Recent Results on the Moment Determinacy of Probability Distributions

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The Great Departmental Seminar in Theory of Probability
Mechanics and Mathematics Faculty, Moscow State University
12 March 2014

PLAN:

Discussion on recent works on probability distributions and their characterization as being unique (**M-determinate**) or non-unique (**M-indeterminate**) in terms of the moments.

1. Basics. Most famous Example: $LogN$.
2. Carleman's condition. Krein's condition.
3. Cramér's condition. Hardy's condition.
4. Criteria based on the rate of growth of the moments.
5. Moment problem for Multivariate distributions.
6. Powers and products of r.v.s and their M-determinacy.
7. Open questions.

Basics: $\mathcal{M}$ = all $X \sim F$, f, finite moments $m_k = \mathbf{E}[X^k]$, $k = 1, 2, \ldots$

Question: Knowing that $\{m_k\}$ is the moment sequence of F, we ask:

Is F the only d.f. with these moments?

If **"Yes"**, F is **M-determinate**, unique with these moments (**M-det**).

If **"No"**, F is **M-indet**, there are G, $G \neq F$, same moments.

General: Given $F \in \mathcal{M}$, either F is M-det, or F is M-indet. How ?

There are conditions for uniqueness, difficult to check.

Rational approach: Find **Easy Conditions**, sufficient or necessary.

Criteria depend on the supp(F): $[0, 1]$ (**Hausdorff**);
$\mathbb{R}^+ = [0, \infty)$ (**Stieltjes**); $\mathbb{R}^1 = (-\infty, \infty)$ (**Hamburger**).

Fundamental Result: For any M-indet F, there are infinitely many distributions, continuous and discrete, all with the same moments as F.

Log-normal distribution: $Z \sim \mathcal{N}(0,1)$, $X = e^Z \sim LogN(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp\left[-\frac{1}{2}(\ln x)^2\right], \ x > 0; \ \ f(x) = 0, \ x \le 0.$$

About X: No m.g.f., HT, $X \in \mathcal{M}$, $m_k = \mathbf{E}[X^k] = e^{k^2/2}$, $k = 1, 2, \ldots$

Two infinite sets of r.v.s, **one absolutely continuous, one discrete:**

$X_\varepsilon, \varepsilon \in [-1, 1]$: density $f_\varepsilon(x) = f(x)\,[1 + \varepsilon \sin(2\pi \ln x)]$, $x > 0$ (JRSS'63)

Y_a, $a > 0$: $\mathbf{P}[Y_a = ae^n] = a^{-n} e^{-n^2/2}/A$, $n = 0, \pm 1, \pm 2, \ldots$ (TPA'81)

Shocking property: $\mathbf{E}[X_\varepsilon^k] = \mathbf{E}[Y_a^k] = \mathbf{E}[X^k] = e^{k^2/2}$, $k = 1, 2, \ldots$.

Conclusion: **LogN is M-indet!** So 'many' others, the same moments.

More: Similar statement is valid for **Log-SkewN.** L&S JAP 2009.

Carleman: Known are all moments $m_k = \mathbf{E}[X^k],\ k = 1, 2, \ldots$
For X in $\mathbb{R}^1$ or $\mathbb{R}^+$, define the following **Carleman quantity**:

$$\mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(m_{2k})^{1/2k}}, \qquad \mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(m_k)^{1/2k}}.$$

In both cases, $\mathrm{C} = \infty$ is **sufficient for F to be M-det.**

Krein: Let $X \sim F$, $F \in \mathcal{M}$ with density $f > 0$. For X in $\mathbb{R}^1$ or $\mathbb{R}^+$, define the logarithmic normalized integral, **Krein quantity**:

$$\mathrm{K}[f] \equiv \int_{-\infty}^{\infty} \frac{-\ln f(y)}{1+y^2} \mathrm{d}y, \qquad \mathrm{K}[f] \equiv \int_{a}^{\infty} \frac{-\ln f(y^2)}{1+y^2} \mathrm{d}y,\ a \geq 0.$$

In both cases, $\mathrm{K}[f] < \infty$ is **sufficient for F to be M-indet.**

Remark: There are converses to the above criteria: A. Pakes, G.D. Lin.

Cramér: For a r.v. $X \sim F$ on $\mathbb{R}^1$, let the m.g.f. $M(t) = \mathbf{E}[e^{tX}]$ exist, i.e. $M(t)$ is well-defined for $t \in (-t_0, t_0)$, $t_0 > 0$ (**light tails**). Then:

- $X \in \mathcal{M}$ (finite all moments);
- X, i.e. F, is M-det.

'Good' distributions: Any X, F with finite support; *Exp*, $\mathcal{N}$,, *Laplace*

If F has **heavy tail(s)** (no m.g.f.), two possibilities:

(a) $F \notin \mathcal{M}$, not all moments are finite. Not our case! Many!
(b) $F \in \mathcal{M}$, then either F is M-det, or it is M-indet.

Question: When is a heavy tailed F unique? And when nonunique?

Answer: Suppose F has f, e.g. $f(x) \sim a \exp(-b|x|^c)$, $a, b, c > 0$.
If F is on $\mathbb{R}^1$, $c = 1$ is the boundary; if F is on $\mathbb{R}^+$, the boundary is $c = \frac{1}{2}$.

"New" Criterion: **G. Hardy (1917/1918).** *The Math. Messenger*

Statement (Hardy): r.v. $X > 0$, $X \sim F$. Suppose $\sqrt{X}$ has m.g.f.:

$\mathbf{E}[e^{t\sqrt{X}}] < \infty$ for $t \in [0, t_0)$, $t_0 > 0$ (H)=Hardy's condition; $\frac{1}{2}$-Cramér.

Then $X \in \mathcal{M}$ and X is M-det: all moments $m_k = \mathbf{E}[X^k]$, $k = 1, 2, \dots$ are finite and F is the only d.f. with the moment sequence $\{m_k\}$.

Proofs: (a) The original. (b) Titchmarch's book. (c) S&L TPA (2012)
Condition (H) $\Longleftrightarrow m_k(X) \le c^k (2k)! \Rightarrow$ $\mathrm{C}[\{m_k\}] = \infty \Rightarrow$ X is M-det.

Notice: The condition is on $\sqrt{X}$ but the conclusion is for X.

Corollary: If a r.v. $X > 0$ has a m.g.f., then its square X^2 is M-det.

Result: In (H), $\frac{1}{2}$ is the best possible constant for X to be M-det.
For each $\rho \in (0, \frac{1}{2})$ there is a r.v. Y with $\mathbf{E}[e^{tY^\rho}] < \infty$ s.t. Y is M-indet.

Comment: Hardy's condition is sufficient but not necessary for M-det.

Rate of growth of the moments and (in)determinacy

Given a r.v. $X > 0$ with moments $m_k,\ k = 1, 2, \ldots$.

Assume $m_1 \geq 1$. Then m_k increase in k. Define the ratio

$$\Delta_k = \frac{m_{k+1}}{m_k} \ \text{ and let } \ \Delta_k = \mathcal{O}((k+1)^{\gamma}) \ \text{ as } k \to \infty.$$

The number $\gamma =$ rate of growth of the moments of X.

Statement 1: If $\gamma \leq 2$, then X is M-det.

Statement 2: $\gamma = 2$ is the best possible constant for which X is M-det.
Equiv: If $\Delta_k = \mathcal{O}((k+1)^{2+\delta}),\ \delta > 0$, there is a r.v. Y which is M-indet.

Remark 1: Similar statments hold for r.v.s on $\mathbb{R}^1$, with $m_{2(k+1)}/m_{2k}$ ...

Remark 2: If $\gamma > 2$, add one condition and show that X is M-indet.

Example: $\xi \sim Exp(1)$, density e^{-x}, $x > 0$, m.g.f.

Result: ξ^r is M-det for $0 \le r \le 2$ and M-indet for $r > 2$.

Proofs: (1) Use Krein for $r > 2$ and Krein-Lin techniques for $0 \le r \le 2$.
(2) Take $r = 2$, $X = \xi^2$, $m_k(X) = (2k)! \Rightarrow X$ is M-det by Carleman.
Also, since $\sqrt{X} = \xi$ is Cramér $\Rightarrow X = (\sqrt{X})^2$ is M-det, by Hardy.
More, $X^{r/2}$ is Cramér for $r \in (0,2) \Rightarrow X^r$ is M-det for $r \in (0,2]$.
(3) Write $\mathbf{E}[\xi^r]$ via gamma-function and use the rate growth results.

Case r = 3: For $X = \xi^3$, $m_k = \mathbf{E}[X^k] = (3k)!$, fast $\nearrow$; density f of X is

$$f(x) = \frac{1}{3} x^{-2/3} e^{-x^{1/3}}; \text{ take perturbation } h(x) = \sin\left(\frac{\pi}{6} - \sqrt{3}x^{1/3}\right), \ x > 0.$$

Stieltjes class $\mathbf{S}(f,h) = \{f_\varepsilon = f[1 + \varepsilon h], \varepsilon \in [-1,1]\}$, with these f, h.
In $\mathbf{S}$, f_ε is density $\Rightarrow$ r.v. $X_\varepsilon \sim f_\varepsilon$: $\mathbf{E}[X_\varepsilon^k] = m_k = (3k)!$.

Remark: Stieltjes classes can be written for any power ξ^r, $r > 2$.

Example: $Z \sim \mathcal{N}(0,1)$, Z^2, Z^3, Z^4, $|Z|^r$.

Easy: Z is Cramér $\Rightarrow$ Z is M-det. More, Z^2 is M-det, by Hardy.
However, $Z^2 = \chi_1^2$ (light tail) is also Cramér $\Rightarrow$ Z^4 is M-det, by Hardy.

Comment: To apply twice Cramér, and twice Hardy, is the shortest way to prove that power 4 of the normal r.v. Z, Z^4 is M-det.

General Result: $|Z|^r$ is M-det for $0 \le r \le 4$, and M-indet for $r > 4$.
Proof: Use Krein for $r > 4$ and Krein-Lin techniques for $0 \le r \le 4$.
Alternatively, use the rate growth conditions.

Delicate Case: The cube, $X = Z^3$, on $\mathbb{R}^1$, two heavy tails, $m_{2k-1} = 0$, $m_{2k} = (6k-1)!!$, $\mathrm{C} < \infty(?)$ Density g of Z^3, by Krein $\Rightarrow$ Z^3 is M-indet.
Same conclusion by using our rate growth result.
Stieltjes class: For some h, $\mathbf{S}(g,h) = \{g_\varepsilon = g[1 + \varepsilon h], \varepsilon \in [-1,1]\}$.

Shocking Fact: $X = Z^3$ is M-indet, however $|X|$ is M-det. Why?

Multidimensional Moment Problem: Work Going ... [full of traps]

Picture Today: Not too much done for multivariate distributions ...
analytic: Petersen (1982), Berg-Thill (1991), Schmüdgen-Putinar (2008)
probability/statistics: K&S (2011–2013) + a few references therein.

Random vector $X = (X_1, ..., X_n) \in \mathbb{R}^n$ with arbitrary distribution F.
Finite are all multi-indexed moments

$$m_{k_1,\dots,k_n} = \mathbf{E}[X_1^{k_1}\cdots X_n^{k_n}],\ k_j \ge 0,\ k_1 + \ldots + k_n = k,\ k = 1, 2, \ldots$$

Same kind of questions and terminology as in dim. 1.
Tools: Cramér, n-dim. m.g.f.; Carleman, next slide; but ... **no Krein**.
Result: $X = (X_1, \ldots, X_n) \sim F$, finite multi-indexed moments; $F_1, \ldots, F_n$ the marginals, marginal moment seq. $\{m_{k_1}^{(1)}\}, \ldots, \{m_{k_1}^{(1)}\}$.
There are many n-dim. d.f.s F with these marginal moment sequences.

Carleman Condition in Dimension n

We need the numbers M_{2k} and M_k, for F on $\mathbb{R}^n$ and $\mathbb{R}^n_+$:

$$M_{2k} = m_{2k,0,\dots,0} + m_{0,2k,0\dots,0} + \dots + m_{0,0,\dots,0,2k} \quad \text{(Hamburger)},$$

$$M_k = m_{k,0,\dots,0} + m_{0,k,0\dots,0} + \dots + m_{0,0,\dots,0,k} \quad \text{(Stieltjes)}.$$

Now the n-Carleman quantity is defined, respectively, as follows:

$$\mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(M_{2k})^{1/2k}} \quad \text{and} \quad \mathrm{C} = \sum_{k=1}^{\infty} \frac{1}{(M_k)^{1/2k}}.$$

$\mathrm{C} = \infty \Rightarrow$ the vector X, or equiv. its n-dimensional d.f. F, is M-det.

If n-Carleman holds for X, then 1-Carleman holds for each X_j. Converse not in general true. There are c.e.s; related to Müntz theorem.

Known Result: Given $X \sim F$ in $\mathbb{R}^n$, marginals $F_1, \ldots, F_n$.

(a) If each of $F_1, \ldots, F_n$ is M-det, then the n-dim. d.f. F is M-det.

(b) If $X_1, \ldots, X_n$ are independent, and F is M-det, then each F_j is M-det.

Comments:

- In (a) we do not say in which way F_j are M-det.
- In (b), $F = F_1 \cdots F_n$, this is used for the converse.
- There are M-det n-dim. d.f.s with M-indet marginals. [Illustrate!]

Strange, counter-intuitive, but true. Quite analytic.

New Result: Again, $X \sim F$ in $\mathbb{R}^n$, marginals $F_1, \ldots, F_n$ with densities $f_1, \ldots, f_n$ which are positive. Assume for each j, Krein condition and Lin condition hold for f_j, $j = 1, \ldots, n$.

Then for any indep/dep structure of X, the n-dim. d.f. F is M-det.

Recent Result: S&L, TPA (2012).

Given is a random vector $X \sim F$ with arbitrary distribution in $\mathbb{R}^n$ and finite all multi-indexed moments $m_{k_1,\dots,k_n} = \mathbf{E}[X_1^{k_1} \cdots X_n^{k_n}], \dots$
Consider the lenght of X: $\|X\| = \sqrt{\|X\|^2} = \sqrt{X_1^2 + \dots + X_n^2}$.

Suppose: 1-dim. non-neg. r.v. $\|X\|$ is Cramér: $\mathbf{E}[e^{c\|X\|}] < \infty, \ c > 0$.
Then the n-dim. Hamburger moment problem for F has a unique solution. Or, we say, the random vector $X \in \mathbb{R}^n$ is M-det, also that F is the only n-dim. d.f. with the set of multi-indexed moments $\{m_{k_1,\dots,k_n}\}$.

Proof: We follow two steps.
Step 1: Cramér for $\|X\| \Rightarrow \|X\|^2$ is M-det, by Hardy (Stieltjes case).
Step 2: Amazing statement by Putinar-Schmüdgen: If $\|X\|^2$ is M-det (1-dim. Stieltjes), then F is M-det (n-dim. Hamburger).

Related Topic: Asymmetry and M-indeterminacy

Easy: $X \sim F$, symmetric: $(-X) \stackrel{d}{=} X$, $f(-x) = f(x)$, real ch.f.
If $X \in \mathcal{M}$, then $\Rightarrow$ $\mathbf{E}[X^{2k-1}] = 0$, $k = 1, 2, \ldots$

Converse Question: We know $X \in \mathcal{M}$, and $\mathbf{E}[X^{2k-1}] = 0, k = 1, 2, \ldots$
Does this imply that X is symmetric?

Answer: 'Yes', if X is M-det, **'No'**, if X is M-indet.

Statement: If F is M-indet, there are 'many' d.f.s on $\mathbb{R}^1$, same moments as F, one is symmetric, 'center', all others are asymmetric.

Idea: Think of $\mathbf{S}(f, h) = \{f_\varepsilon = f[1 + \varepsilon h],\ \varepsilon \in [-1, 1]\}$.

More: If η is Laplace, $\frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}^1$, $Y = \eta^3$ is M-indet and symmetric, but Y_ε, $\varepsilon \neq 0$ is not! How 'much' is the asymmetry in the class $\mathbf{S}(f, h)$?

Answer: $\frac{1}{2}D(f, h)$ (= max total variation distance from the center).

Another Interesting Question: Let $X \in \mathcal{M}$, $\mathbf{E}[X^{2k-1}] = 0, k = 1, 2, \dots$ and we know that X is M-det. Is it true that X is symmetric?

Answer: No.

Statement: Let a r.v. $X \sim \chi^2_n$, $n = 1, 2, \dots$ and $Y = X - n$. Then:
$Y \in \mathcal{M}$, $\mathbf{E}[Y^{2k-1}] = 0$, $k = 1, 2, \dots,$ Y is M-det, Y is not symmetric.
Proof: With W a standard BM, we have

$$J = \int_0^1 W_s \, \mathrm{d}W_s = \frac{1}{2}(W_1^2 - 1) \;\Rightarrow\; \xi = W_1^2 \sim \chi^2_1.$$

Hence $\xi - 1 = 2J$, $\mathbf{E}[J^{2k-1}] = 0 \;\Rightarrow\; \mathbf{E}[(\xi - 1)^{2k-1}] = 0$, $k = 1, 2, \dots$
Use this with n independent BMs. Done.

Comment: The ch.f. of Y, $\psi(z) = \mathbf{E}[\mathrm{e}^{zY}] = \mathrm{e}^{-inz}(1 - 2iz)^{-n/2}$, real z.
Show that all odd order derivatives of $\psi(z)$ at zero are equal to zero.

Question: Are there other ch.f.s with such a property? Perhaps yes.

Products and Powers of Random Variables: ξ and $\perp\!\!\!\perp\ \xi_1, \dots, \xi_n$

When are $Y_n = \xi_1 \cdots \xi_n$ and $X_n = \xi^n$ M-det, and when M-indet? Same?
Stieltjes case: The moments of X_n dominate those of Y_n; we 'expect':
M-det of $X_n \Rightarrow$ M-det of Y_n, M-indet of $Y_n \Rightarrow$ M-indet of X_n.
Strangely enough, in general this is not true; there are counterexamples.
Generalized gamma-distributions: $GG(a,b,c), a,b,c > 0$. Density
$f(x) = Kx^{a-1}\mathrm{e}^{-bx^c}, x > 0$. Here: Exp, gamma, half-normal, χ^2, half-Bessel.

Result 1: $\xi \sim Exp(1)$. Then Y_2 is M-det, $Y_n = \xi_1 \cdots \xi_n$ for $n = 3, 4, \dots$, are all M-indet. Recall ξ^2 is M-det, while ξ^n for $n = 3, 4, \dots$, are all M-indet.

Result 2: $|Z|$, half-normal. The product of 2, 3 or 4 $\perp\!\!\!\perp$ half-normals is M-det, while the product of 5 or more $\perp\!\!\!\perp$ half-normals is M-indet.

Result 3: Half-logistic, $2\mathrm{e}^{-x}/(1+\mathrm{e}^{-x})^2,\ x > 0$. Product of 3 or more half-logistic r.v.s is M-indet.

Result 4: Product of 2 or more χ^2 r.v.s is M-indet.

Hamburger case: r.v. ξ on $\mathbb{R}^1$ and $\perp\!\!\!\perp$ copies $\xi_1, \dots, \xi_n$

Result 1: $Z \sim \mathcal{N}$. Then, $Z_1 Z_2$ is M-det, while $Y_n = Z_1 \cdots Z_n$ for $n = 3, 4, \dots,$ are all M-indet. Recall, Z^2 is M-det, Z^3 is M-indet, Z^4 is M-det, any next power, $5, 6, \dots,$ is M-indet.

Result 2: The product of two $\perp\!\!\!\perp$ Laplace r.v. $(\frac{1}{2}e^{-|x|})$ is M-indet. (Above: product of two *Exp* is M-det.) Square of Laplace r.v. is M-det.

Result 3: Logistic: $1/(2 + e^x + e^{-x})$. Products of 3 or more is M-indet.

Result 4: Product of Laplace r.v. and logistic r.v. is M-indet.

Result 5: Symmetric X on $\mathbb{R}^1$ and $Y > 0$, $T = XY$. If X or Y is M-indet, T is M-indet. If X and Y have rates of growth of moments a and b, and $a + 2b \le 2$, T is M-det. If $a + 2b > 2$ + cond get M-indet T.

Result 6: N = r.v. in $\mathbb{N}_0$, $\tilde{X} = \xi^N$, $\tilde{Y} = \xi_1 \cdots \xi_N$. All r.v.s $\perp\!\!\!\perp$.
We have conditions for M-det and M-indet. E.g., if $N \sim$ Poisson, $Z \sim \mathcal{N}$, both $\tilde{X}$ and $\tilde{Y}$ are M-indet.

Q1: How to construct discr. distr. with moments $\{(3k)!, k = 1, 2, \dots\}$?

Q2: How to define Krein integral in dimension 2 or more?
$(K[f] = \int \frac{-\ln f(x)}{1+x^2} dx < \infty$ or $-\ln f(x^2) \dots)$

Q3: How to define Lin's condition for discrete distributions?
(Smooth continuous case: $-xf'(x)/f(x) \to \infty$, large x.)

Q4: $X \sim F, F' = f$, finite moments, inf.div., M-indet. Stieltjes class
$\mathbf{S}(f, h) = \{f_\varepsilon = f(1 + \varepsilon h),\ \varepsilon \in [-1, 1]\}$, any perturbation h.
Conjecture: f_ε with $\varepsilon \neq 0$ is not inf.div.

Q5: Continuous r.v. $X \Rightarrow$ discrete (rounded) r.v. $\lfloor X \rfloor$ or $\lfloor X + \frac{1}{2} \rfloor$.
Conjecture: X and $\lfloor X \rfloor$ share the same M-det/indet property.

Q6: $X \sim F$ on $\mathbb{R}^+$ with $\mathbf{E}[X^k] = e^{k^2/2}$, $k = 1, 2, \dots$.
Conjecture: If F is unimodal, F is unique and $F = LogN(0, 1)$.

Q7: F – inf.div. d.f., L – the Lévy measure in the Lévy-Khintchine formula for the ch.f. of F. Known: F and L have the same number of moments: F finite moments $\iff$ L finite moments.
Conjecture: F is M-det $\iff$ L is M-det.

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