Theorem (K) Supercritical region: Let $p = \frac{1+\varepsilon}{n}$ with $\varepsilon \gg n^{-1/3}$, and $1 \ll \alpha \ll (\varepsilon^3 n)^{1/2}$. Then

$$\Pr\left[|W(n,p) - \theta_\varepsilon n| \geq \alpha \left(\frac{n}{\varepsilon}\right)^{1/2} \right] \leq 2e^{-\Omega(\alpha^2)}.$$

Łuczak, 1990: With probability $1 - O((\varepsilon^3 n)^{-1/9})$,

$$|W(n,p) - \theta_\varepsilon n| \leq 0.2 n^{2/3}.$$

(Notice that $(\frac{n}{\varepsilon})^{1/2} \ll n^{2/3}$.)

Theorem (K) Subcritical region: Suppose $\lambda = 1 - \varepsilon$ with $n^{-1/3} \ll \varepsilon \ll 1$, then,

$$\Pr\left[W(n,p) \geq \frac{\log(\varepsilon^3 n) - 2.5 \log\log(\varepsilon^3 n) + c}{-(\varepsilon + \log(1-\varepsilon))} \right] \leq 2e^{-\Omega(c)},$$

and

$$\Pr\left[W(n,p) \leq \frac{\log(\varepsilon^3 n) - 2.5 \log\log(\varepsilon^3 n) - c}{-(\varepsilon + \log(1-\varepsilon))} \right] \leq 2e^{-e^{\Omega(c)}},$$

for a positive constant $c > 0$.

improving

$$(2-\alpha)\frac{\log(\varepsilon^3 n)}{\varepsilon^2} \leq W(n,p) \leq (2+\alpha)\frac{\log(\varepsilon^3 n)}{\varepsilon^2},$$

for $\alpha \gg \max\{\varepsilon, \log^{-1/2}(\varepsilon^3 n)\}$.

Ding, K, Lubetzky, Peres (2010)

Completely describe the giant component of $G(n,p)$ when it emerges.

Theorem (Ding, K, Lubetzky, Peres) Let $p = \frac{1+\varepsilon}{n}$ with $n^{-1/3} \ll \varepsilon \ll n^{-1/4}$. Then, with high probability, **the 2-core of the giant component** of $G(n,p)$ may be described as follows : Let X be a Gaussian random variable with mean $\frac{2}{3}\varepsilon^3 n$ and variance $\varepsilon^3 n$, and let K be a random 3-regular graph on $2\lfloor X \rfloor$ vertices. Replace each edge of K by a path, where the path lengths are i.i.d geometric random variables with mean $1/\varepsilon$.

Moreover, the giant component of $G(n,p)$ may be described by attaching an independent Poisson $(1-\varepsilon)$-branching process to each vertex of the 2-core.

Application: Diameter

Subcritical Region: (Łuczak, 1998) The largest diameter of a component of $G(n,p)$ with $p = \frac{1-\varepsilon}{n}$ is

$$\frac{\log 2\varepsilon^3 n}{-\log(1-\varepsilon)} + O(1),$$

including the limiting distribution of $O(1)$-term.

Inside Window: (Nachmias & Peres, 08)

$$\operatorname{diam} L(n,p) = \Theta(n^{1/3}).$$

Supercritical Region: (Fernholz & Ramachandran '07)
For $pn = 1 + \Theta(1)$,

$$\operatorname{maxdiam} G(n,p) = \operatorname{diam} L(n,p) = (\alpha(n,p) + o(1)) \log n,$$

where $\alpha(n,p) = \cdots$. In particular, for $p = \frac{1+\varepsilon}{n}$ with a constant $\varepsilon > 0$,

$$\alpha(n,p) = \frac{3 - o_\varepsilon(1)}{\varepsilon}$$

(see also Bollobás, Janson & Riordan '07, and Chung & Lu '01).

(Riordan & Wormald, '11) For most of ε with $\varepsilon^3 n \gg 1$ and $\varepsilon \ll 1$,

$$\operatorname{diam} L(n,p) = \frac{3+o(1)}{\varepsilon} \log(\varepsilon^3 n)\Big(= \operatorname{maxdiam} G(n,p)\Big),$$

and more.

(Łuczak and Seierstad, '11) Upper and lower bounds for the entire supercritical regime that differ by a factor of $\frac{1000}{7}$.

(Ding, K, Lubetzky, Peres, '11) For all ε with $\varepsilon^3 n \gg 1$ and $\varepsilon \ll 1$,

$$\operatorname{diam} L(n,p) = \frac{3+o(1)}{\varepsilon} \log(\varepsilon^3 n)\Big(= \operatorname{maxdiam} G(n,p)\Big).$$

Mixing Time $\tau(n,p)$ of $L(n,p)$

(Fountoulakis and Reed '08 and Benjamini, Kozma, and N. C.Wormald 10^+) For $pn = 1 + \Theta(1)$,

$$\tau(n,p) = \Theta(\log^2 n).$$

Inside Window: (Nachmias & Peres, 08)

$$\tau(n,p) = \Theta(n).$$

(Ding, Lubetzky, Peres, '10^+)

$$\tau(n,p) = \Theta\Big(\frac{1}{\varepsilon^3}\log^2(\varepsilon^3 n))\Big) = \Big(\text{max mixing time}\, G(n,p)\Big).$$

Theorem

$$K(n,p) \sim \widetilde{K}(n,p) \text{ and } \mathcal{C}(n,p) \sim \widetilde{C}(n,p).$$

Proof. Use Poisson cloning model with careful analysis.

Theorem

$$L(n,p) \sim \widetilde{L}(n,p).$$

Proof. Suppose $C(n.p) = C$. For each vertex $u_i \in C$, $i = 1, ..., |C|$, let T_i be the tree attached to u_i in C. Then, for positive integers t_i with $\sum_{i=1}^{|C|} t_i = \ell$,

$$\begin{aligned}
& \Pr\left[|T_i| = t_i \text{ for all } i \middle| C\right] \\
= \; & \Pr\left[|L(n,p)| = \ell \middle| C\right] \Pr\left[|T_i| = t_i \text{ for all } i \middle| C, \ell\right] \\
= \; & \Pr\left[|L(n,p)| = \ell | C\right] \frac{\binom{\ell}{t_1 \cdots t_{|C|}} \prod_{i=1}^{|C|} t_i^{t_i - 1}}{\sum_{(r_i): \sum r_i = \ell} \binom{\ell}{r_1 \cdots r_{|C|}} \prod_{i=1}^{|C|} r_i^{r_i - 1}}
\end{aligned}$$

$$= \quad \Pr\left[|L(n,p)| = \ell | C\right] \frac{\prod_{i=1}^{|C|} \frac{t_i^{t_i - 1}}{t_i!}}{\sum_{(r_i): \sum r_i = \ell} \prod_{i=1}^{|C|} \frac{r_i^{r_i - 1}}{r_i!}}$$

$$= \Pr\left[|L(n,p)| = \ell | C\right] \frac{\prod_{i=1}^{|C|} \frac{t_i^{t_i - 1}}{\mu t_i!} (\mu e^{-\mu})^{t_i}}{\sum_{(r_i):\sum r_i = \ell} \prod_{i=1}^{|C|} \frac{r_i^{r_i - 1}}{\mu r_i!} (\mu e^{-\mu})^{r_i}}$$

$$= \Pr\left[|L(n,p)| = \ell | C\right] \frac{\Pr[\tilde{T}_i = t_i \text{ for all } i | C]}{\Pr[\sum_{i=1}^{|C|} \tilde{T}_i = \ell | C]}$$

for any $\mu > 0$. We take $\mu < 1$ satisfying $\mu e^{-\mu} = (1+\varepsilon)e^{-(1+\varepsilon)}$. Enough to show that

$$\frac{\Pr\left[|L(n,p)| = \ell | C\right]}{\Pr[|\widetilde{L}(n,p)| = \ell | C]} = 1 + o(1),$$

uniformly in ℓ.

However, $\Pr\left[|L(n,p)| = \ell | C\right]$ is more or less known due a theorem of Pittel and Wormald ('05).

• Idea of the Proof for the super critical region

Easy:

$$A(\theta) \geq M(\theta) - N(\theta)$$

where

$$M(\theta) = \text{number clones of vertices that have a clone larger than } \theta\lambda.$$

$$N(\theta) = \text{number of matched clones until } \theta\lambda.$$

Then

$$\Pr\left[\max_{\theta:\theta_1 \leq \theta \leq 1} \left| M(\theta) - (1-\theta e^{-(1-\theta)\lambda})\lambda n \right| \geq \Delta\right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta_1)n}\})},$$

$$\Pr\left[\max_{\theta:\theta_1 \leq \theta \leq 1} |N(\theta) - (1-\theta^2)\lambda n| \geq \Delta\right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta_1)n}\})}.$$

Therefore,

$$\Pr\left[\max_{\theta:\theta_1\leq\theta\leq}\left|M(\theta)-N(\theta)-(\theta-e^{-(1-\theta)\lambda})\theta\lambda n\right|\geq\Delta\right]$$

$$\leq 2e^{-\Omega(\min\{\Delta,\frac{\Delta^2}{(1-\theta_{-1})n}\})}.$$

We may also define the **Poisson Cloning Models** for

Random k-uniform hypergraphs

Random k-SAT Problems

Random Directed Graphs

Similar analyses are possible using PCM and COLA for

- The k-core problem for hypergraph
(Pittel-Spencer-Wormald, ...)

- Structures of the giant component:

$$\text{e.g.}\quad \# \text{ of vertices of degree } i \geq 2$$

(Łuczak, Pittel, ...)

- Strong component of the random directed graph

(Karp, ...)

- Pure literal rule for random k-SAT problems
(Broder-Frieze-Upfal, ...)

And more

- Unit clause algorithm for random k-SAT problems (Chao-Franco, ...)
- Karp-Sipser Algorithm

  (Karp-Sipser, Aronson-Frieze-Pittel, ...)
- Giant Component of

$$H \cup G(n, p)$$

for a fixed graph H. (K-Spencer)

The k-core Problem

A k-core of a graph is a largest subgraph
with minimum degree at least k

(due to **Bollobás**).

Pittel, Spencer & Wormald ('96):
For random graph $G(n, p)$ and

$$\lambda_k = \min_{\rho > 0} \frac{\rho}{P(\rho, k-1)},$$

where

$$P(\rho, k-1) := \Pr\Big(\mathrm{Poi}(\rho) \geq k-1\Big) = e^{-\rho} \sum_{l=k-1}^{\infty} \frac{\rho^l}{l!},$$

if $k \geq 3$,

$$\Pr\left[G(n, \lambda/(n-1)) \text{ has a } k\text{-core }\right] \to \begin{cases} 0 & \text{if } \lambda < \lambda_k - n^{-\delta} \\ 1 & \text{if } \lambda > \lambda_k + n^{-\delta}, \end{cases}$$

for any $\delta \in (0, 1/2)$, and

$$\Pr\left[\text{either } \exists \text{ no } k\text{-core or } \exists \ k\text{-core of size } \geq (1 - n^{-\delta})\lambda_k^* n\right]$$

$$\to 1.$$

Recall,

$$\lambda_k = \min_{\rho > 0} \frac{\rho}{P(\rho, k-1)}.$$

(improving Łuczak's result).

C. Cooper ($\geq$ '02): Simpler proof for

$$\Pr\Big[G(n, \lambda/(n-1)) \text{ has a } k\text{-core}\Big] \to \begin{cases} 0 & \text{if } \lambda < (1-\varepsilon)\lambda_k \\ 1 & \text{if } \lambda > (1+\varepsilon)\lambda_k. \end{cases}$$

(K) For Poisson Cloning Model $G_{PC}(n,p)$ and $k \geq 3$,

$$\Pr\Big[G_{PC}(n, \lambda/(n-1)) \text{ has a } k\text{-core}\Big] \to$$

$$\begin{cases} 0 & \text{if } \lambda_k - \lambda \gg n^{-1/2} \\ 1 & \text{if } \lambda - \lambda_k \gg n^{-1/2}, \end{cases}$$

and

$$\Pr\Big[\text{either } \exists \text{ no } k\text{-core or}$$

$$\exists\, k\text{-core of size} \geq (1-(\omega(n)n)^{-1/2})\lambda_k^* n\Big] = 1 - o(1),$$

for any $\omega(n) \to \infty$. Recall,

$$\lambda_k = \min_{\rho>0} \frac{\rho}{P(\rho, k-1)}.$$

We will be

solving the problem as well as generating $G_{PC}(n,p)$.

(Recall $p = \lambda/(n-1)$).

At step 0,

A vertex v is light if $d(v) < k$,
or the number of v-clones is less than k.

It is heavy, otherwise.

Take a light clone w and
then choose the largest clone except w.

We will be

solving the problem as well as generating $G_{PC}(n, p)$.

In general, at step i,

A vertex v is light if
the number of unmatched v-clones is less than k.

It is heavy, otherwise.

Take a unmatched light clone w and
then choose the largest unmatched clone except w.

- **Parameters**

$$N_i = \text{the number of unmatched clones},$$

$$N_0 \approx \lambda n.$$

$$\lambda_i = \text{the length of the interval},$$

$$\lambda_0 = \lambda.$$

$$H_i = \text{the number of unmatched heavy clones},$$

$$H_0 = \sum_{v \in V} d(v) 1(d(v) \geq k),$$

$$E[H_0] = \lambda P(\lambda, k-1) n,$$

where $d(v)$'s are i.i.d. Poi(λ). (Recall

$$P(\rho, k-1) := \Pr\left(\text{Poi}(\rho) \geq k-1\right) = e^{-\rho} \sum_{l=k-1}^{\infty} \frac{\rho^l}{l!}.)$$

Generally, if λ_i is given,

$$H_i \approx \sum_{v \in V} d_i(v) 1(d_i(v) \geq k)$$

and

$$E[H_i] \approx \lambda_i P(\lambda_i, k-1) n,$$

where $d_i(v)$'s are i.i.d. Poi(λ_i). (Recall

$$P(\rho, k-1) := \Pr\Big(\text{Poi}(\rho) \geq k-1\Big) = e^{-\rho} \textstyle\sum_{l=k-1}^{\infty} \frac{\rho^l}{l!}.)$$

Since $N_i - H_i$=the total number of light clones,

$$\exists \text{ no } k\text{-core} \quad \text{iff} \quad N_i - H_i > 0 \text{ for all } i \text{ with } N_i > 0$$

In particular,

the pair (N_i, λ_i) tells everything we want.

Trivially,

$$N_i = N_0 - 2i.$$

In expectation,

$$\lambda_1 = \Big(1 - \frac{1}{N_0 - 1}\Big)\lambda$$

since we took the largest number among $N_0 - 1$ i.i.d uniform random numbers from 0 to λ. Similarly, in expectation,

$$\lambda_i = \lambda \prod_{j=0}^{i-1} \Big(1 - \frac{1}{N_i - 1}\Big).$$

Precisely,

$$\lambda_i = \lambda \prod_{j=0}^{i-1} \Big(1 - T_j\Big),$$

where T_i are mutually independent and

$$T_j = \text{min. of } N_j - 1 \text{ uniform random numbers in } [0,1],$$

especially,

$$\Pr[T_i \geq t] \approx e^{-(N_{i-1}-1)t}.$$

Thus, for $\theta_i^2 := N_i/N_0$

$$\lambda_i \approx \lambda \exp\Big(-\sum_{j=0}^{i-1} \frac{1}{N_p - 2j - 1}\Big) \approx \theta_i \lambda.$$

In terms of θ_i,

$$N_i = \theta_i^2 N_0 \approx \theta_i^2 \lambda n,$$

and

$$H_i \approx \lambda_i P(\lambda_i, k-1) n \approx \theta_i \lambda P(\theta_i \lambda, k-1) n.$$

Since $\exists$ no k-core $\quad$ iff $\quad N_i - H_i > 0$ for all i,

$$\exists \text{ no } k\text{-core} \quad \text{iff} \quad \theta - P(\theta\lambda, k-1) > 0 \quad \forall \theta \in (0,1),$$

or

$$\exists\ k\text{-core} \quad \text{iff} \quad \theta - P(\theta\lambda, k-1) = 0 \quad \text{for some } \theta \in (0,1).$$

EASY: If

$$\lambda < \lambda_k = \min_{\rho>0} \frac{\rho}{P(\rho, k-1)},$$

then

$$\forall\ \theta \in (0,1), \quad \theta - P(\theta\lambda, k-1) > 0,$$

and if

$$\lambda > \lambda_k,$$

then

$$\exists \theta \in (0,1), \quad \theta - P(\theta\lambda, k-1) = 0.$$

.

Hypergraph

In terms of θ_i,

$$N_i = \theta_i^h N \approx \theta_i^h \lambda n,$$

and

$$H_i \approx \lambda_i P(\lambda_i, k-1)n \approx \theta_i^{h-1} \lambda P(\theta_i^{h-1}\lambda, k-1).$$

For

$$f_h(\theta) = \theta - P(\theta^{h-1}\lambda, k-1)$$

$$\exists \text{ no } k\text{-core} \quad \text{iff} \quad f_h(\theta) > 0 \quad \forall \theta \in (0,1),$$

or

$$\exists \ k\text{-core} \quad \text{iff} \quad f_h(\theta) = 0 \quad \text{for some } \theta \in (0,1).$$

For h-uniform hypergraph,

$$\Pr\left[H_h(n, \lambda/(n-1)) \text{ has a } k\text{-core}\right] \to$$

$$\begin{cases} 0 & \text{if } \lambda < \lambda_k^{(h)} - n^{-1/2}\log n \\ 1 & \text{if } \lambda > \lambda_k + \omega(n) n^{-1/2}, \end{cases}$$

where

$$\lambda_k^{(h)} = \min_{\rho > 0} \frac{\rho}{(P(\rho, k-1))^{h-1}}.$$

For 2-core for 3-uniform hypergrah,

$$\lambda_k^{(h)} \approx 2.45542\cdots, \quad \text{or} \quad m_k^{(h)} = 1.63694\cdots.$$

(Note that $2 \times 0.818 = 1.636$.)

Random k-SAT

Random Digraph

Random 2-SAT

Boolean Variables: $x_1, ..., x_n \in \{0, 1\}$

Negation of x: $\bar{x} = 1 - x$

$2n$ literals: $x_1, \bar{x}_1, ..., x_n, \bar{x}_n$

x and y are *strictly distinct (s.d.)*
if $x \neq y$ and $x \neq \bar{y}$

k-clause:

$$C = v_1 \vee \cdots \vee v_k$$

where $v_1, ..., v_k$ are s.d. literals

How many k-clauses??

Take k Boolean variables out of n.

Then $\exists$ two choices (negation or not) for each variable.

$$2^k \binom{n}{k}$$

k-SAT Formula:

$$F = F(x_1, ..., x_n) = C_1 \wedge \cdots \wedge C_m$$

where $C_1, ..., C_m$ are k-clauses.

F is *satisfiable* if

$$F(x_1, ..., x_n) = 1$$

for some $x_1, ..., x_n \in \{0, 1\}$

k-SAT problem: NP-Complete if $k \geq 3$

(P if $k = 2$)

- Random 2-SAT $F(n,p)$:

Each 2-clause appears in F
with probability p

Expected # of clauses

$$m = 2^2 p \binom{n}{2}$$

(Goerdt '92, Chvátal & Reed '92, F. de la Vega '92) For $k = 2$,

$$\Pr[F_2 \text{ is SAT }] \to \begin{cases} 1 & \text{if } m/n \to c < 1 \\ 0 & \text{if } m/n \to c > 1 \end{cases}$$

- **Poisson Cloning Model** $F_{PC}(n, p)$

For each literal v, independently
take $d(v) \in \mathrm{Poi}(p(2n-1))$ clones and
match all clones uniformly at random.

E.g.

For each clone w, assign a uniform random (real) number from 0 to $\lambda := 2p(n-1)$.

We say that a clone is larger than another clone if the assigned numbers are so.

Take two largest clones and match them
and repeat it for the remaining, or unmatched, clones.

Or,

choose the first unmatched clone
according to a certain selection rule (SR)
without looking assigned numbers,
then match it to the largest unmatched clone.

$F(n,p)$ vs. $F_{PC}(n,p)$

Lemma. If $pn^2 = O(n)$, then there are positive constants c_1 and c_2 so that for any collection $\mathcal{F}$ of (SIMPLE) formulae

$$c_1 \Pr[F_{PC}(n,p) \in \mathcal{F}] - e^{-\Omega(pn^2)} \leq \Pr[F(n,p) \in \mathcal{F}]$$

and

$$\Pr[F(n,p) \in \mathcal{F}] \leq c_2 \Big(\Pr[F_{PC}(n,p) \in \mathcal{F}] \Big)^{1/2} + e^{-\Omega(pn^2)}.$$

- Pure Literal of a formula F

  a literal x is pure iff x appears in F but not $\bar{x}$

- Light Clone

  clones of pure literals are called light.

- Pure Literal Rule for $F_{PC}(n, p)$

  take a light clone and match it to the largest (unmatched) clone and remove them

- **Parameters**

$$N_i = \text{the number of unmatched clones,}$$

$$N_0 \approx 2\lambda n.$$

$$\lambda_i = \text{the length of the interval,}$$

$$\lambda_0 = \lambda.$$

$$H_i = \text{the number of unmatched heavy clones,}$$

$$H_0 = \sum_{j=1}^{n} (X_j + Y_j) 1(X_j Y_j \geq 1),$$

$$E[H_0] = 2\lambda(1 - e^{-\lambda}),$$

where X_i, Y_i's are i.i.d. Poi(λ).

Generally, if λ_i is given,

$$H_i = \sum_{j=1}^{n} (X_j^{(i)} + Y_j^{(i)}) 1(X_j^{(i)} Y_j^{(i)} \geq 1),$$

and

$$E[H_i] = 2\lambda_i (1 - e^{-\lambda_i}),$$

where $X_j^{(i)}, Y_j^{(i)}$'s are i.i.d. Poi(λ_i).

Note that

$$N_i - H_i = \text{the total number of light clones.}$$

Thus

$$\exists \text{ no } (1,1)\text{-core} \quad \text{iff} \quad N_i - H_i > 0 \text{ for all } i.$$

Trivially,

$$N_i = N_0 - 2i.$$

In expectation,

$$\lambda_1 = \Big(1 - \frac{1}{N_0 - 1}\Big)\lambda$$

since we took the largest number among $N_0 - 1$ i.i.d uniform random numbers from 0 to λ. Similarly, in expectation,

$$\lambda_i = \lambda \prod_{j=0}^{i-1} \Big(1 - \frac{1}{N_i - 1}\Big).$$

Actually,

$$\lambda_i = \lambda \prod_{j=0}^{i-1} \Big(1 - T_i),$$

where T_i are mutually independent with

$$\Pr[T_i \geq t] \approx e^{-(N_{i-1}-1)t}.$$

Thus, for $\theta_i^2 := N_i / N_0$

$$\lambda_i \approx \lambda \exp\Big(- \sum_{j=0}^{i-1} \frac{1}{N_p - 2j - 1}\Big) \approx \theta_i \lambda.$$

In terms of θ_i,

$$N_i = \theta_i^2 N \approx 2\theta_i^2 \lambda n,$$

and

$$H_i \approx 2\lambda_i(1 - e^{-\lambda_i})n \approx 2\theta_i\lambda(1 - e^{-\theta_i\lambda})n.$$

Since $\exists$ no $(1,1)$-core iff $N_i - H_i > 0$ for all i,

for $f(\theta) := \theta - (1 - e^{-\theta\lambda})$,

$$\exists \text{ no } (1,1)\text{-core iff } f(\theta) > 0 \quad \forall \theta \in (0,1),$$

or

$$\exists\ (1,1)\text{-core iff } f(\theta) = 0 \quad \text{for some } \theta \in (0,1).$$

Since $f(0) = 0$ and

$$f'(\theta) = 1 - \lambda e^{-\theta\lambda} = \begin{cases} > 0 \ \text{ for all } \theta \in (0,1) & \text{if } \lambda < 1 \\ < 0 \ \text{ for all } \theta \in (0,\delta_\lambda) & \text{if } \lambda > 1 \end{cases}$$

for some $\delta_\lambda > 0$,

$f(\theta) = 0$ has a solution in $\theta \in (0, 1)$ iff $\lambda > 1$.

After Pure Literal Alg. Stops??

For a variable x with

$$(d(x), d(\bar{x})) = (1, 1)$$

if there is $(x \vee \bar{x})$, then remove it.

Otherwise, take the corresponding two clauses

$$(x \vee y), (\bar{x} \vee z)$$

and replace them

by $(y \vee z)$.

Thus

x makes no difference.

After Pure Literal Alg. Stops??

For all remaining variables x, $(d(x), d(\bar{x})) \geq (1, 1)$.

Let R be the number of variables with

$$(d(x), d(\bar{x})) > (1, 1).$$

(Case I) If $R = 0$ or 1,

(Case II) For large R,

(Case III) If $R = c \geq 2$,

After Pure Literal Alg. Stops??

For all remaining variables x, $(d(x), d(\bar{x})) \geq (1, 1)$.

Let R be the number of variables with

$$(d(x), d(\bar{x})) > (1, 1).$$

(Case I) If $R = 0$ or 1, $\Pr[F \text{ is SAT} | R] = 1$.

(Case II) For large R,

(Case III) If $R = c \geq 2$,

After Pure Literal Alg. Stops??

For all remaining variables x, $(d(x), d(\bar{x})) \geq (1,1)$.

Let R be the number of variables with

$$(d(x), d(\bar{x})) > (1,1).$$

(Case I) If $R = 0$ or 1, $\Pr[F \text{ is SAT} | R] = 1$.

(Case II) For large R, $\Pr[F \text{ is SAT} | R] = e^{-\Theta(R)}$.

(Case III) If $R = c \geq 2$,

After Pure Literal Alg. Stops??

For all remaining variables x, $(d(x), d(\bar{x})) \geq (1, 1)$.

Let R be the number of variables with

$$(d(x), d(\bar{x})) > (1, 1).$$

(Case I) If $R = 0$ or 1, $\Pr[F \text{ is SAT}|R] = 1$.

(Case II) For large R, $\Pr[F \text{ is SAT}|R] = e^{-\Theta(R)}$.

(Case III) If $R = c \geq 2$, $0 < \Pr[F \text{ is SAT}|R] < 1$.

If the are R variables with

$$(d(x), d(\bar{x}) > (1, 1),$$

then there are

$$\geq 3R \text{ clones}$$

or equivalently, there are

$$\geq 3R/2 \text{ clauses.}$$

Theorem (in progress) If $m/n = 1 - \mu n^{-1/3}$, then

$$\lambda_{final} = \Theta(n^{-1/3}/\mu^{1/2}).$$

If $m/n = 1 + \mu n^{-1/3}$, then

$$\lambda_{final} = \Theta(\mu n^{-1/3}).$$

Corollary If $m/n = 1 - \mu n^{-1/3}$, then

$$\Pr[R \geq 2] = \Theta(1/\mu^3).$$

If $m/n = 1 + \mu n^{-1/3}$, then with probability $1 - e^{-\Theta(\mu^3)}$

$$R = \Theta(\mu^3).$$

This reprove

Theorem (Bollobás, Borgs, Chayes, K, Wilson)

If $m/n = 1 - \mu n^{-1/3}$, then

$$\Pr[F_2 \text{ is SAT }] = 1 - \Theta(1/\mu^3)$$

If $m/n = 1 + \mu n^{-1/3}$, then

$$\Pr[F_2 \text{ is SAT }] = \exp(-\Theta(\mu^3)).$$

Pure Literal Alg. for k-SAT

In terms of θ_i,

$$N_i = \theta_i^k N \approx 2\theta_i^k \lambda n,$$

and

$$H_i \approx 2\lambda_i(1 - e^{-\lambda_i})n \approx 2\theta_i^{k-1}\lambda(1 - e^{-\theta_i^{k-1}\lambda}).$$

For

$$f_k(\theta) := \theta - (1 - e^{-\theta^{k-1}\lambda})) = \theta - P(\theta^{k-1}\lambda, 1),$$

$$\exists \text{ no } (1,1)\text{-core} \quad \text{iff} \quad f(\theta) > 0 \quad \forall\theta \in (0,1),$$

or

$$\exists\ (1,1)\text{-core} \quad \text{iff} \quad f(\theta) = 0 \quad \text{for some } \theta \in (0,1).$$

Thus

$$\lambda_k = \min_{\rho>0} \frac{\rho}{(1-e^{-\rho})^{k-1}}, \quad \text{or} \quad m_k = \min_{\rho>0} \frac{2\rho}{k(1-e^{-\rho})^{k-1}}$$

E.g.,

$$m_3 = 1.63694\cdots, \quad m_4 = 1.54456\cdots, \quad m_5 = 1.40356\cdots.$$

(Mitzenmacher('97), Molloy & Wormald(?))