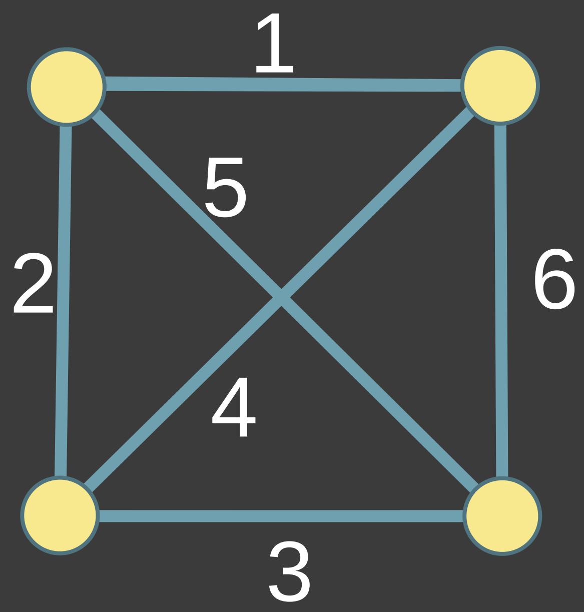
each of $\binom{\binom{4}{2}}{3} = \binom{6}{3}$ =20 graphs



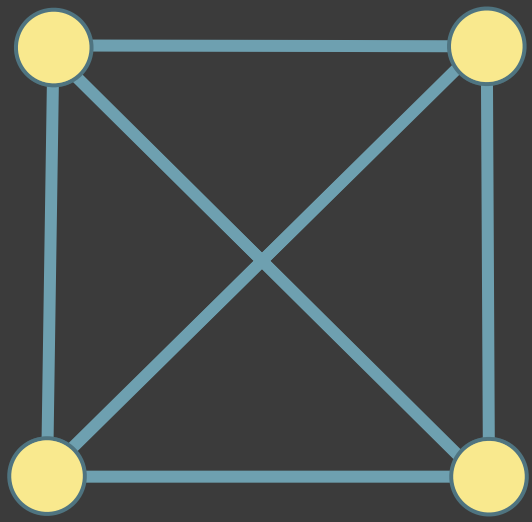
is equally likely to be $G(4,3)$.

Graph Process $G(n,1),\ldots,G(n,m),\ldots$

- Randomly order all $\binom{n}{2}$ edges of K_n so that each of $\binom{n}{2}!$ ordereings is equally likely.
- The graph $G(n,m)$ is the graph consisting of the first m edges in the random ordering.
- Notice that $G(n,m)$ in the graph process has the same distribution as $G(n,m)$ defined earlier.

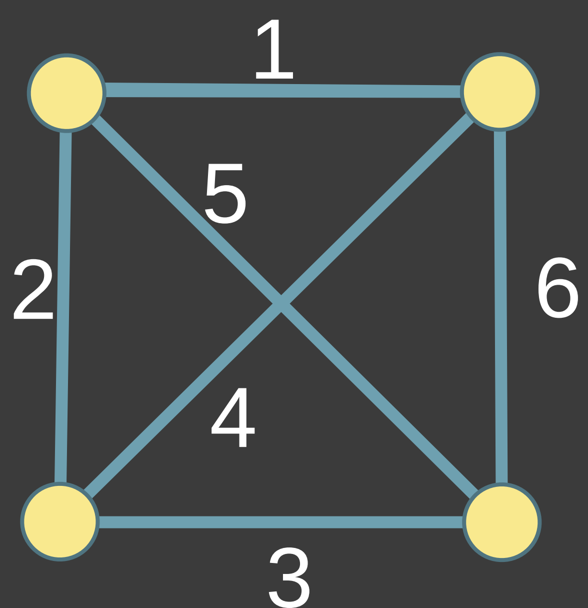


G_1 G_2 G_3 G_4 G_5 G_6

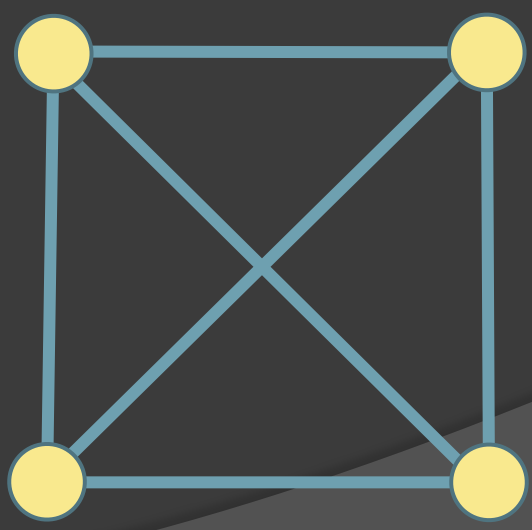


Graph process under constraint

Triangle Free Process: no triangle is allowed



G_1 G_2 G_3 G_4

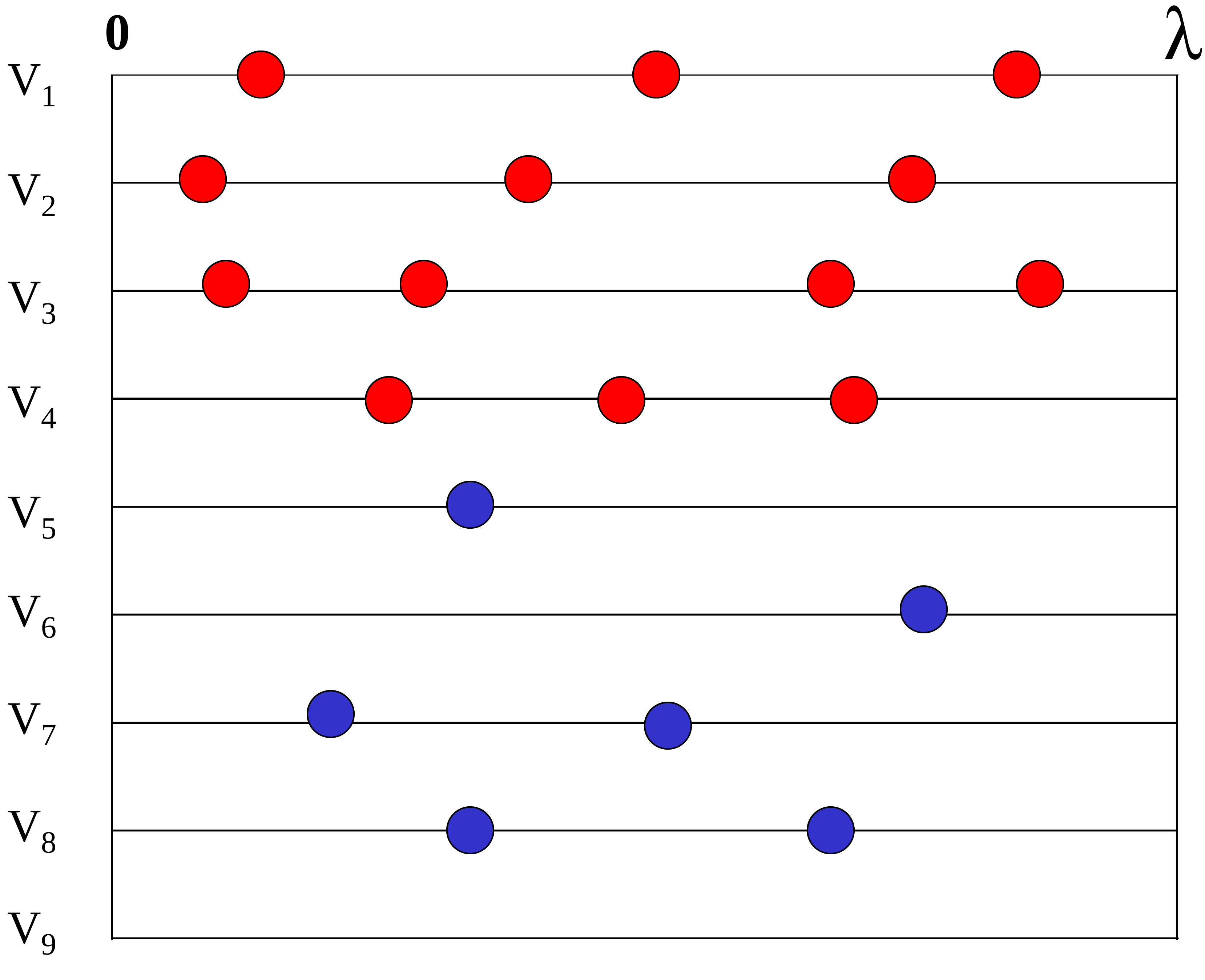


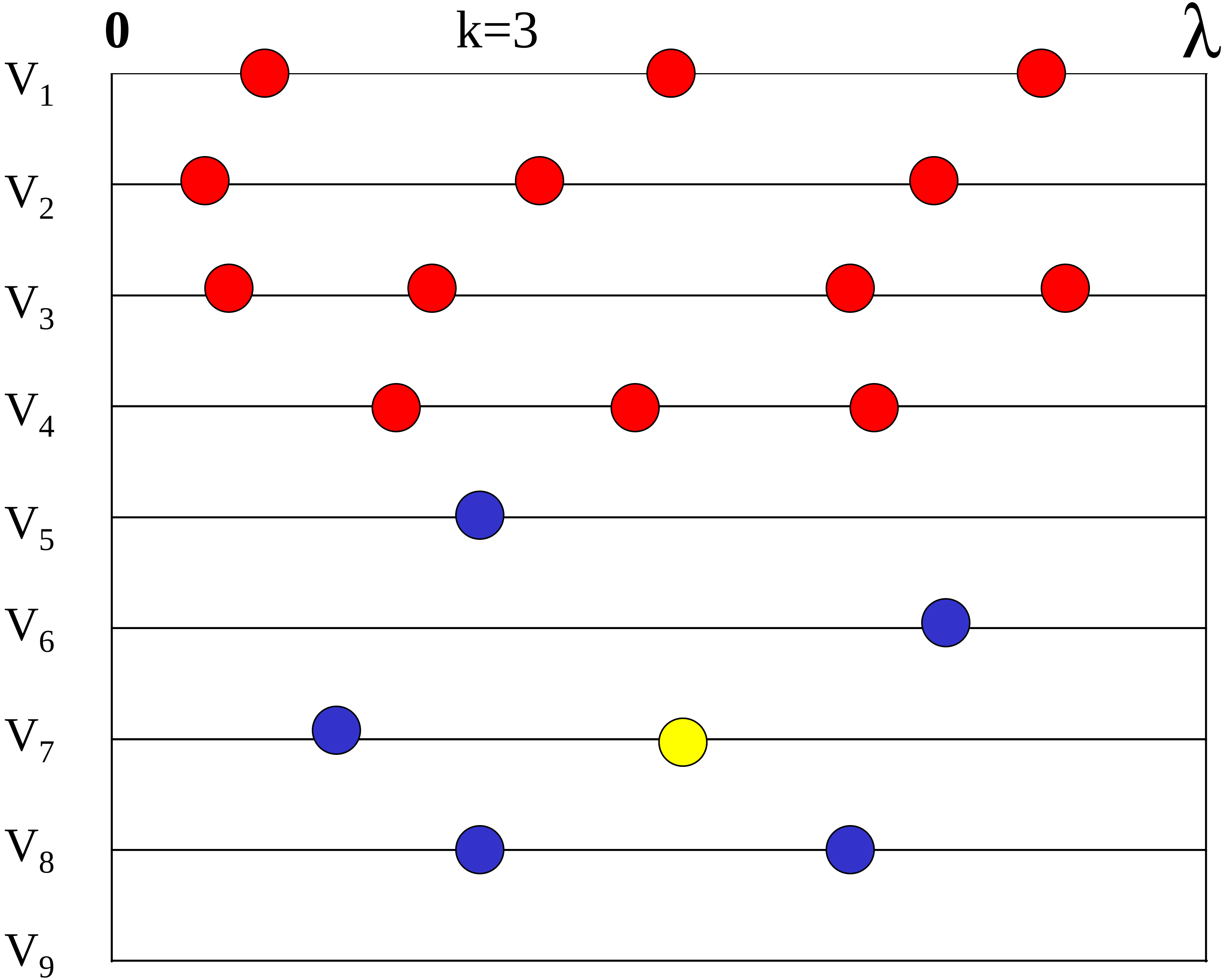
This is called the triangle-free process.

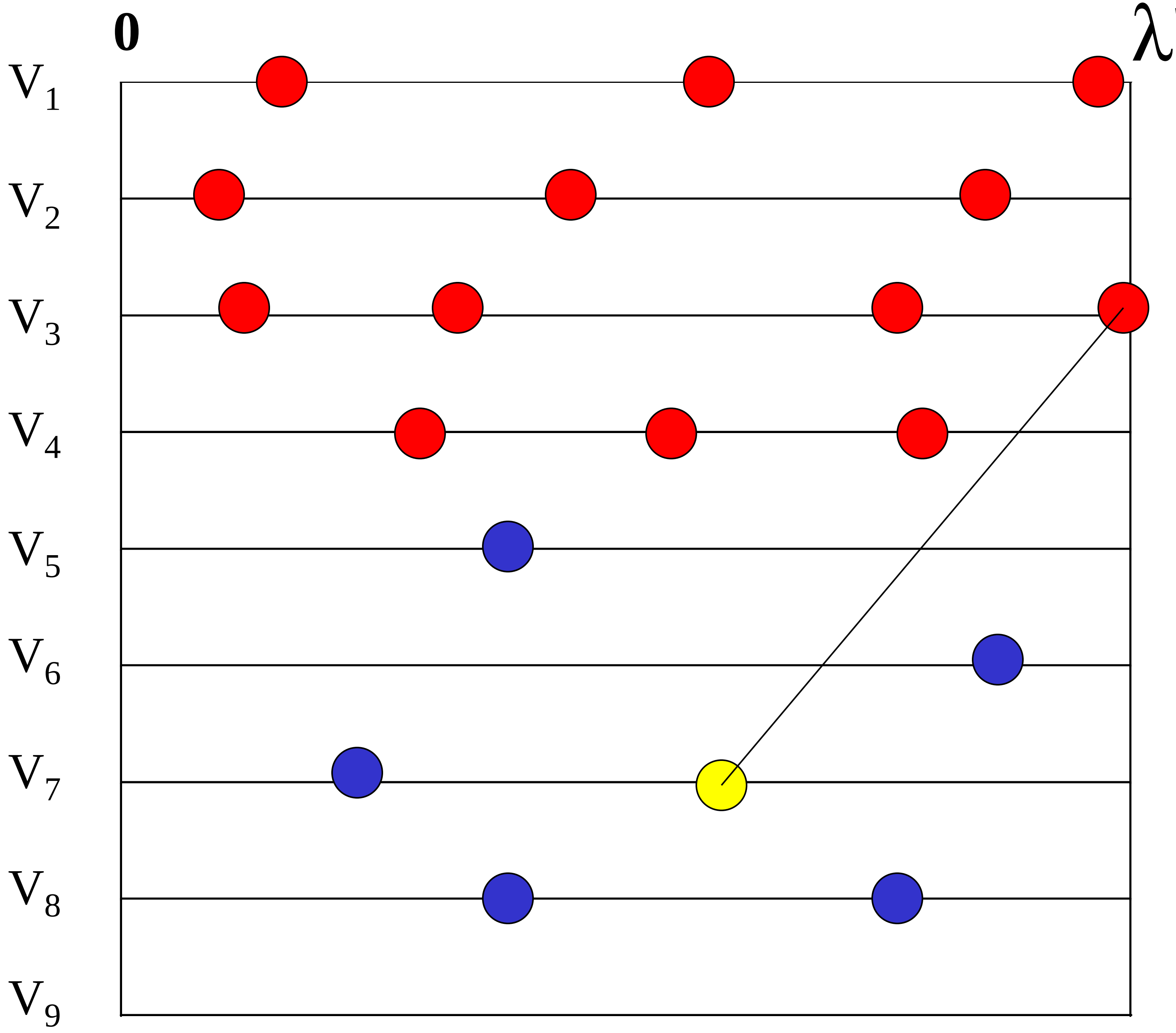
Graph process under constraint

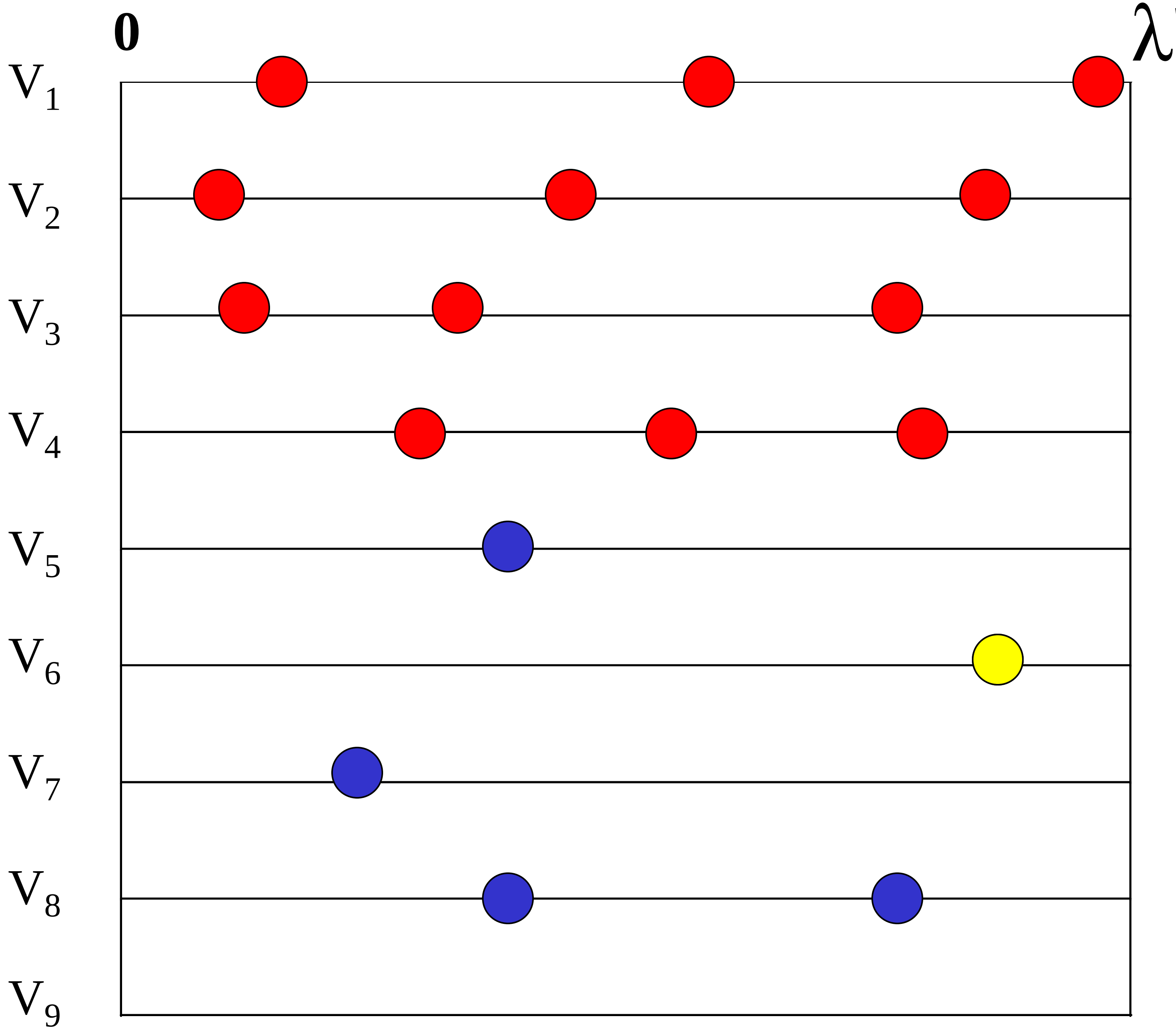
- Many applications to construct graphs satisfying certain properties.
- It is generally not easy to analyze the process
- Studied by many researchers including Ajtai, Komlós, Szemerédi, Rodl, Kahn, and more.

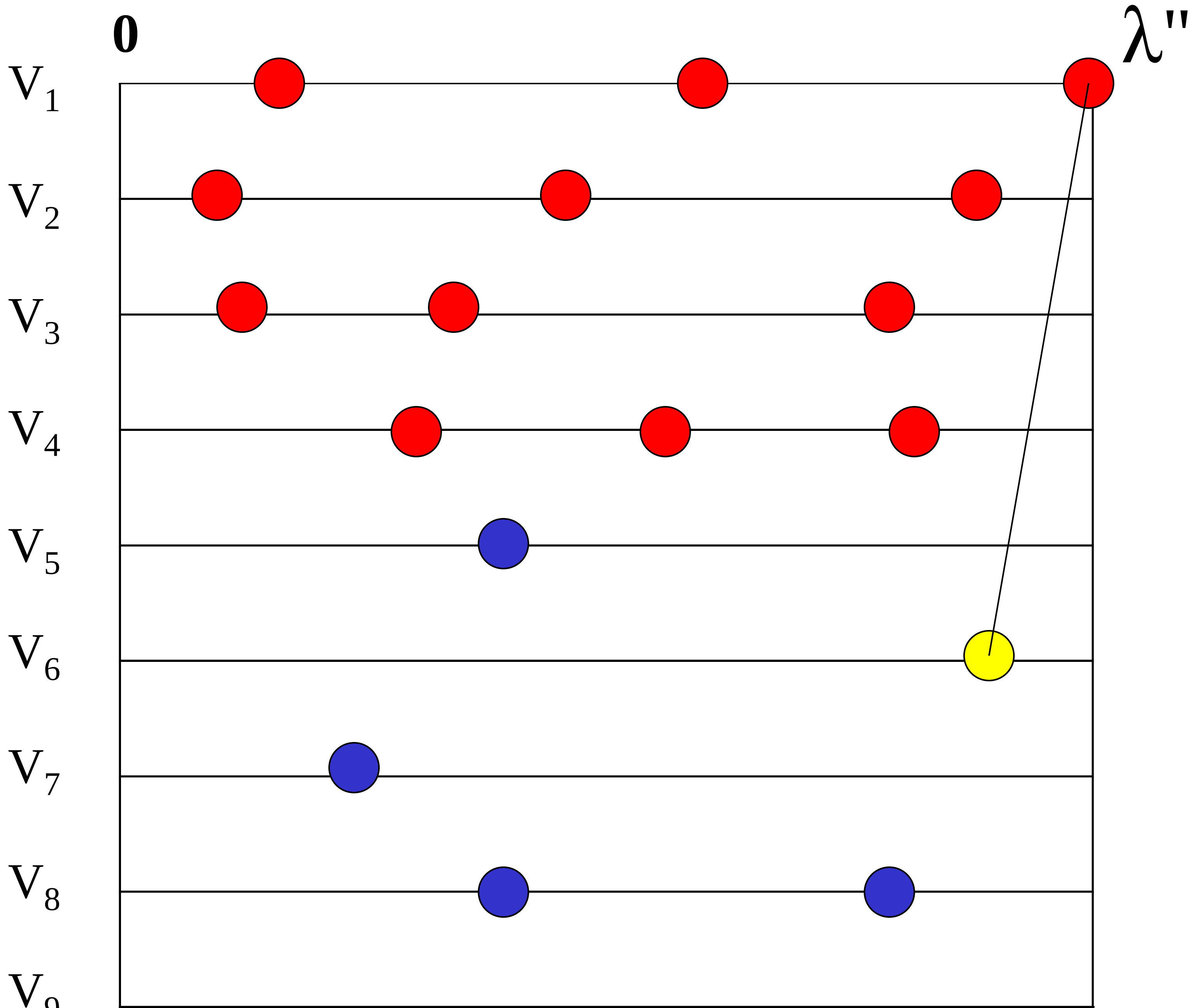
Application: Ramsey number $R(3, t)$

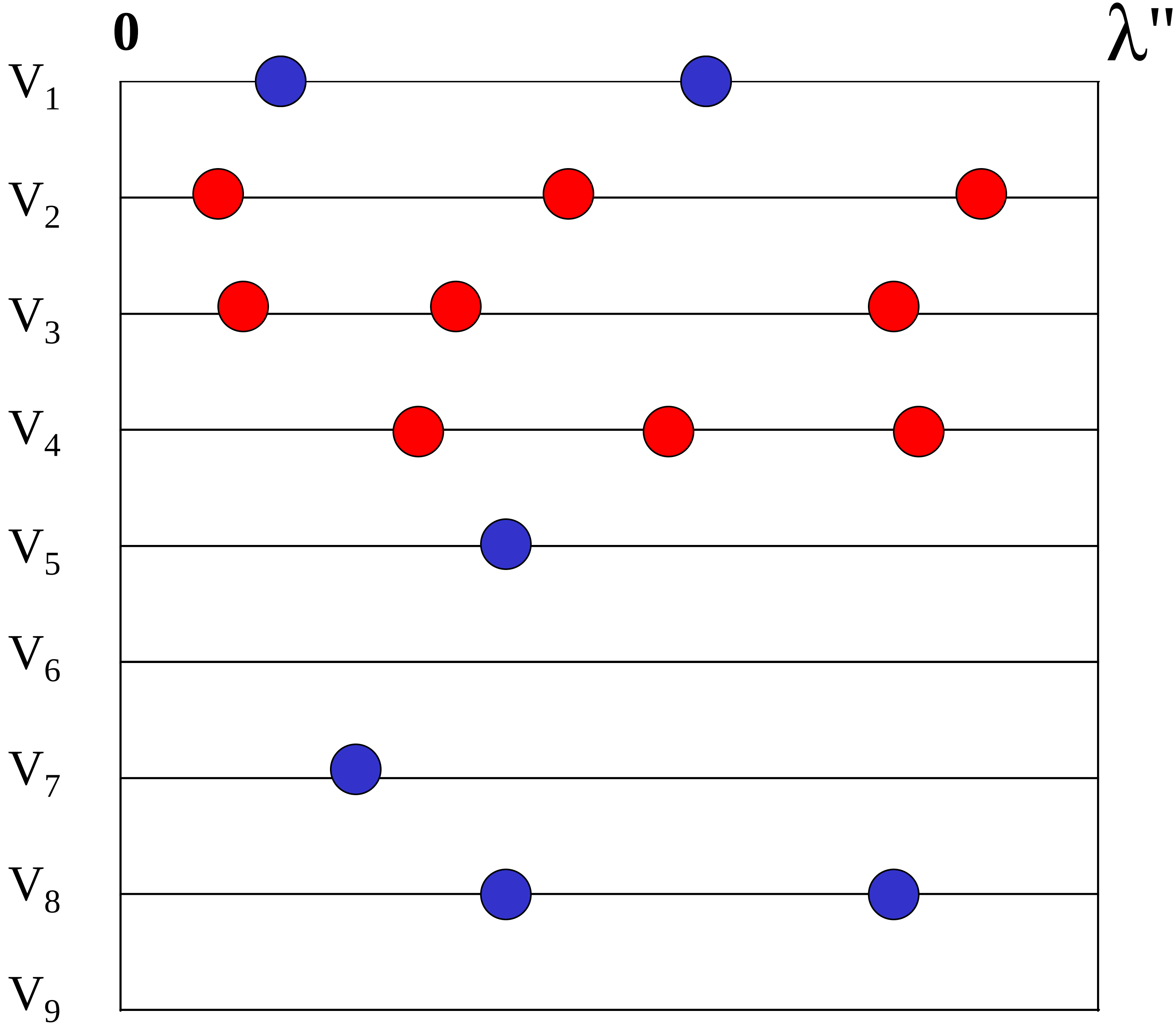

0
λ
V1
V2
V3
V4
V5
V6
V7
V8
V9

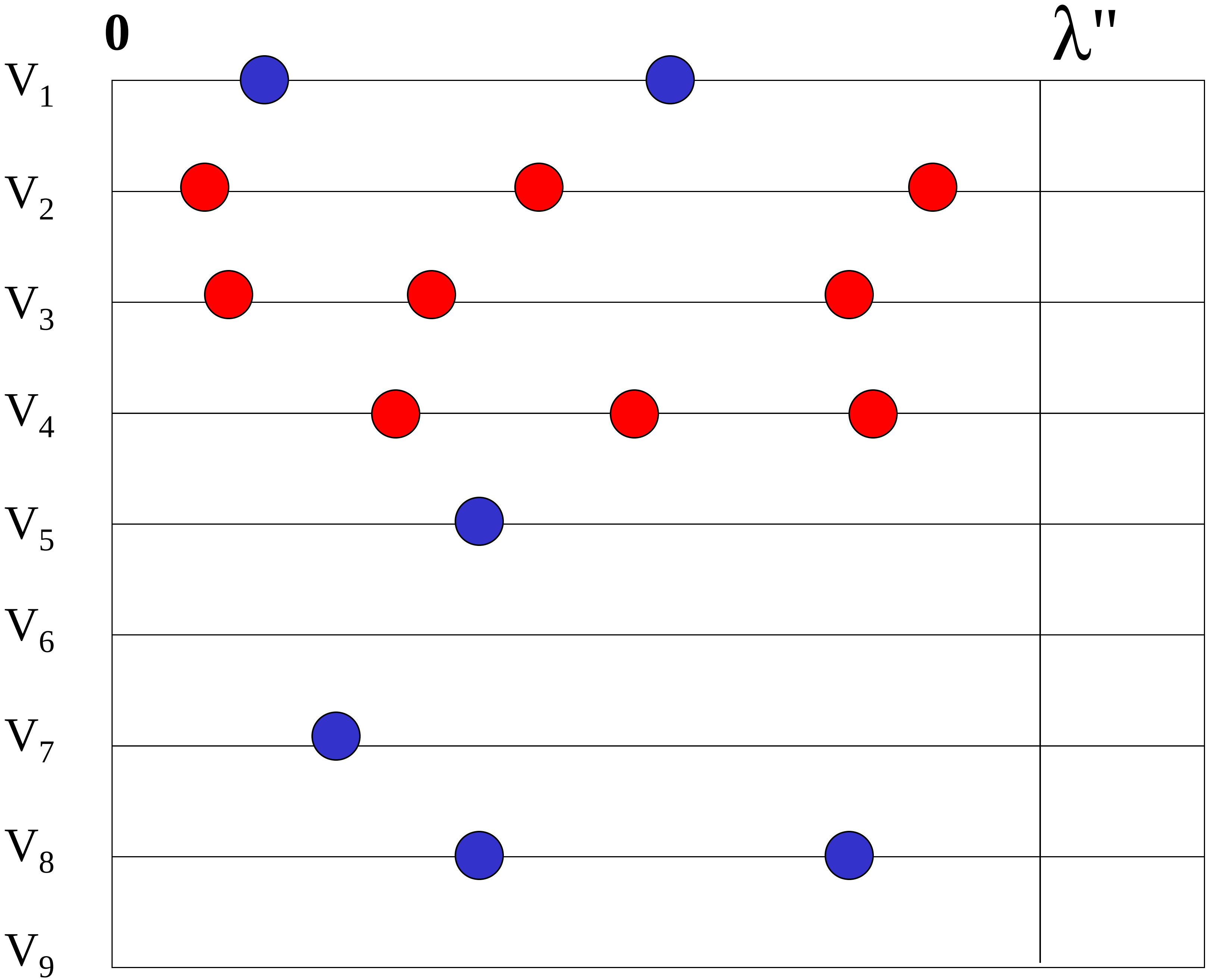

0
k=3
λ
V1
V2
V3
V4
V5
V6
V7
V8
V9


0
λ'
V1
V2
V3
V4
V5
V6
V7
V8
V9


0
λ'
V1
V2
V3
V4
V5
V6
V7
V8
V9

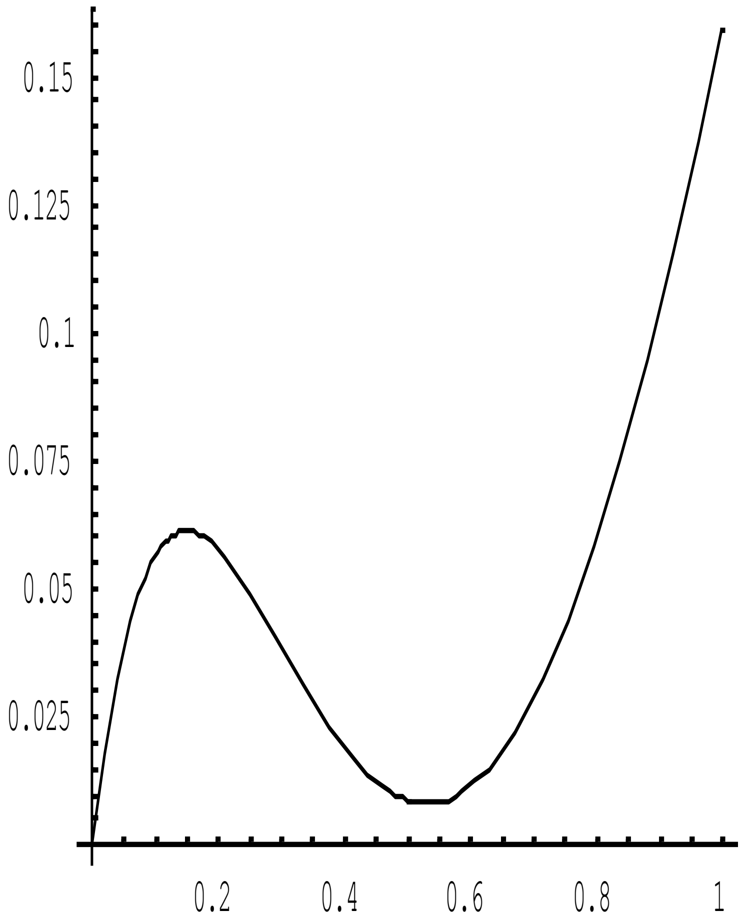

0
λ"
V_1
V_2
V_3
V_4
V_5
V_6
V_7
V_8
V_9


0
λ''
V1
V2
V3
V4
V5
V6
V7
V8
V9

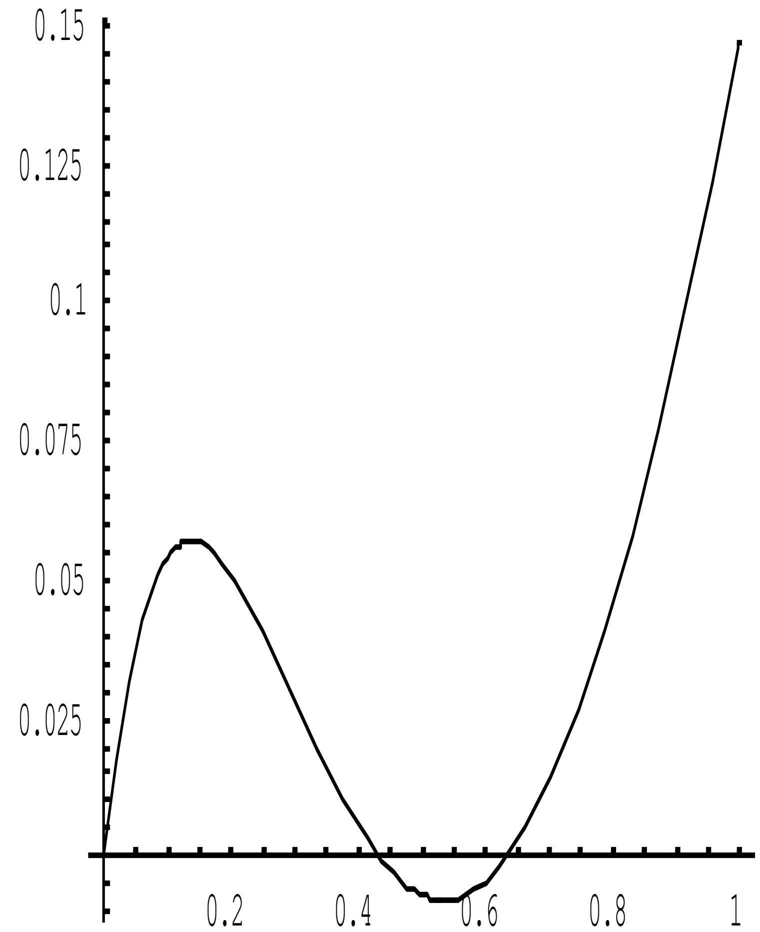

0
λ''
V1
V2
V3
V4
V5
V6
V7
V8
V9

Number of light clones

k=3, λ=3.3



k=3, λ=3.4



k=3, λ=3.35