Rotor-router model

V.B. Priezzhev

References

Physics (arXiv)

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2.its development, P.,Dhar, et al, 9611019

3.some details, Povolotsky,P.,Shcherbakov, 9802070

4.recent papers, Poghosyan,P.,1310.1225; Dandekar, Dhar, 1312.6888.

Mathematics (arXiv)

1. alternate model, Cooper, Spencer, 0402323

2. important review, Holroyd, Levine, Meszaros, Peres, Propp, Wilson, 0801.3306

3. unicycles:conjecture “5/16”, Levine, Peres, 1106.2226

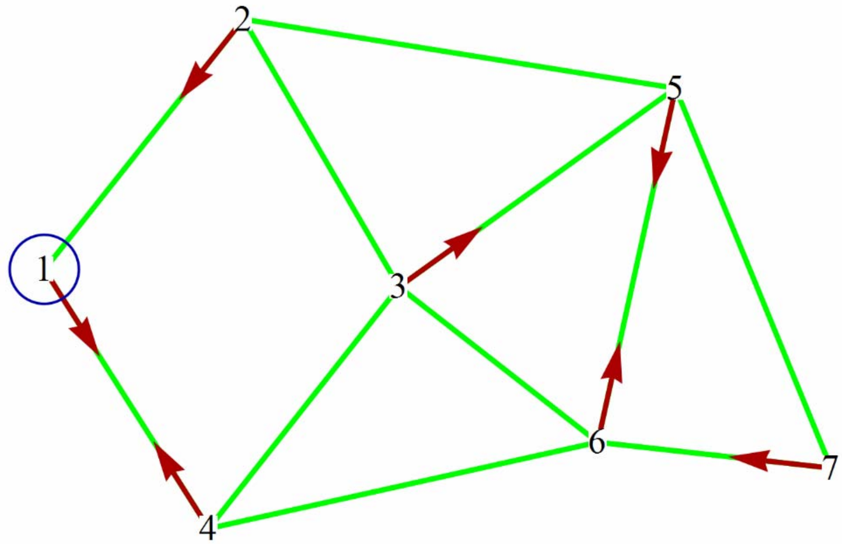
4. proof of conjecture “5/16”, Poghosyan, P., Ruelle, 1106.5453

5. Kenyon’s proof, Kenyon, Wilson, 1107.3377

Content

- Euler walk
- Subdiffusion
- Derandomization
- Unicycles
- Conjecture “5/16”
- Proof of conjecture

Euler walk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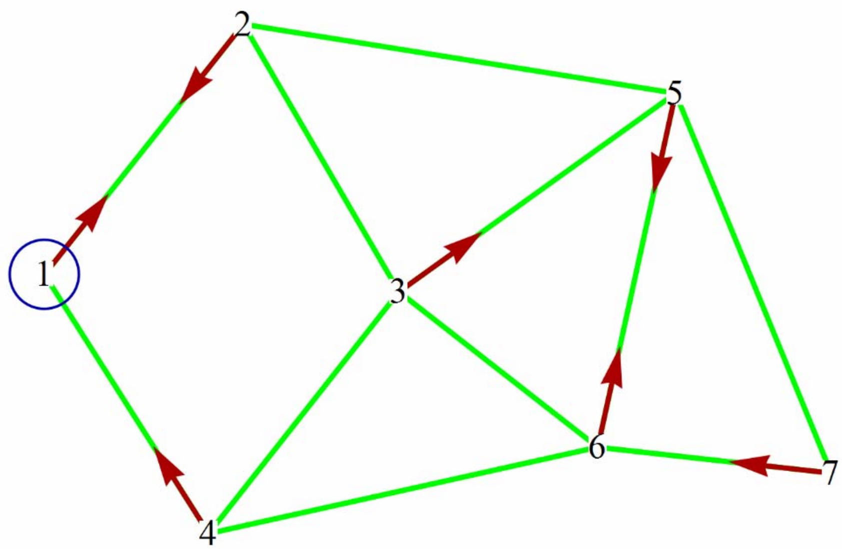
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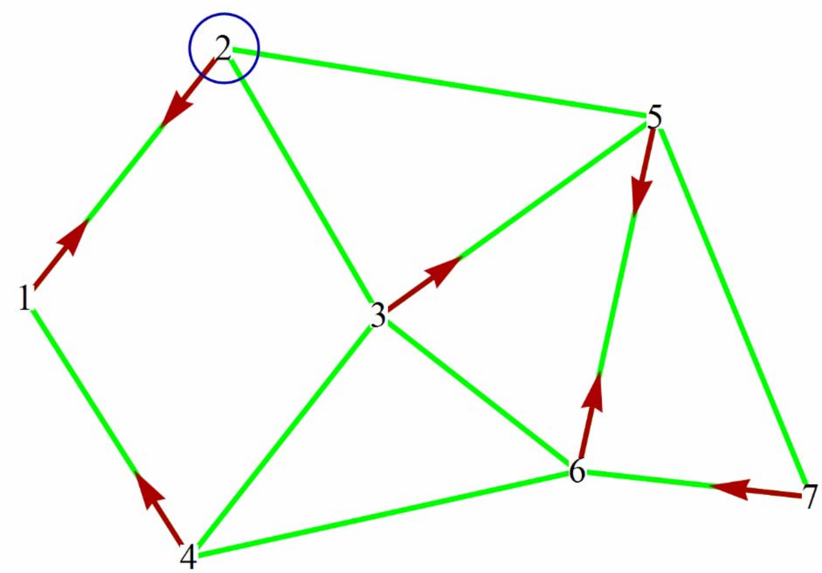
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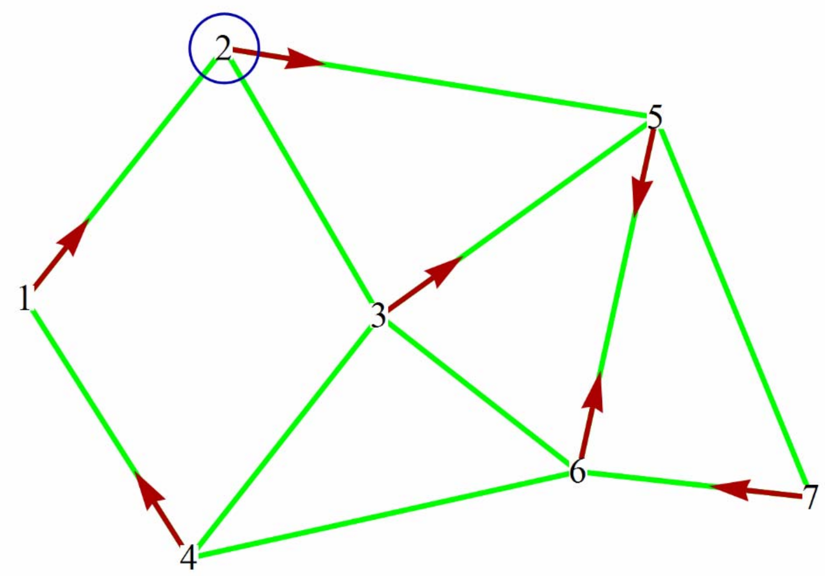
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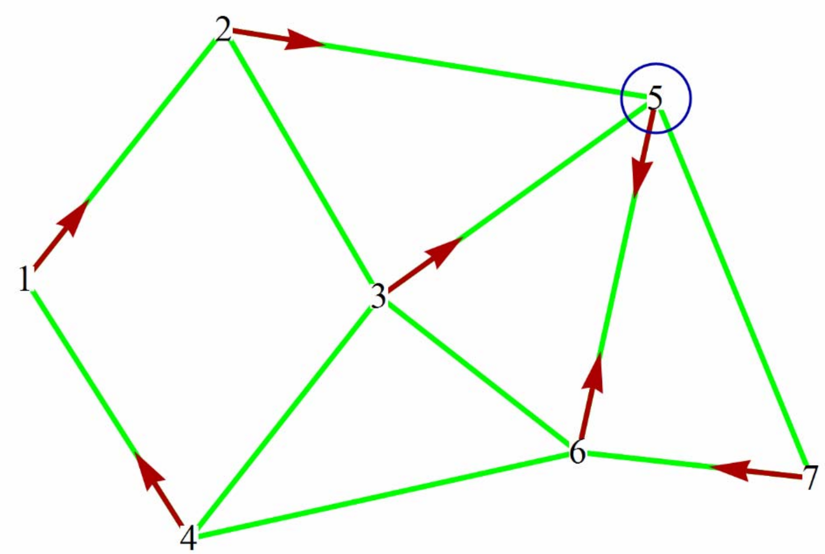
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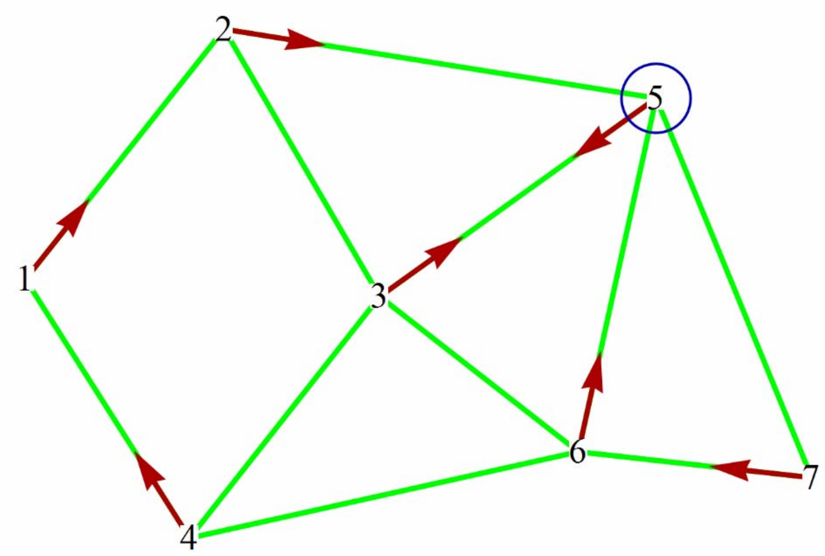
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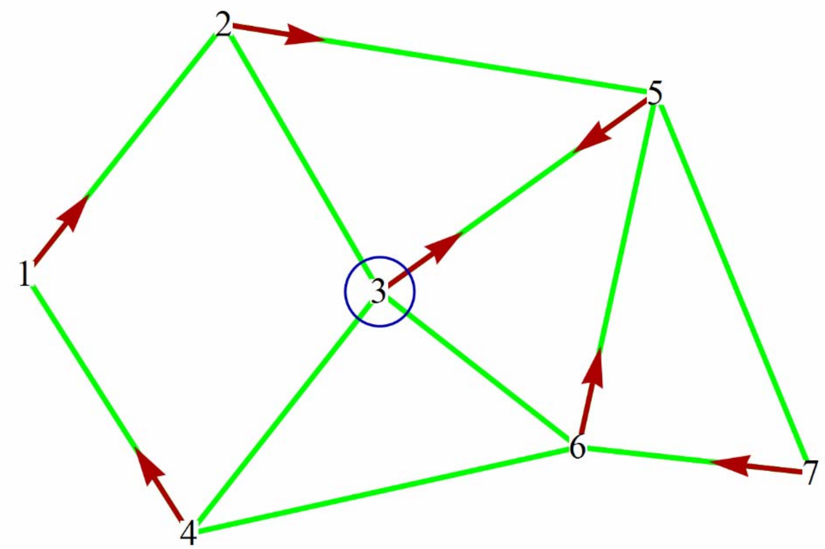
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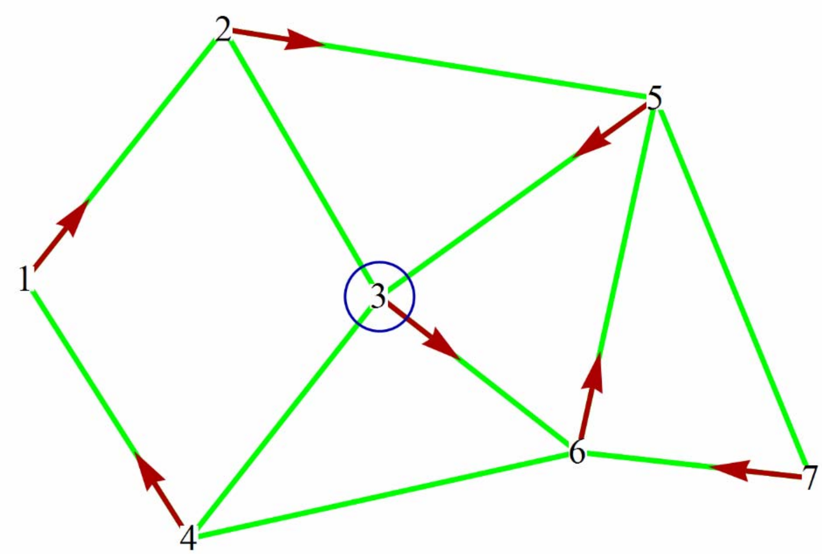
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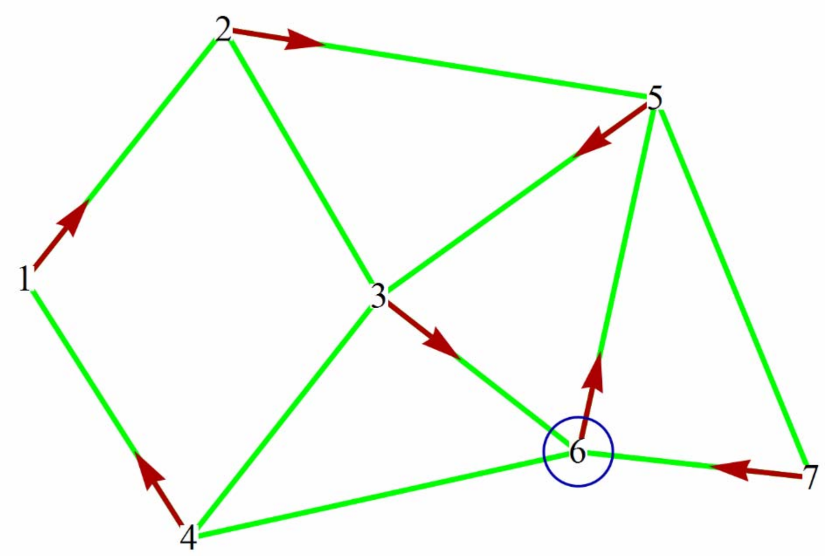
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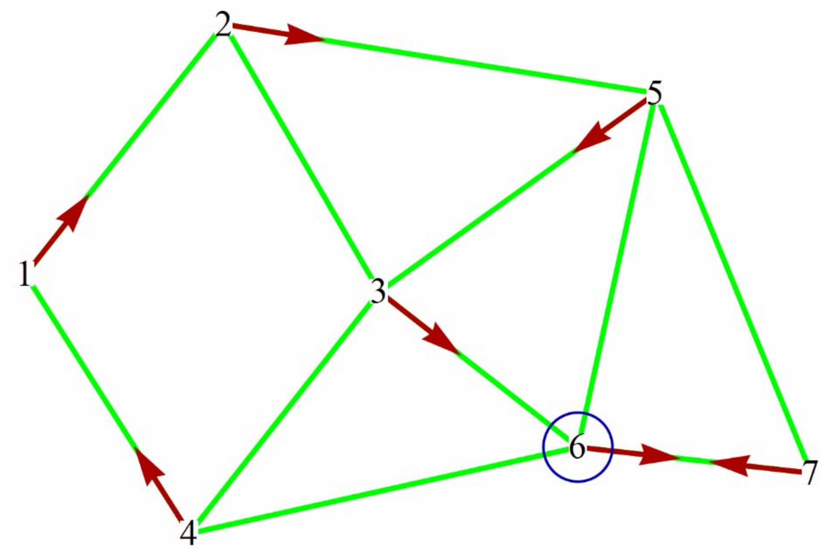
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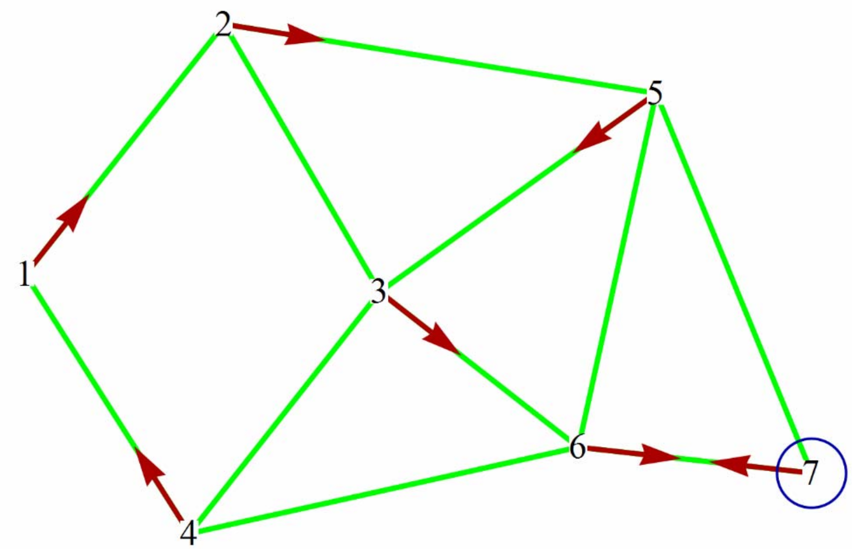
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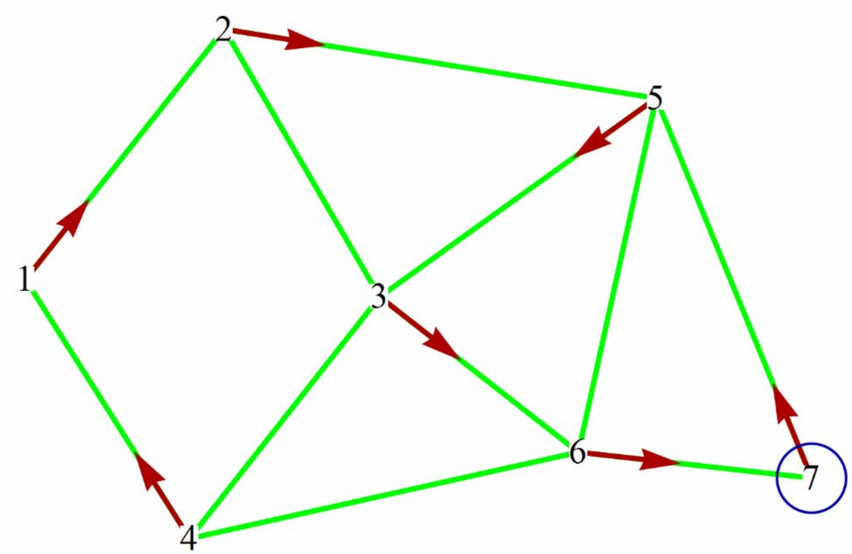
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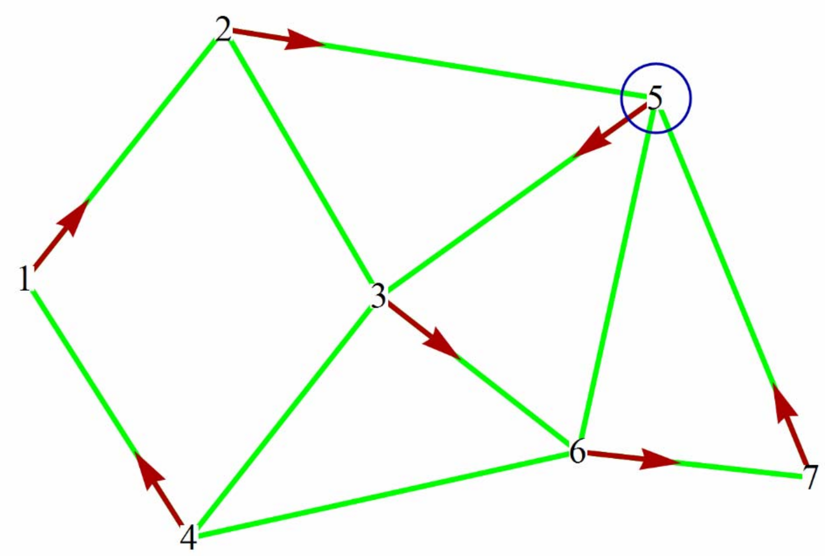
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Theorem: Every limit cycle is an Euler tour

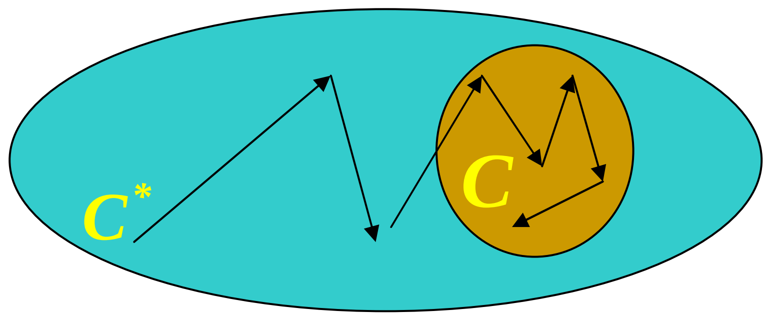
Idea of proof:

The walker leaves the point i along some bond b_1. We evolve it till after time T it returns to b_1 for the first time. We can show that no other bond in this path is visited twice.

Assume the contrary and suppose that during the T steps the bond c, originating from the point j, is the first bond that is visited twice.

Each successive exit from j is along a different direction so there will be deg(j) + 1 exits. But the number of visits to j equals number of exits. Hence there must exist some bond going into j which is also passed more than once. This contradicts the fact that c was taken to be the first bond to be passed twice. Thus all b_i, i = 1 to T are distinct .

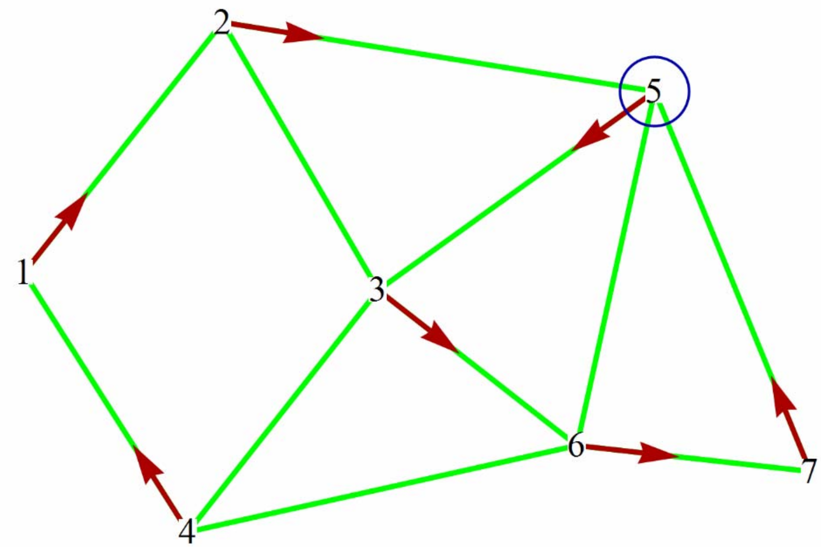
Set of initial arrow configurations $|C^*| = \prod_{i=1}^{N} \deg(i)$



Set of recurrent configurations C

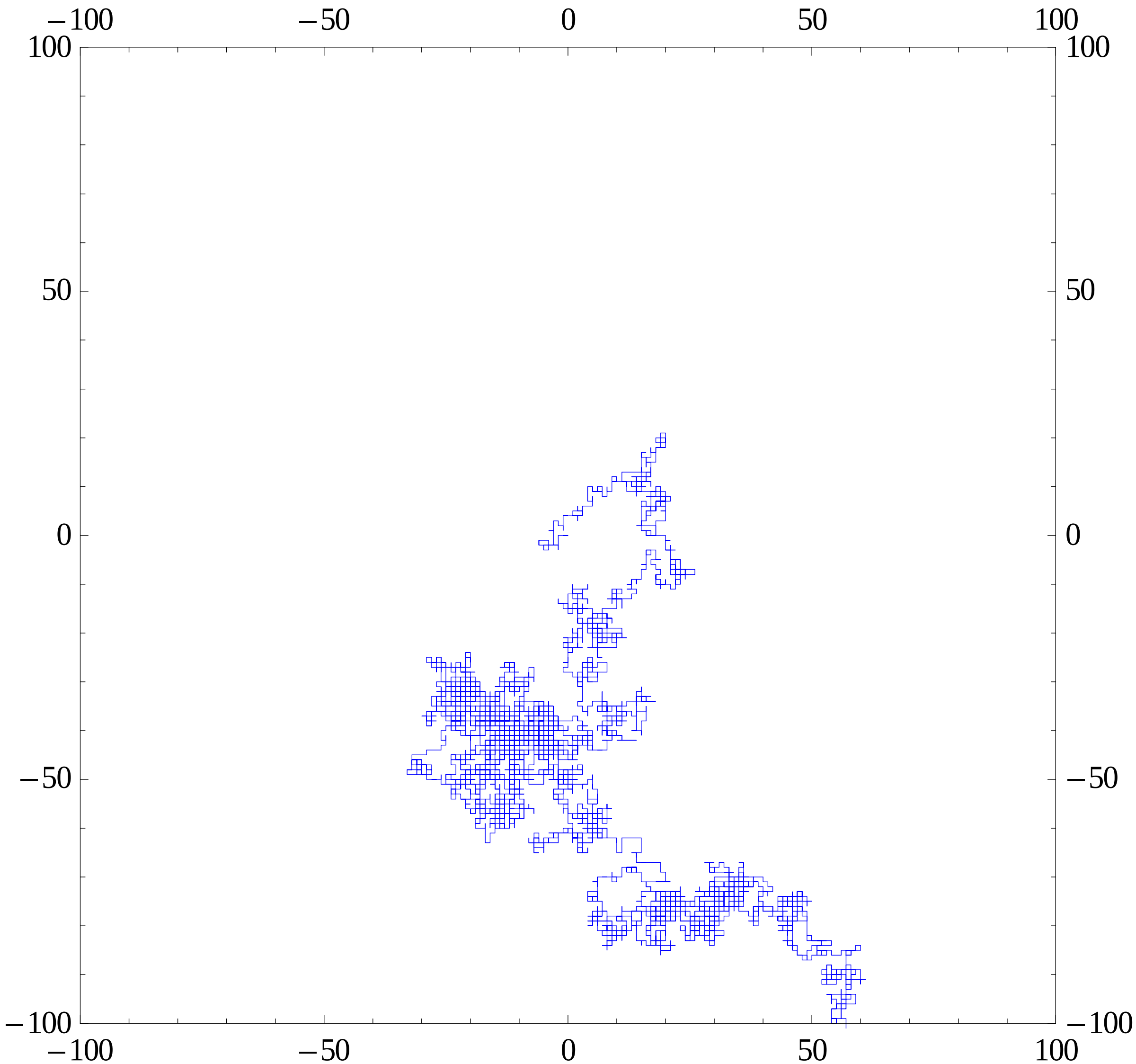
$|C|$ = number of unicycles of graph G

Theorem: Let G be a connected digraph. Then element c from C is a recurrent state if and only if it is a unicycle.



Unicycle: 3-6-7-5 cycle, 4-1-2-5 tree

Subdiffusion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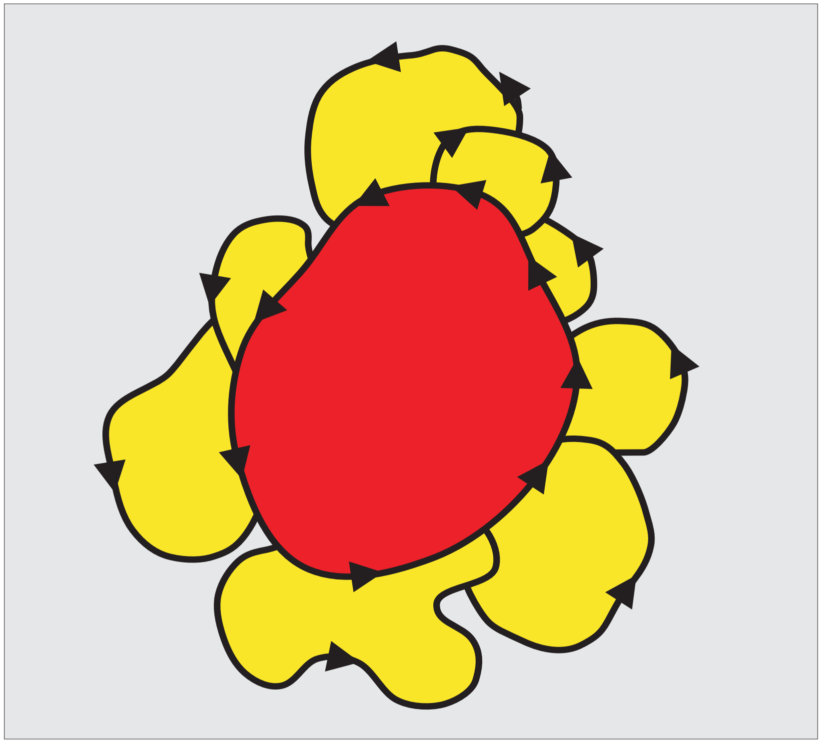
Rotor-router walk on infinite lattice

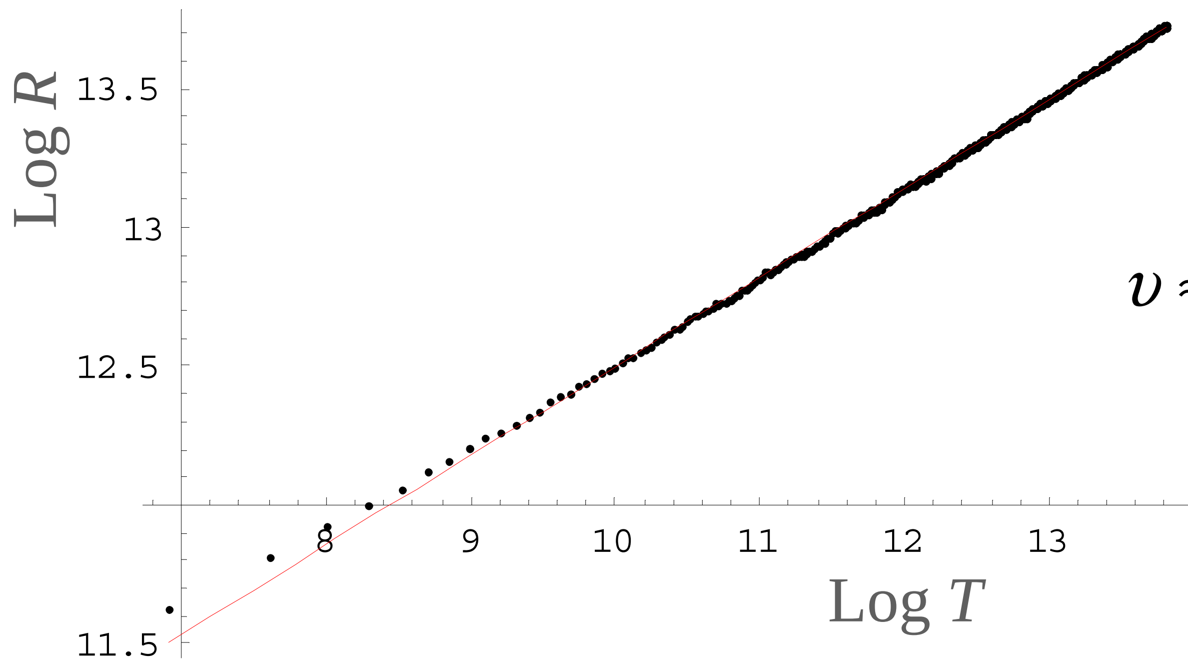


Walker returns many times to red area before adding a new yellow part to the cluster of visited sites

$$\frac{dR}{dT} \sim \frac{1}{R^2}$$

$$\langle R^2 \rangle \sim T^{2\nu}, \; \nu = \frac{1}{3}$$

Mean-square distance for time T



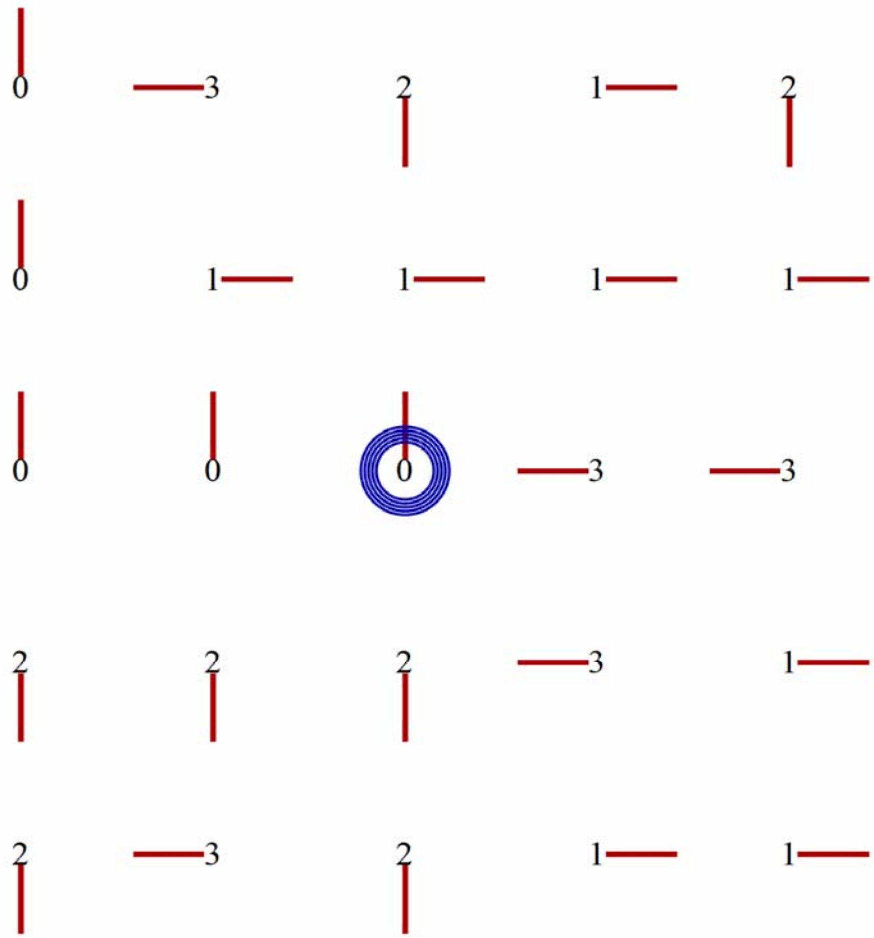
$$\upsilon \approx \frac{1}{3}$$

$$\langle R^2 \rangle \simeq T^{2\upsilon}$$

Number of steps – 10^6

Averaged over 10000 runs

Derandomization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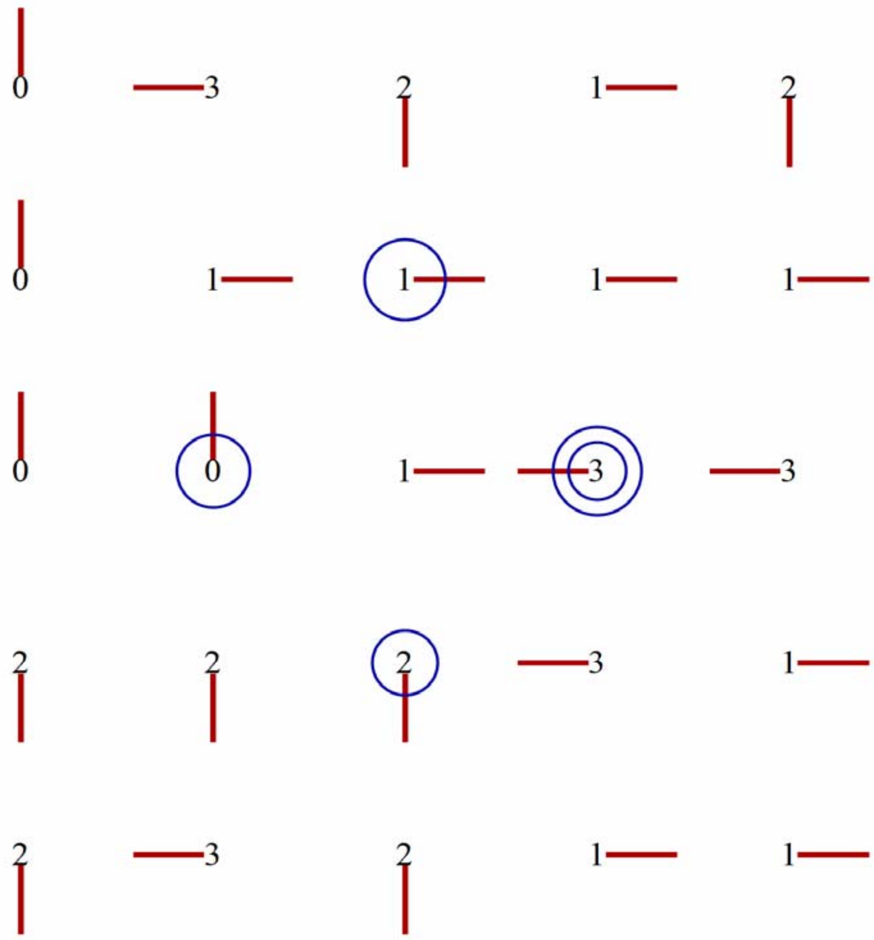
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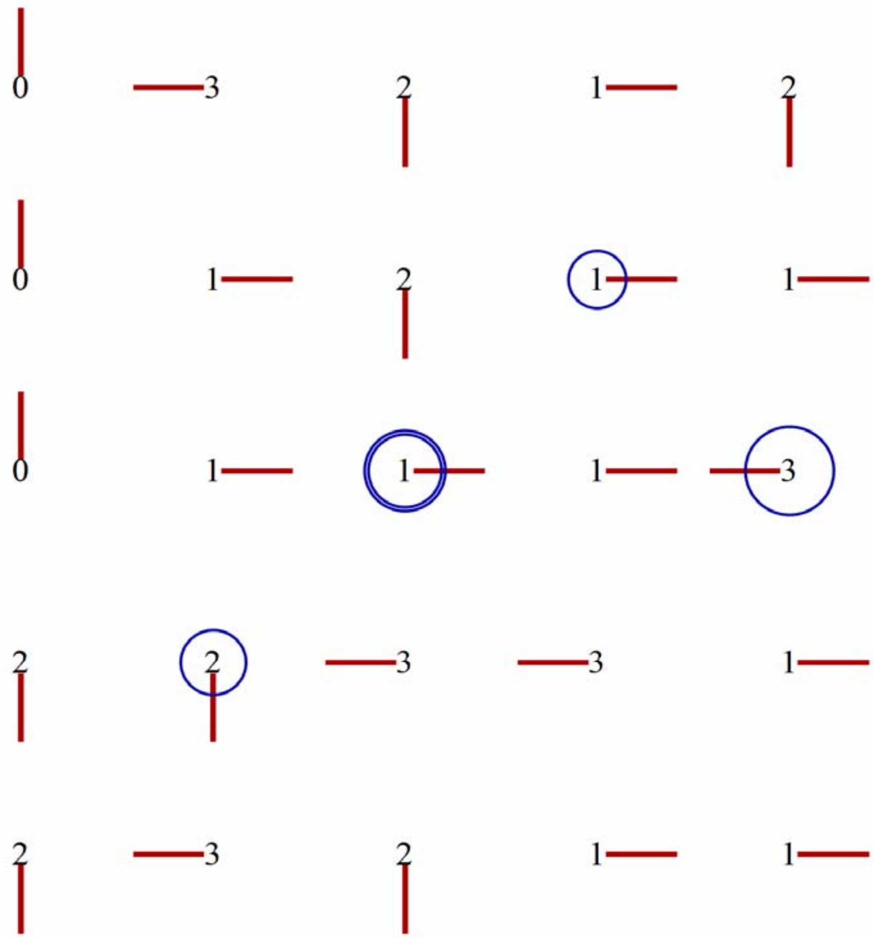
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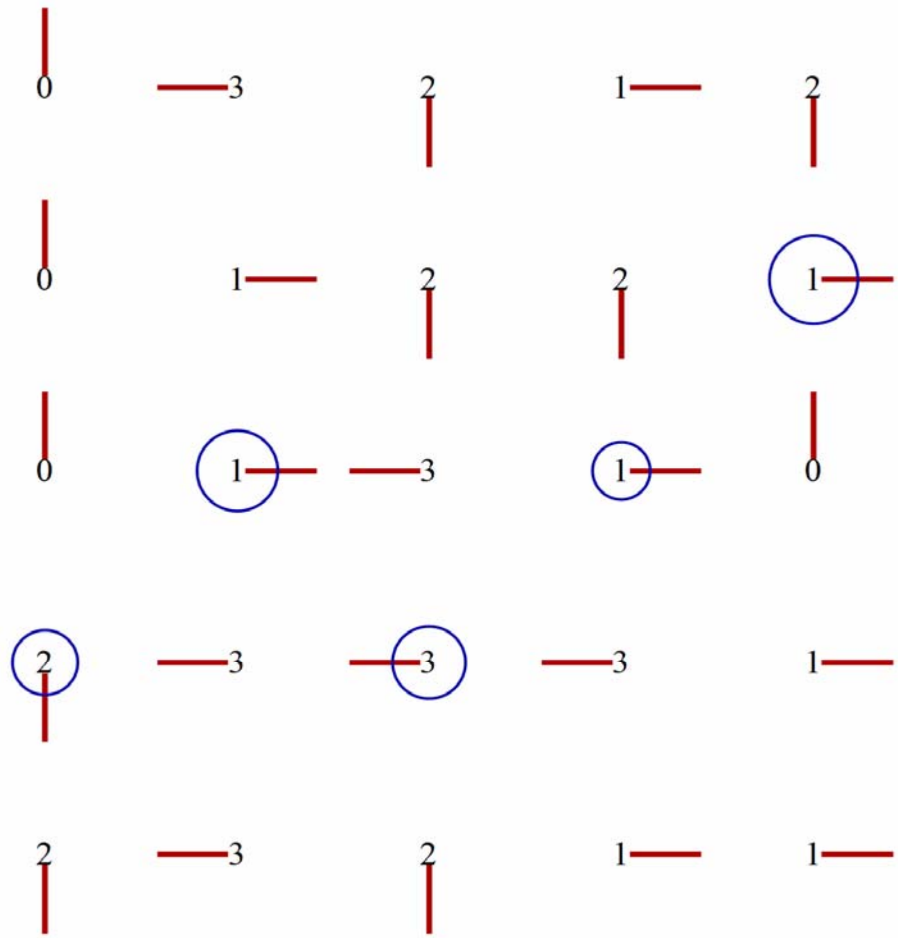
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J.N. Cooper and J. Spencer, Simulating a random walk with constant error. Combin. Probab. Comput.,15(6):815, (2006)

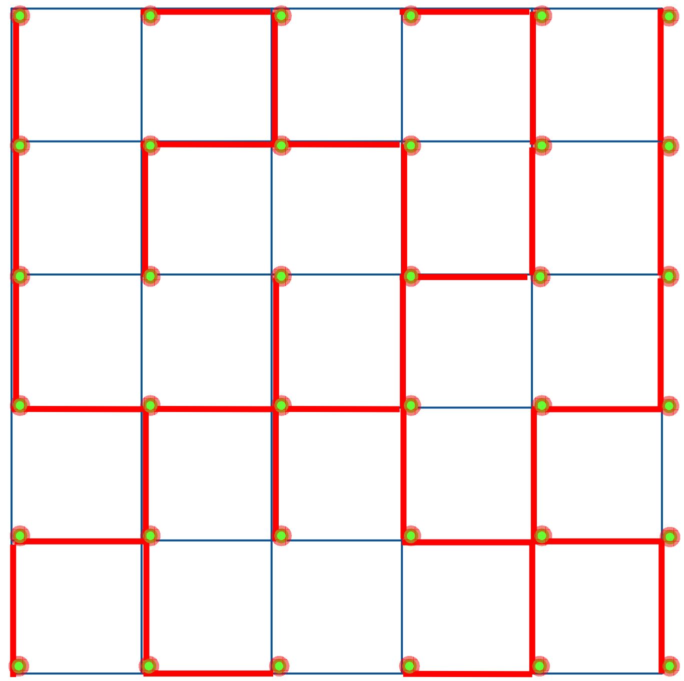
Let N be number of particles in the origin of square lattice at initial moment t=0.

- $W_N(x,t)$ - expected number of particles distributed by rotor-router in x after t steps
- $\hat{W}_N(x,t)$ - expected number of particles distributed by simple random walk in x after t steps

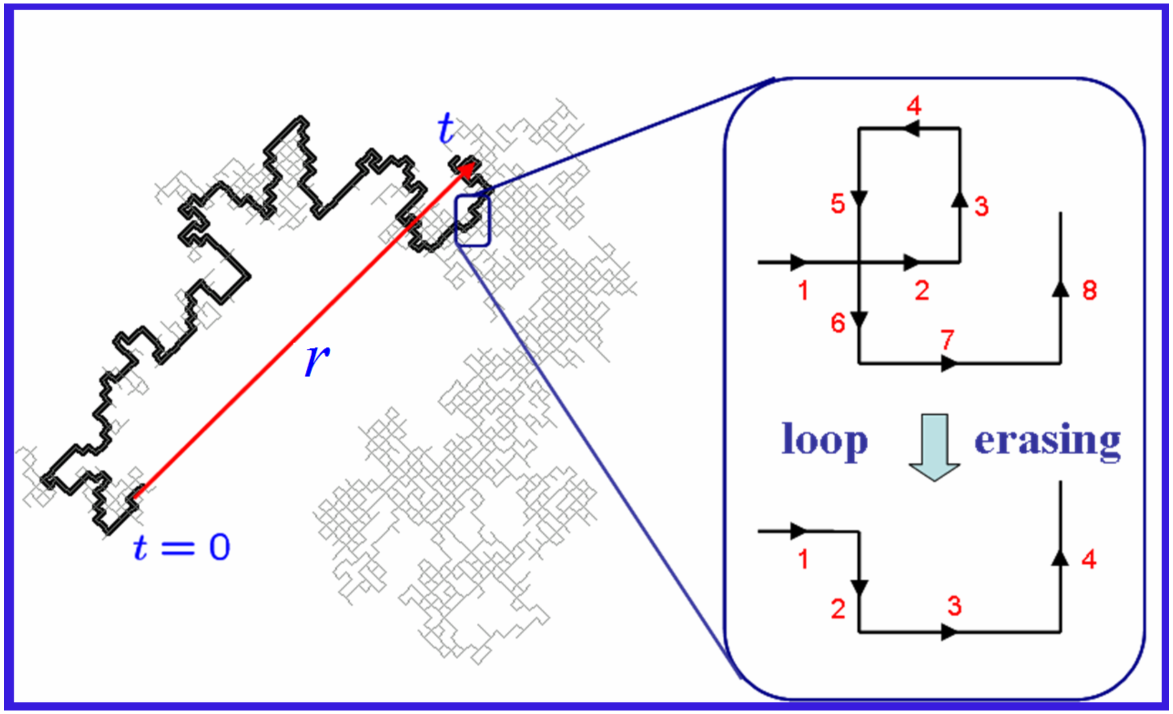
Theorem: $|\hat{W}_N(x,t) - W_N(x,t)| \leq C$ where C doesn't depend on number of particles N and number of steps t.

Unicycles

Graph: $|V|=|E|+1$ (spanning tree)



The Loop-Erased Random Walk (LERW) is a simple random walk where the loops are erased in chronological order.



Return probability P_{ret} is probability that the LERW visits a neighboring vertex of the origin.

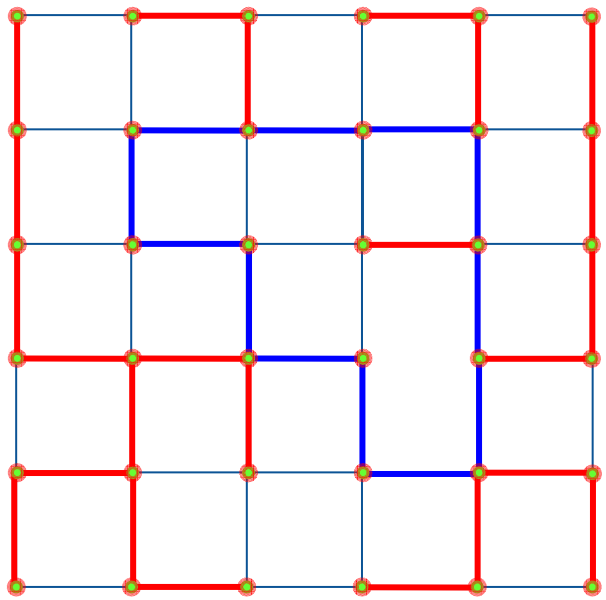
- The LERW has been introduced by G.F. Lawler in 1980 to modify the uniform measure of the usual self-avoiding random walk.

- The connection between the LERW and spanning trees has been discussed in several works (Broder (1989), Pemantle (1989) and so on) with the result that the probability measure of LERW paths coincides with the uniform measure of paths on spanning trees.
- Moreover, the probability measure exists in the limit of infinite lattice.

- Using Coloumb gas theory, Majumdar (1992) found the fractal dimension of the LERW equal to 5/4. Kenyon (2000) obtained this result rigorously.

- Lawler, Schramm and Werner (2004) proved that LERW in the continuum limit converges to SLE curve.

Graph: |V|=|E| (unicycle)



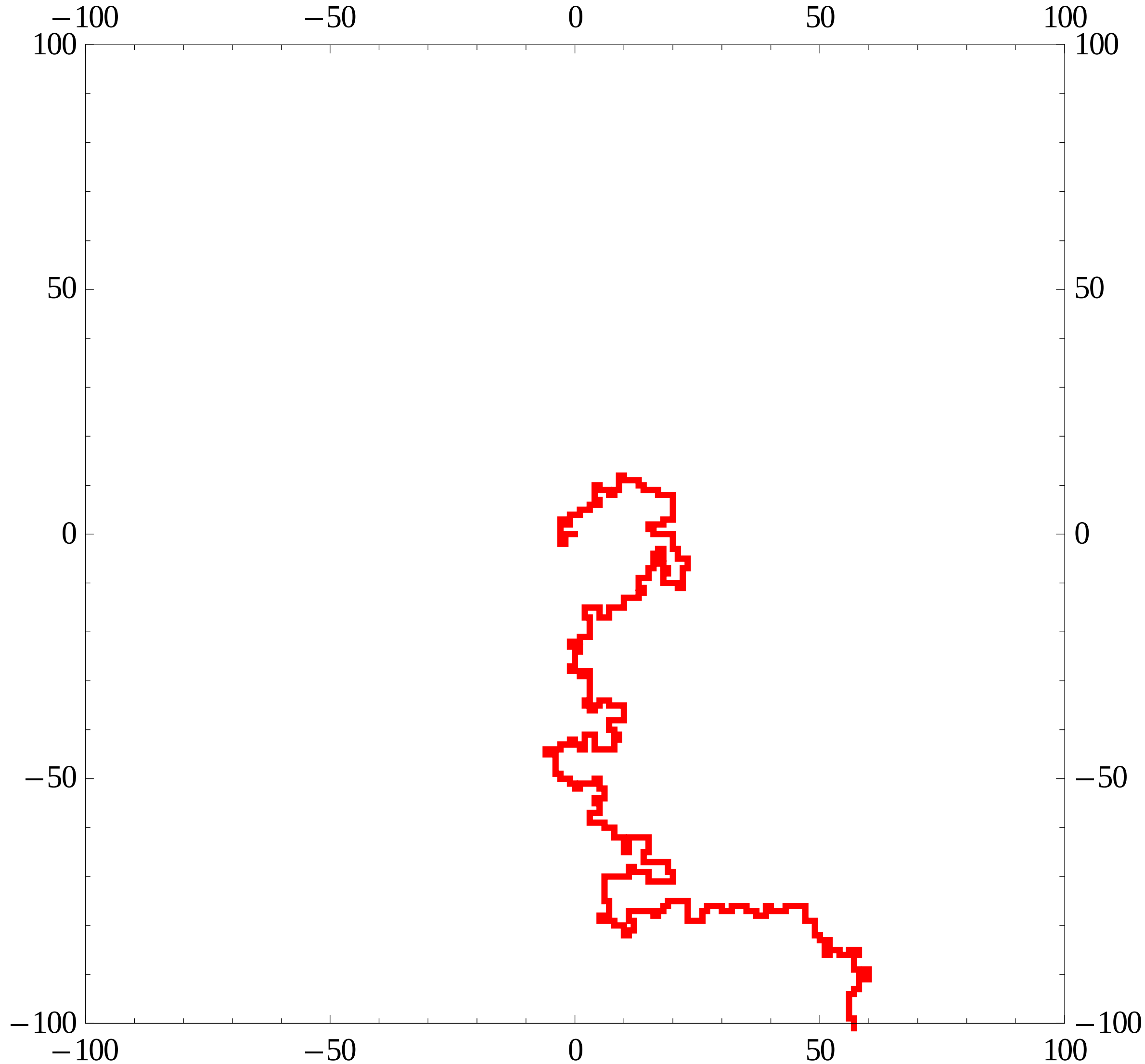
- Levine Peres (2011): if the return probability is 5/16 in the limit of large lattice, then

$$\frac{N_{\text{unicycles}}}{N_{\text{ST}}} = \frac{1}{8}$$

$$\langle L_{cycle} \rangle = 8$$

L. Levine, Y. Peres, arXiv: 1106.2226

Conjecture “5/16”



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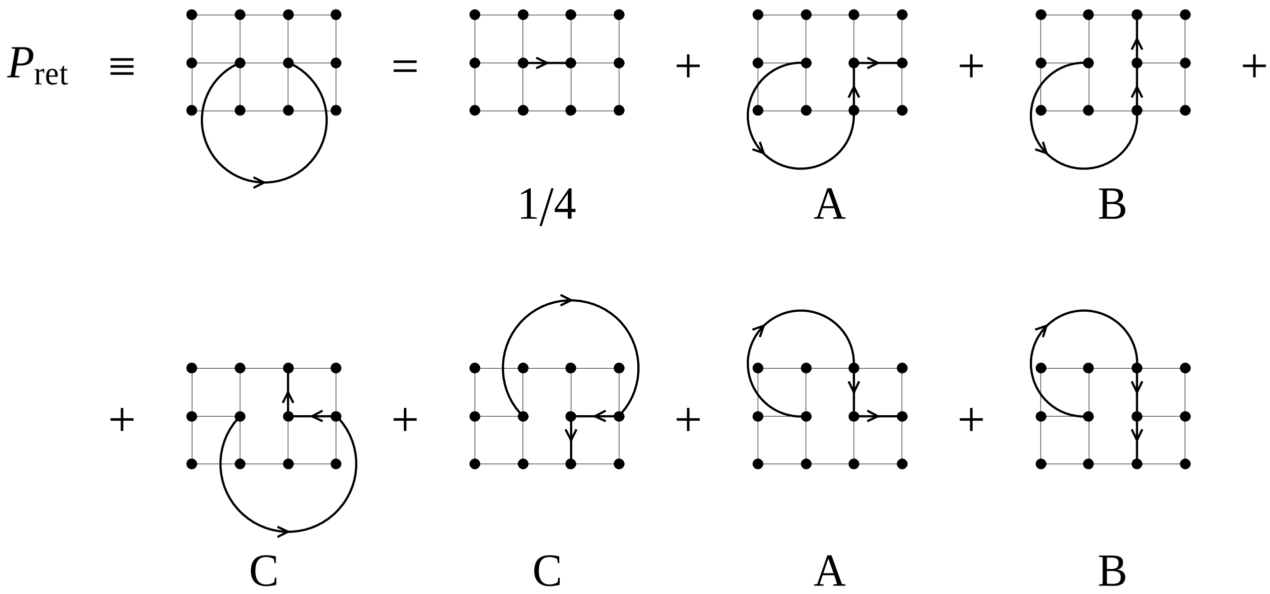
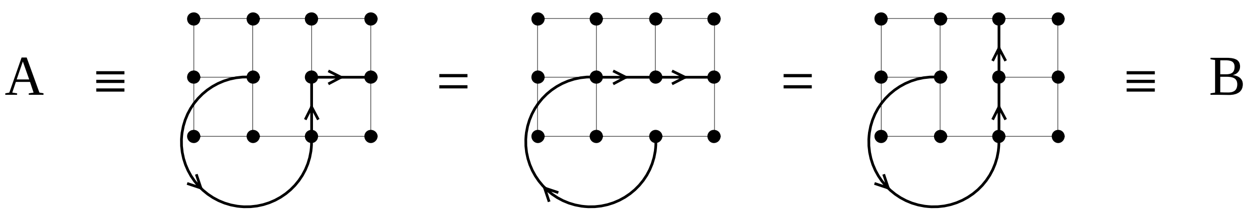
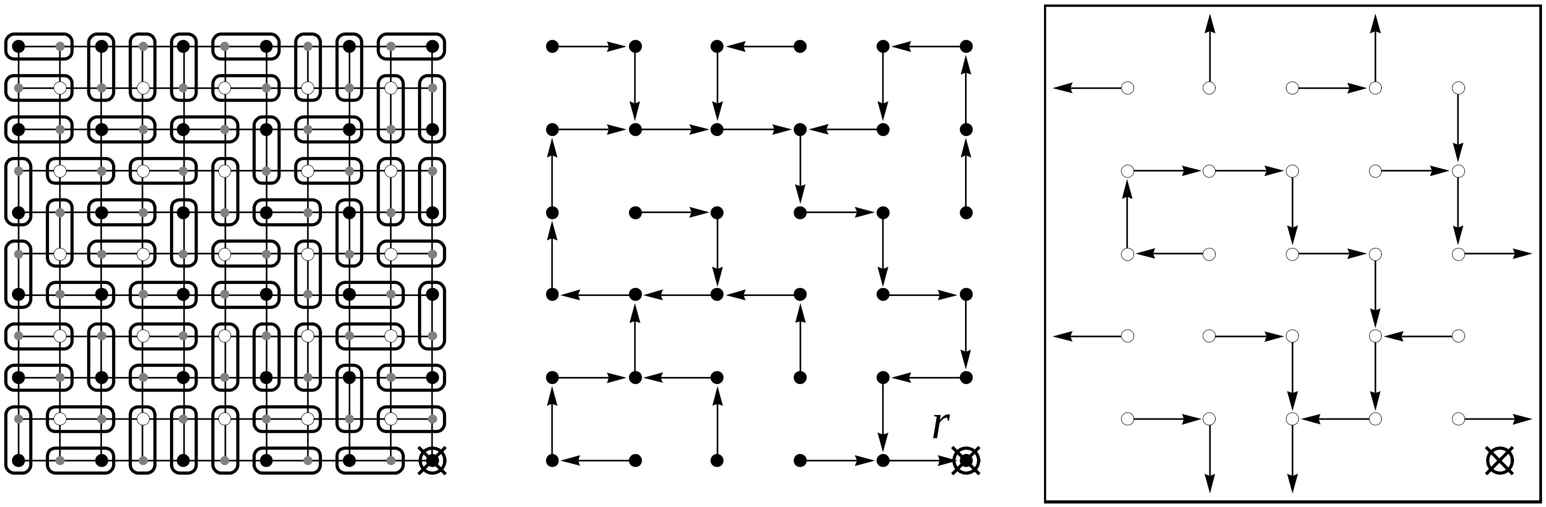
Proof of conjecture



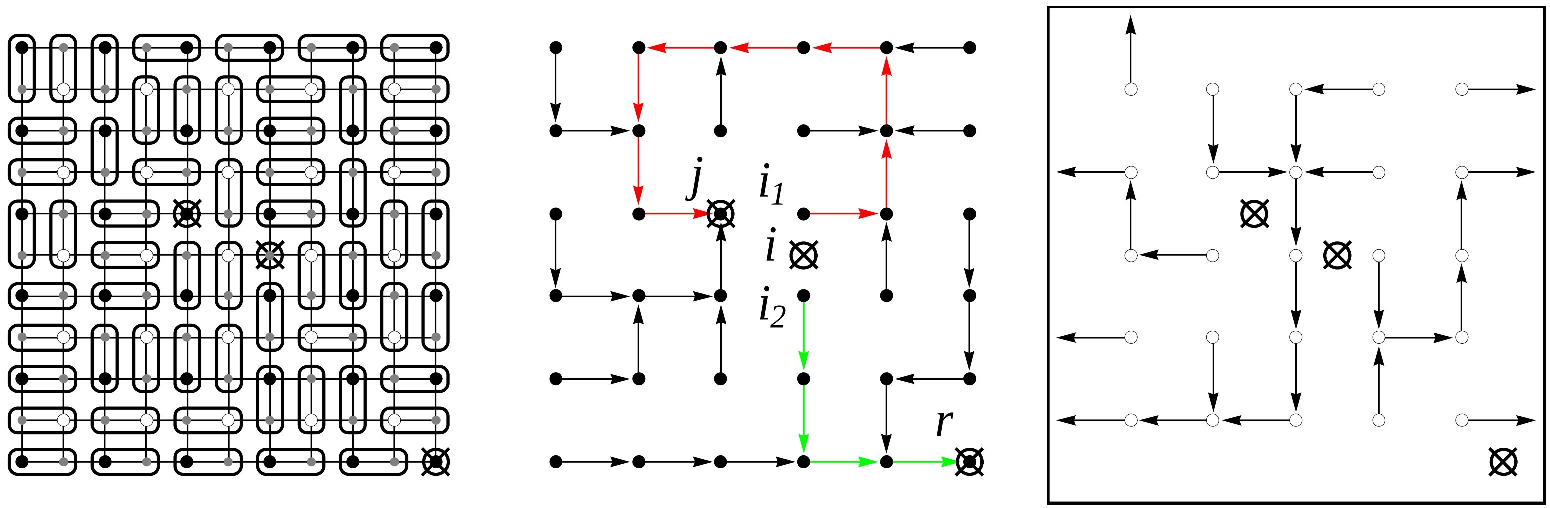
$$P_{ret} = \frac{1}{4} + 2\left(A + B + C\right)$$



Dense packed dimers with one monomer



(a) (b) (c)



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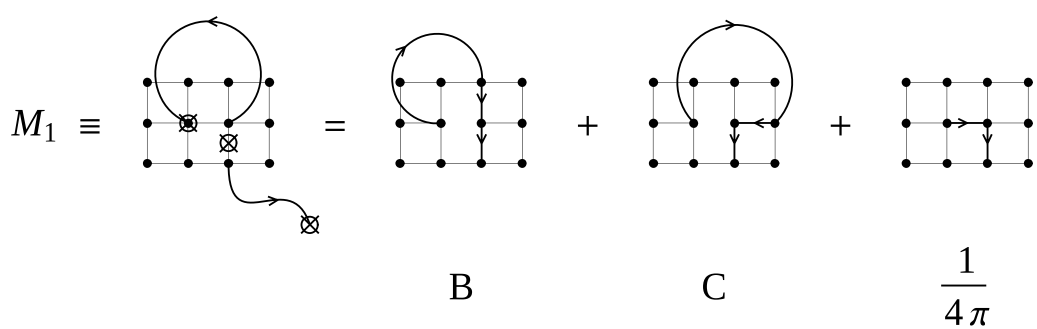

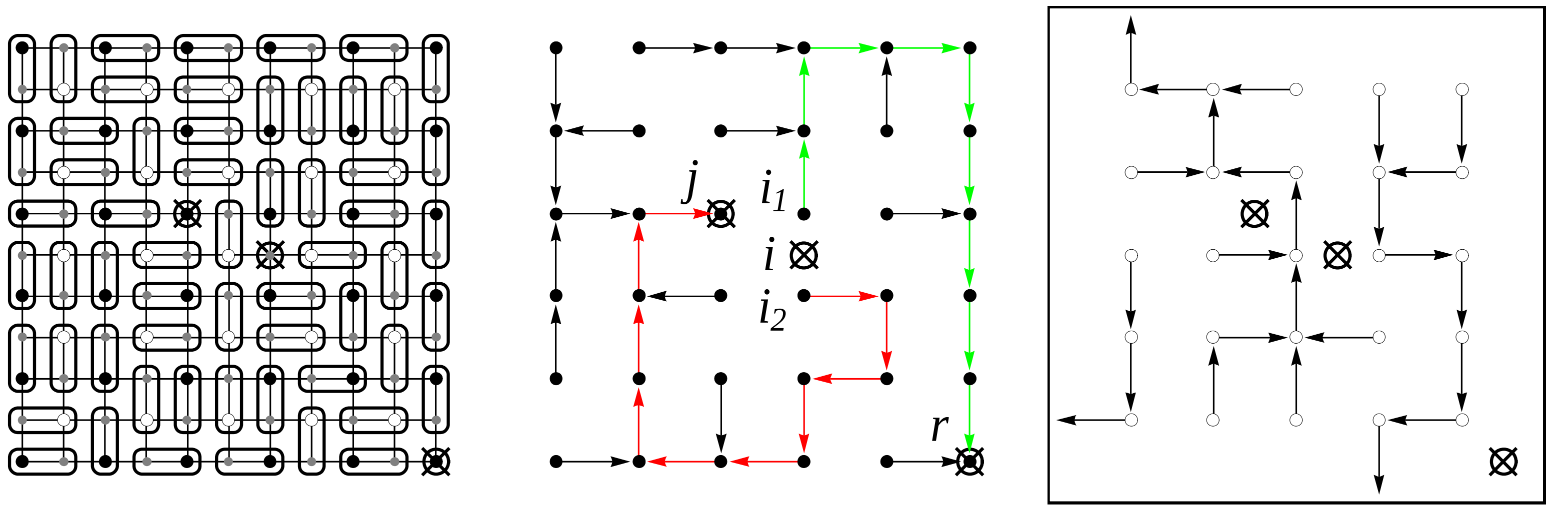

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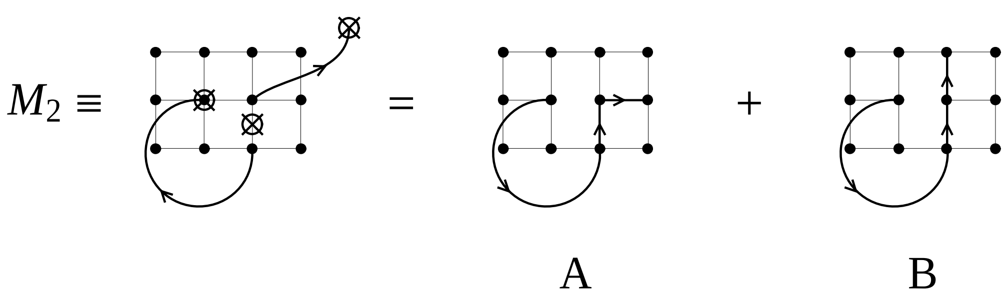

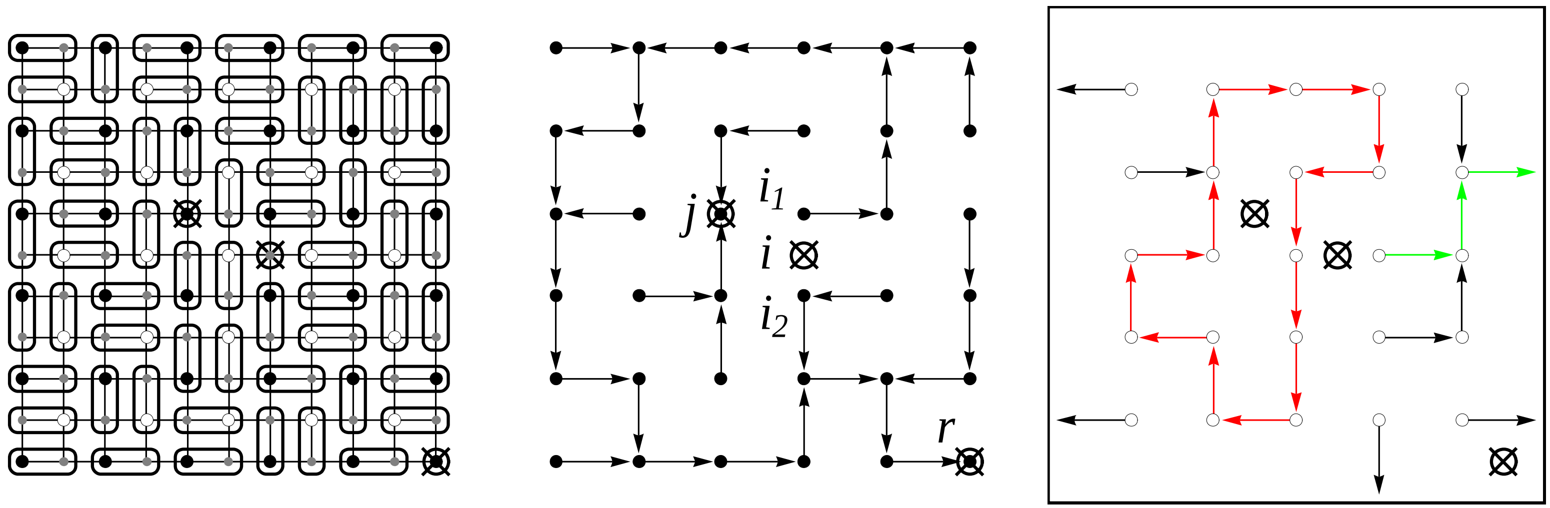

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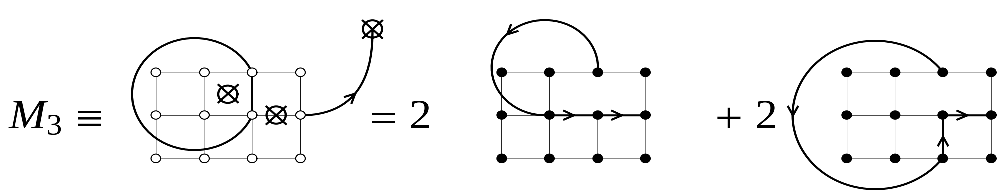

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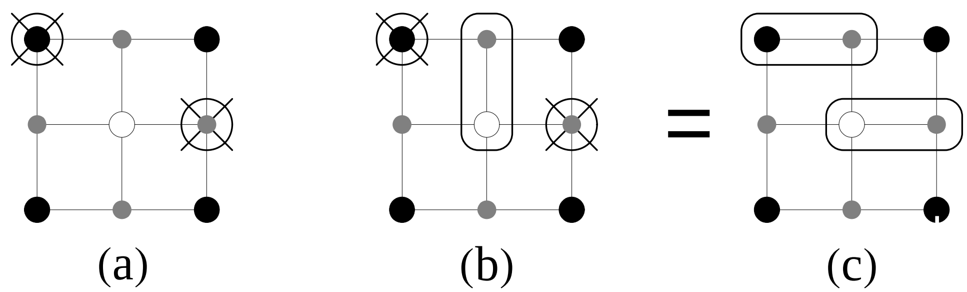
Monomer-monomer correlations and return probability of the LERW



(a) $$P_{\mathrm{mm}} = M_1 + M_2 + M_3 = \frac{1}{4\pi} + A + 4B + 3C \qquad P_{\mathrm{mm}} = \frac{1}{2\pi}$$

(b) = (c) $$P_{\mathrm{mdm}} = M_2 + \frac{1}{2} M_3 = A + 2B + C \qquad P_{\mathrm{mdm}} = \frac{1}{8} - \frac{1}{4\pi}$$

$$\Downarrow$$

$$A = B = \frac{3}{32} - \frac{1}{4\pi} \qquad C = \frac{1}{2\pi} - \frac{5}{32}$$

$$\Rightarrow$$

$$P_{\mathrm{ret}} = \frac{1}{4} + 2(\mathrm{A} + \mathrm{B} + \mathrm{C}) = \frac{5}{16}$$

Discrete Laplace Equation

$$\sum_k G_{i,k}\Delta_{k,j} = \delta_{i,j}$$

$i,\ j, k$ - lattice points

Discrete Lattice Laplacian

$$\Delta_{i,j} = \begin{cases} \deg(i), & i = j \\ -1, & (i,j)-bond \\ 0, & otherwise \end{cases}$$

$$G = \Delta^{-1}$$

Green function

For two dimensional square lattice $r = (x, y), \quad r' = (x', y')$

$$G_{r,r'} = G_{r,r} - \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{1-\cos\left[(x'-x)\alpha+(y'-y)\beta\right]}{2-\cos\alpha-\cos\beta}\,d\alpha\,d\beta$$

Let $R = |j-i|$ - distance between vertices i and j. Then for $R \gg 1$

$$G_{i,j} = G(R) = G_{i,i} - \frac{1}{2\pi}\ln R + \ldots$$

The Generalized Laplacian Matrix

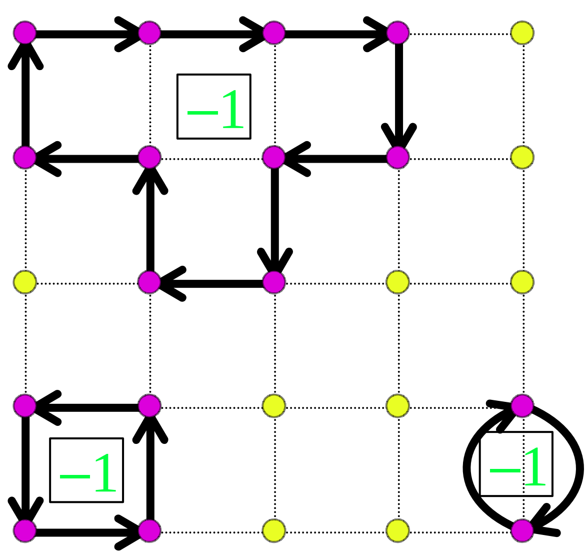
$$\Delta_{i,j} = \begin{cases} z_i, & i = j \\ -x_{i,j}, & (i, j) - bond \\ 0, & otherwise \end{cases}$$

i, j, k - lattice points

$$\Lambda = \det \Delta = \sum_{\Gamma} \chi(\Gamma)$$

Λ - partition function

Γ - Configuration of oriented closed loops



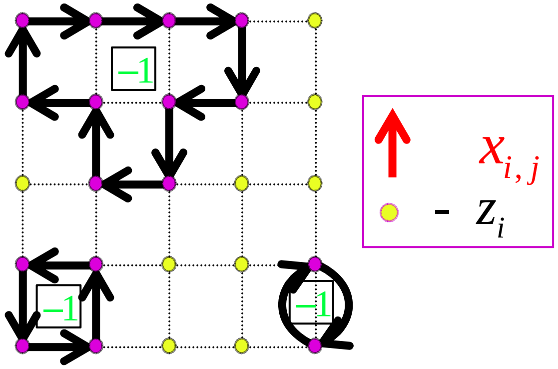
χ - Fermionic weight

each oriented bond $i \to j$: $x_{i,j}$

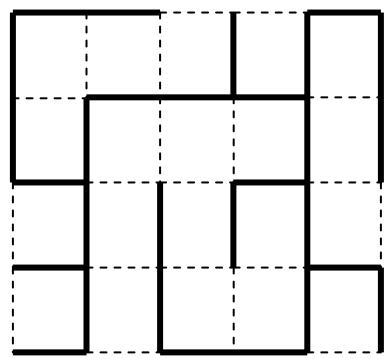
each free site $i \bullet$: z_i

each closed loop (-1)

Spanning tree is a covering graph containing no cycles.
Given a root in arbitrary point, ST becomes oriented.



Configurations of oriented closed loops



ST

Inclusion-Exclusion Principle

N elements and n Properties.

$N_{i_1,i_2,\dots,i_r}$ = # elements with $P(i_1), P(i_2), \dots, P(i_r)$.

N_0 = # elements having non of $P(1), P(2), \dots, P(n)$.

$$N_0 = N - \sum_{i_1} N_{i_1} + \sum_{i_1<i_2} N_{i_1,i_2} - \cdots + (-1)^n N_{1,2,\dots,n}$$

Kirhhoff theorem

Let $P(i)$ be an i-th cycle

Put $z_i = \deg(i)$ $\quad x_{i,j} = 1$

$$\det \Delta = \text{\# Spanning Trees}$$

Correlations

Let Δ' be a matrix which differs from Δ by a finite number of elements, so that $\Delta' - \Delta = B$ consists of several non zero elements, then

$$\frac{\det \Delta'}{\det \Delta} = \det\left(I + BG\right)$$

can be expressed as a determinant of finite size matrix

where $G = \Delta^{-1}$ is the Green function